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A TREATISE ON ELECTRICITY

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A TREATISE ON ELECTRICITY

BY

F. B. PIDDUCK

Fellow of Queen's College, Oxford

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PREFACE

IN the present book I have tried to meet the needs of those readers who require an advanced text covering both the theoretical and practical sides of the subject, as far as this can conveniently be done in a single volume. Though complete in itself, the book is not intended for beginners, who may be supposed to have read one of the excellent elementary treatises available, such as Whetham's *Theory of Experimental Electricity*. Space saved from the detailed treatment of elementary topics has been used to give an account of some of the more recent developments of electricity. It thus comes about that the book falls naturally into two sections; the first eight chapters having been kept fairly simple and containing all the principles necessary for a right appreciation of the subject, while the remaining chapters form introductory accounts of special subjects, which may be consulted by students before starting on treatises devoted to them alone.

I am indebted to Prof. J. S. Townsend for kindly criticising a portion of the manuscript, and to Mr E. W. B. Gill, with whom the book was originally planned, for placing his material at my disposal. Some of the experimental arrangements described are also due to Prof. Townsend. Prof. J. Zenneck kindly sent some photographs of oscillatory discharges, while Mr C. T. R. Wilson lent a number of negatives of his cloud photographs showing the tracks of ionising particles in gases. Blocks for the illustrations have kindly been supplied by Messrs Baird and Tatlock, the Cambridge Scientific Instrument Co., the Chloride Electric Storage Co., Messrs J. J. Griffin and Sons, Messrs Longmans, Green and Co., Messrs Nalder Bros., Messrs Newton and Wright, Messrs W. G. Pye and Co., the British Westinghouse Co., and the Weston Electrical Instrument Co.

F. B. P.

February, 1915.

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ERRATUM

Page 279, line 13. *For* detecting, but not measuring, *read* detecting or measuring.

CHAPTER I

MATHEMATICAL INTRODUCTION

1. Right-handed co-ordinate axes. If Ox, Oy, Oz are the positive directions of a set of rectangular axes in three dimensions, and Ox, Oy are fixed, the third line Oz may be drawn at right angles to the plane xOy in two directions. Let the axes be moved so that Ox points due east, and Oy due north. Then the axes are called *right-handed* if Oz points vertically upwards. Fig. 1 shows a right-handed set in perspective, the axis of y pointing

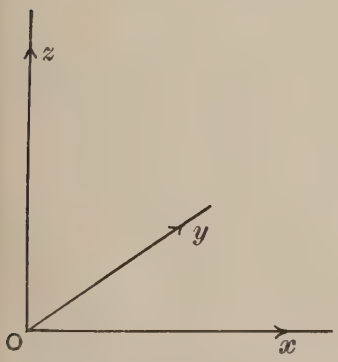


Fig. 1

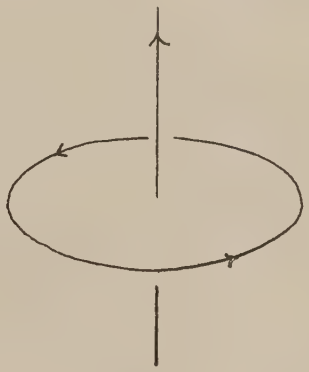


Fig. 2

away from the observer. Only right-handed axes are used in this book.

Imagine an ordinary (right-handed) corkscrew to be pointed along Ox and turned so as to travel in the positive direction of Ox. A quarter-turn will then bring Oy to the position Oz, and similarly a quarter-turn about Oy brings Oz to Ox, and a quarter-turn about Oz brings Ox to Oy. The rule of derivation of the rotation of the right-handed corkscrew from the direction in which it points is called the *right-handed screw law*. Fig. 2 gives another representation of this law.

2. Vectors. A vector may be described briefly as a directed magnitude. It must be capable of being represented completely by a line of suitable length drawn in a suitable direction, and must further obey the "parallelogram law"; i.e. the joint effect of two vectors represented by the lines PA, PB must be a vector represented by PD the diagonal of the parallelogram drawn with PA, PB as adjacent sides. The velocity and acceleration of a particle, and the force acting on it, are examples of vectors. By analogy we speak of the *components* u_x, u_y, u_z of a vector u in the directions of the co-ordinate axes.

If u, v are two vectors, the quantity $u_x v_x + u_y v_y + u_z v_z$ is called their *scalar product*. If u is the value of a single vector at the point (x, y, z), $\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$ is called the *divergence* of u, and written div u.

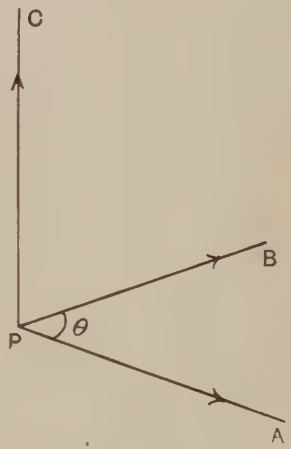


Fig. 3

Returning to the vectors u, v, let us represent them by the lines PA, PB in Fig. 3, in which PB is shown receding from the observer. A line PC is drawn perpendicular to the plane APB in the direction shown, and of length

$$PA \,.\, PB \sin\theta.$$

Then PC represents a vector called the *vector product* of u and v. The direction of PC is easily remembered by observing that, although PA, PB, PC are not all mutually perpendicular, they still have the same general configuration as a set of right-handed axes in space. In fact, if the line Oy in Fig. 1 was bent towards Ox till the angle xOy became θ, the lines Ox, Oy, Oz could be moved so as to coincide with PA, PB, PC in this order.

The reader should notice the following important theorem:

If (u_x, u_y, u_z) and (v_x, v_y, v_z) are the components of two vectors u, v, then the components of their vector product are

$$(u_y v_z - u_z v_y, \quad u_z v_x - u_x v_z, \quad u_x v_y - u_y v_x),$$

or symbolically

$$\left\| \begin{matrix} u_x, & u_y, & u_z \\ v_x, & v_y, & v_z \end{matrix} \right\|.$$

Writing

$$w_x = u_y v_z - u_z v_y, \quad w_y = u_z v_x - u_x v_z, \quad w_z = u_x v_y - u_y v_x,$$

we have

$$w_x u_x + w_y u_y + w_z u_z = w_x v_x + w_y v_y + w_z v_z = 0.$$

Hence the vector $w = (w_x, w_y, w_z)$ is perpendicular to u and v and therefore parallel to PC. Let W be its magnitude and let also $U = PA$ be the magnitude of u and $V = PB$ the magnitude of v. If the *true* direction-cosines of PA, PB are (l, m, n) and (l', m', n') respectively, we have

$$u_x = lU, \quad u_y = mU, \quad u_z = nU,$$
$$v_x = l'V, \quad v_y = m'V, \quad v_z = n'V.$$

Hence by definition we have three equations of the type

$$w_x = UV\,(mn' - m'n).$$

Hence

$$W^2 = U^2V^2\{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2\}$$
$$= U^2V^2 \sin^2\theta$$

by a well-known theorem of solid geometry. It remains to verify that w is drawn in the direction specified above: and to settle this it is sufficient to consider a special case, namely when u coincides with the axis of x and v with the axis of y. Taking $u_x = 1$, $u_y = u_z = 0$, $v_x = 0$, $v_y = 1$, $v_z = 0$, we find $w_x = w_y = 0$ and $w_z = +1$. The direction of w is therefore correct.

If u is a single vector, the vector whose components are

$$\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}, \ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}, \ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

is called the *curl* of u.

3. Gauss' transformation*. Let S be a closed surface, enclosing a space that will be called τ. Take three quantities u_x, u_y, u_z, functions of the co-ordinates x, y, z of a point in the regions considered. Let $d\tau$ be an element of volume of τ, dS an element of area of S, and denote by l, m, n the direction-cosines of the normal to dS, drawn so as to point away from τ. Then

$$\int (lu_x + mu_y + nu_z)\, dS = \int \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) d\tau \ldots (1),$$

* This is often attributed in English text-books to Green: it was, however, discovered by Gauss in 1813.

the surface-integrals extending over the whole of S, and the volume-integrals throughout τ.

Take a point P $(0, y, z)$ and round it draw the element of area (dy, dz) in the plane $x = 0$. Draw the prism PQ on this base with its generators parallel to the axis of x, cutting S in the elements of surface dS_1, dS_2, ..., of which for simplicity six are taken.

We start by finding the contribution of the parts of the prism within S to the integral $\int \frac{\partial u_x}{\partial x} d\tau$ or $\iiint \frac{\partial u_x}{\partial x} dx\, dy\, dz$. This is

$$dy\,dz \left[\int \frac{\partial u_x}{\partial x} dx \right],$$

or

$$dy\,dz\,(u_{x2} - u_{x1} + u_{x4} - u_{x3} + u_{x6} - u_{x5}),$$

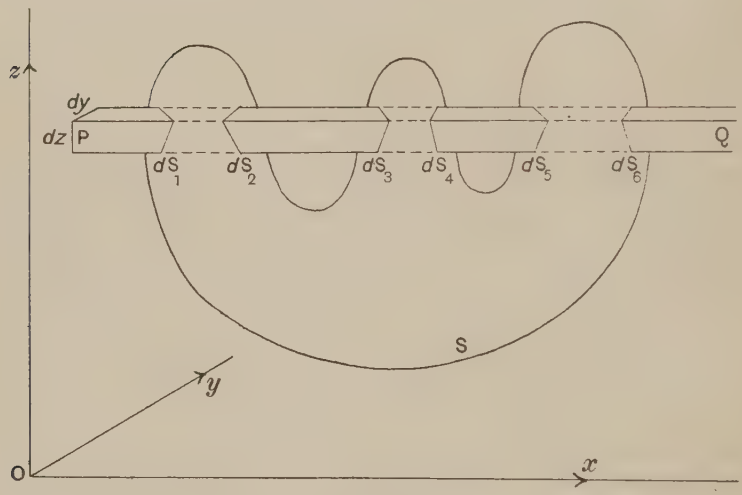


Fig. 4

u_{xr} standing for the value of u_x at dS_r. Let (l_r, m_r, n_r) be the corresponding outward normal. The projection of dS_1 on the plane $x = 0$ is clearly $dydz$, and it is evident from the figure that l_1 is negative, otherwise the prism would not enter the surface. Hence $dydz = -l_1 dS_1$. Similarly we have $dydz = -l_2 dS_2$. Thus the contribution of the prism to $\int \frac{\partial u_x}{\partial x} d\tau$ is

$$l_2 u_{x2} dS_2 + l_1 u_{x1} dS_1 + l_4 u_{x4} dS_4 + l_3 u_{x3} dS_3 + l_6 u_{x6} dS_6 + l_5 u_{x5} dS_5,$$

which is the same as its contribution to $\int l u_x dS$. By moving the

prism about we can include the whole surface and volume, and we have therefore

$$\int l u_x \, dS = \int \frac{\partial u_x}{\partial x} \, d\tau.$$

Similarly

$$\int m u_y \, dS = \int \frac{\partial u_y}{\partial y} \, d\tau,$$

and

$$\int n u_z \, dS = \int \frac{\partial u_z}{\partial z} \, d\tau.$$

Adding, we have the theorem. The method of proof is evidently general. Also no doubt can arise as to its validity provided that u_x, u_y, u_z are finite and continuous, and that $\partial u_x/\partial x$, $\partial u_y/\partial y$, $\partial u_z/\partial z$ are finite, for all points on or inside the surface.

If we regard u_x, u_y, u_z as the components of a vector, $l u_x + m u_y + n u_z$ is the normal component. Hence the surface-integral of the normal component of a vector over a surface is equal to the volume-integral of its divergence extended throughout the enclosed space.

From Gauss' transformation we can deduce *Green's reciprocal theorem*, namely that

$$\int (U \Delta V - V \Delta U) \, d\tau = \int \left(U \frac{\partial V}{\partial \nu} - V \frac{\partial U}{\partial \nu} \right) dS,$$

where Δ stands for the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ (Laplace's operator) and $\partial/\partial \nu$ for $l\partial/\partial x + m\partial/\partial y + n\partial/\partial z$, an operation known as "differentiation along the normal." Here U and V may be any two functions of x, y, z provided that the first derivates are finite and continuous and the second derivates finite in the region considered. Putting in Gauss' transformation

$$u_x = U \frac{\partial V}{\partial x}, \quad u_y = U \frac{\partial V}{\partial y}, \quad u_z = U \frac{\partial V}{\partial z}$$

we have

$$\int U \frac{\partial V}{\partial \nu} \, dS = \int U \Delta V \, d\tau + \int \left(\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} + \frac{\partial U}{\partial z} \frac{\partial V}{\partial z} \right) d\tau.$$

Interchanging U and V, we have

$$\int V \frac{\partial U}{\partial \nu} \, dS = \int V \Delta U \, d\tau + \int \left(\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} + \frac{\partial U}{\partial z} \frac{\partial V}{\partial z} \right) d\tau,$$

from which Green's reciprocal theorem follows by subtraction. If the quantity under the surface-integral vanishes at infinity in the order $1/r^3$ or more, Green's theorem may be applied to the whole of space, giving $\int(U\Delta V - V\Delta U)\, d\tau = 0$.

4. Stokes' Theorem. Let C be a closed curve in space. Draw any unclosed surface S having C as its rim; that is, having the same relation to C that the membrane of a drum has to its boundary. Let l, m, n be the direction-cosines of the normal at the surface-element dS of S. Then

$$\int u_x dx + u_y dy + u_z dz$$
$$= \int \left[l\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + m\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + n\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right] dS \quad (2),$$

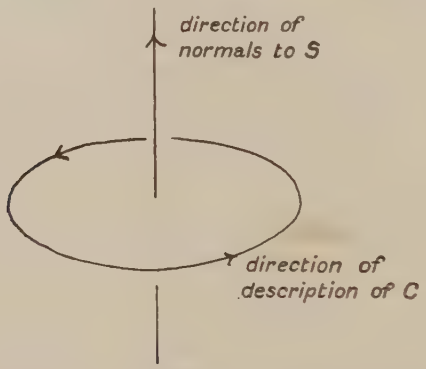


Fig. 5

where the line-integral is taken completely round the circuit C and the surface-integral all over the surface S.

Here dS cannot be said to have an outward normal. The proper direction of the normal is defined by the following *right-handed screw rule*:

The direction in which the circuit is traversed is derived from the direction of the normals to the surface by the right-handed screw law.

The proof of the theorem can be reduced to the simpler one in which the contour C is small by dissecting the cap S into any number of smaller ones (Fig. 6). Consider a number of the resulting segments S_1, S_2, S_3, S_4, forming a composite segment $LMNP$. Evidently the surface-integral over $LMNP$ is the sum of those over S_1, S_2, S_3, S_4. The same is true of the line-integral round

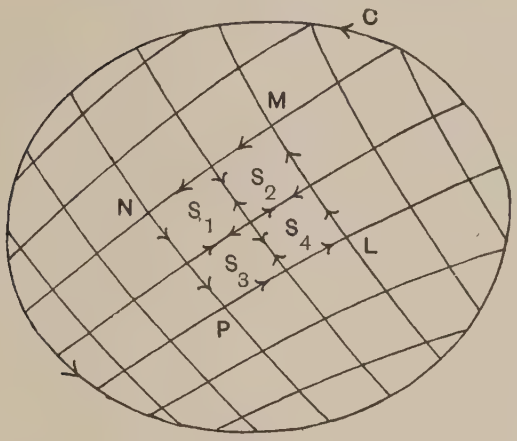


Fig. 6

the edge, because the common boundaries of S_1, S_2, S_3, S_4 are described in opposite directions in the separate integrals, and disappear from the sum. Thus Stokes' theorem will be true in general if it is true for a small contour, which we may take in the vicinity of the point (x, y, z).

At a point (x', y', z') on the contour, the value of u_x is approximately

$$u_x' = u_x + (x' - x)\frac{\partial u_x}{\partial x} + (y' - y)\frac{\partial u_x}{\partial y} + (z' - z)\frac{\partial u_x}{\partial z}.$$

The line-integral in Stokes' theorem is $\int u_x'dx' + u_y'dy' + u_z'dz'$. Terms like $\int dx'$ and $\int x'dx'$ taken round the whole contour vanish; hence

$$\int u_x'dx' = \frac{\partial u_x}{\partial y}\int y'dx' + \frac{\partial u_x}{\partial z}\int z'dx'.$$

Further, $\int y'dz' + z'dy' = 0$ since $y'z'$ resumes its original value after traversing the contour. Thus

$$\begin{aligned}\int u_x'dx' + u_y'dy' + u_z'dz' &= \frac{\partial u_x}{\partial y}\int y'dx' + \frac{\partial u_x}{\partial z}\int z'dx' \\ &\quad + \frac{\partial u_y}{\partial z}\int z'dy' + \frac{\partial u_y}{\partial x}\int x'dy' \\ &\quad + \frac{\partial u_z}{\partial x}\int x'dz' + \frac{\partial u_z}{\partial y}\int y'dz' \\ = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\int y'dz' &+ \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\int z'dx' + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\int x'dy'.\end{aligned}$$

Now $\int y'dz'$ is the area of projection of dS on the plane $x = 0$, and is therefore $\pm ldS$. We have to show that, with our rule of directions, the positive sign is to be taken. In Fig. 7, if l is positive the contour is described so that the area of the projected curve is always kept to the left. Hence l and $\int y'dz'$ are positive together. Since $\int y'dz' = ldS$, $\int z'dx' = mdS$, and $\int x'dy' = ndS$, we have proved the theorem for the small contour.

5. Flux of a vector. We shall often have occasion to deal with a portion of a surface with a definite rim, such as that

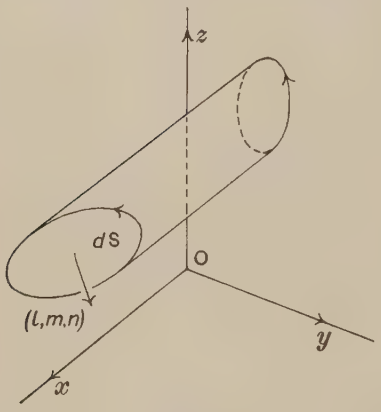


Fig. 7

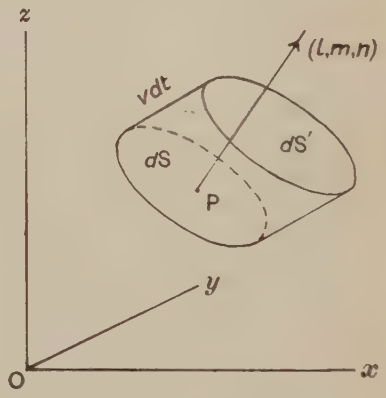


Fig. 8

occurring in Stokes' theorem. If S is such a portion of a surface, the surface-integral of the normal component of a vector extended over S is called the *flux of the vector through S*. The term *flux* does not always express a flow of a quantity; but the name is derived from the notion of flow, as may be seen by the following. Take a point $P(x, y, z)$ in a fluid (Fig. 8), and round it describe an element of surface dS whose normal is (l, m, n). In the neighbourhood of P let ρ be the density and (v_x, v_y, v_z) the velocity of the fluid. The fluid which at time t was at dS will have moved in the short interval dt a distance vdt, so as to lie on a new surface-element dS'. If θ is the angle between v and (l, m, n) the amount of fluid that passes through dS in time

dt is equal to $\rho \times$ the volume of the cylinder between the two elements

$$= \rho \,.\, dS \,.\, v dt \cos\theta,$$
$$= \rho\,(lv_x + mv_y + nv_z)\, dS dt.$$

Hence if we have a finite surface S with a definite rim, the rate at which fluid is crossing it is $\int_S (l\rho v_x + m\rho v_y + n\rho v_z)\, dS$, that is, the *flux of the vector* ρv. This result is important in the theory of electric currents.

In some cases the flux of a vector through a surface S depends on the form of its bounding edge only, being the same for all surfaces S with a given rim. For example, let S and S' be two surfaces with a rim C, and let the components of the vector u satisfy the identical relation $\dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y} + \dfrac{\partial u_z}{\partial z} = 0$. S and S' together form a closed surface to which Gauss' transformation may be applied; but since the normal of one surface must be reversed in order to make it an outward normal the transformation gives

$$\int_S (lu_x + mu_y + nu_z)\, dS = \int_{S'} (lu_x + mu_y + nu_z)\, dS.$$

Thus the fluxes calculated for S and S' are the same; that is, the flux depends only on the rim C.

Vectors with the above property (the divergence vanishing) are sometimes called *solenoidal*. The vector ρv in steady fluid motion evidently satisfies this condition, for when the motion is steady no accumulation of fluid can go on between any two surfaces S, S', so that the flux through S must equal that through S'.

The curl of every vector is easily seen to be solenoidal, either directly or by considering that Stokes' theorem expresses its flux through a surface in terms of the edge alone.

An important general equation may be noticed in connexion with the flow of fluids. If the motion is not steady the rate at which fluid is entering a surface S, namely

$$-\int_S (l\rho v_x + m\rho v_y + n\rho v_z)\, dS,$$

must be accounted for by the rate of increase of the mass of fluid inside S, that is $\frac{\partial}{\partial t}\int_\tau \rho\, d\tau$. Hence

$$-\int_S (l\rho v_x + m\rho v_y + n\rho v_z)\, dS = \int_\tau \frac{\partial \rho}{\partial t}\, d\tau.$$

Transforming the left-hand side by Gauss' theorem and comparing results, we have

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0,$$

which is known somewhat inappropriately as the *equation of continuity*.

6. The equation of wave-motion. The partial differential equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} \quad \ldots\ldots\ldots\ldots (3)$$

imposes a certain restriction on θ as a function of x, y, z and t. It implies that if the quantity θ is at time $t = 0$ distinct from zero within a certain region of space, this "disturbed" region advances in all directions with velocity c. For this reason the equation is called the "equation of wave-motion," c being the velocity of propagation of all the waves.

The proof of this result in general would occupy too much space. We shall verify the property in two cases. Firstly, let θ depend on x and t only. Then

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}.$$

The integral of this equation is $\theta = f(x - ct) + \phi(x + ct)$, where f and ϕ are arbitrary functions. The term $f(x - ct)$ is unchanged as long as $x - ct$ is, and therefore represents motion with velocity c along the axis of x. Similarly the other term represents motion back along the axis of x, without change of type.

Secondly, let θ depend only on r and t, where $r \equiv (x^2 + y^2 + z^2)^{\frac{1}{2}}$ is the distance of (x, y, z) from the origin. We have $\frac{\partial \theta}{\partial x} = \frac{x}{r}\frac{\partial \theta}{\partial r}$, as may be most easily seen by regarding $-\theta$ as the potential energy of a particle placed at (x, y, z) in a field of force, in which

case the radial force $\frac{\partial\theta}{\partial r}$ is equivalent to the forces $\frac{\partial\theta}{\partial x}$, $\frac{\partial\theta}{\partial y}$, $\frac{\partial\theta}{\partial z}$. Proceeding further, we have

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\theta}{\partial r}\right),$$

and therefore

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{3}{r}\frac{\partial\theta}{\partial r} + r\left(-\frac{1}{r^2}\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial^2\theta}{\partial r^2}\right).$$

The equation of wave-motion becomes

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2},$$

or

$$\frac{\partial^2(r\theta)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(r\theta)}{\partial t^2}.$$

Thus $r\theta = f(r - ct) + \phi(r + ct)$, representing waves diverging and converging radially with velocity c. The solution for waves travelling outwards is

$$\theta = \frac{1}{r}f\left(t - \frac{r}{c}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Still more special solutions of the equation of wave-motion are found by giving particular forms to the function f. Thus if $f(t) = A\cos pt$ (4) becomes

$$\theta = \frac{A}{r}\cos p\left(t - \frac{r}{c}\right),$$

showing how the amplitude of an expanding spherical wave depends on the distance from the origin. This result is valuable in the theory of sound.

7. Bessel functions of zero order. The function

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{1^2 \cdot 2^2}\left(\frac{x}{2}\right)^4 - \frac{1}{1^2 \cdot 2^2 \cdot 3^2}\left(\frac{x}{2}\right)^6 + \ldots \quad (5)$$

is called a Bessel function of zero order. It is a particular case of the general Bessel function

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

which, however, we shall not use. The values of $J_0(x)$ and its derivative $J_0'(x)$ have been tabulated and may be regarded as

known functions. Fig. 9 shows the curve $y = J_0(x)$ from $x = 0$ to $x = 25$. Its form suggests very strongly that J_0 has an infinite number of real roots. We shall have occasion to assume this later.

It is easy to see that $J_0(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0,$$

which is known as Bessel's differential equation. It is generally more convenient to consider, instead of $J_0(x)$, the quantity $J_0(kr)$,

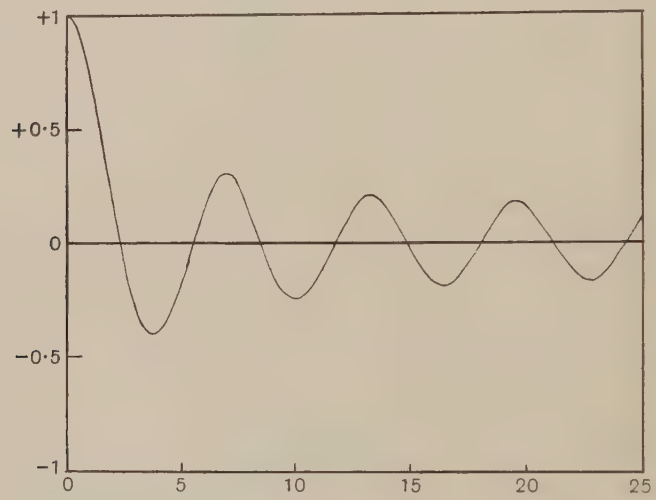


Fig. 9

regarded as a function of r, and call it J_0 for short. It is then necessary to distinguish between J_0', which is $\frac{dJ_0(kr)}{d(kr)}$, and $\frac{dJ_0}{dr}$. The differential equation becomes

$$r\frac{d^2J_0}{dr^2} + \frac{dJ_0}{dr} + k^2rJ_0 = 0 \quad \ldots\ldots\ldots\ldots(6).$$

Writing this in the form $k^2rJ_0 = -\frac{d}{dr}\left(r\frac{dJ_0}{dr}\right)$ and integrating, we have

$$\int_0^r rJ_0\,dr = -\frac{r}{k^2}\frac{dJ_0}{dr} = -\frac{r}{k}J_0' \quad \ldots\ldots\ldots\ldots(7).$$

A second useful formula is

$$(k'^2 - k^2)\int_0^r rJ_0(kr)\,J_0(k'r)\,dr = rJ_0(k'r)\frac{dJ_0(kr)}{dr} - rJ_0(kr)\frac{dJ_0(k'r)}{dr} \quad \ldots(8).$$

To verify this we notice that

$$\frac{d}{dr}\left[rJ_0(k'r)\frac{dJ_0(kr)}{dr} - rJ_0(kr)\frac{dJ_0(k'r)}{dr}\right]$$
$$= J_0(k'r)\frac{dJ_0(kr)}{dr} - J_0(kr)\frac{dJ_0(k'r)}{dr}$$
$$+ rJ_0(k'r)\frac{d^2J_0(kr)}{dr^2} - rJ_0(kr)\frac{d^2J_0(k'r)}{dr^2}$$
$$= -k^2rJ_0(k'r)\,J_0(kr) + k'^2rJ_0(kr)\,J_0(k'r) \text{ from (6).}$$

The result (8) follows at once. Similarly we can verify the formula

$$\left.\begin{aligned} 2\int_0^r rJ_0^2(kr)\,dr &= r^2\left[J_0^2(kr) + \frac{1}{k^2}\left\{\frac{dJ_0(kr)}{dr}\right\}^2\right] \\ &= r^2\left[J_0^2(kr) + J_0'^2(kr)\right] \end{aligned}\right\} \quad \ldots(9).$$

If k is a root of the equation $J_0(kb) = 0$, (9) gives

$$\int_0^b rJ_0^2(kr)\,dr = \tfrac{1}{2}b^2J_0'^2(kb) \ldots\ldots\ldots(10);$$

and similarly if k and k' are two *distinct* roots of the same equation,

$$\int_0^b rJ_0(kr)\,J_0(k'r)\,dr = 0 \ldots\ldots\ldots\ldots(11)$$

from (8).

As an example of these formulae we may determine the coefficients in the expansion

$$f(r) = C_1J_0(k_1r) + C_2J_0(k_2r) + \ldots + C_nJ_0(k_nr) + \ldots \quad (12)$$

in cases in which such an expansion is valid, k_1, k_2 ... being the roots of the equation $J_0(kb) = 0$. Multiply equation (12) by $rJ_0(k_nr)$ and integrate from $r = 0$ to $r = b$. Equation (11) shows that the only term on the right is that arising from $C_nJ_0(k_nr)$, so that

$$\int_0^b rJ_0(k_nr)\,f(r)\,dr = C_n\int_0^b rJ_0^2(k_nr)\,dr$$
$$= \tfrac{1}{2}b^2C_nJ_0'^2(k_nb).$$

Hence $$C_n = \frac{2}{b^2 J_0'^2(k_n b)} \int_0^b r J_0(k_n r) f(r)\, dr \ldots\ldots\ldots (13).$$

In particular, if

$$\left.\begin{aligned} f(r) &= n_0, \text{ a constant, for } r < a \\ &= 0 \text{ for } a < r < b \end{aligned}\right\}$$

we have $$C_n = \frac{2n_0}{b^2 J_0'^2(k_n b)} \int_0^a r J_0(k_n r)\, dr$$

$$= -\frac{2an_0}{b^2} \frac{J_0'(k_n a)}{k_n J_0'^2(k_n b)}.$$

CHAPTER II

PERMANENT MAGNETISM

8. Magnets. There are a number of bodies which are capable of attracting iron to a greater or less degree. These are called *magnets*. The mineral magnetite, which contains considerable quantities of an oxide of iron (Fe_3O_4), is a natural magnet; other substances can be made into magnets, or *magnetised*. Of pure metals, iron, cobalt and nickel alone show this property to a marked degree; iron is the best of the three, the relative efficacy of the three metals being roughly in the ratio 3 : 2 : 1. It is usual to enumerate various methods by which iron can be magnetised by contact either with a natural or with an artificial magnet; but by far the best and simplest way is by means of the electric current. An iron rod can be magnetised very easily by winding it round with cotton-covered copper wire until the wire forms a spiral covering it closely from end to end, and then joining the ends of the wire to the terminals of an electric battery, for example a Daniell cell. While the current is flowing the rod is strongly magnetised, and a portion of its power remains when the current is cut off. Care must be taken to pass a sufficiently strong current or to wind a sufficiently large number of turns of wire round the rod.

The capacity of iron to retain its magnetism after the current is cut off may be called its residual power; and the extent of this depends very much on the treatment the iron has previously received. For those purposes for which small residual power is required wrought iron may be used, as it loses most of the temporary magnetism given it by the current when it is subjected to vibration or rough usage. For permanent magnets, on the other hand, the qualities required are high residual power and

great resistance to demagnetisation by external magnets or by rough usage; and this is secured by using hard steel. As many standard measuring instruments at the present day depend on the constancy of the magnets they contain, special precautions have to be used to secure this. The magnets are often made of tungsten steel, hardened from a temperature of about 800° C. After magnetisation they are exposed to violent shocks and changes of temperature, so that whatever magnetism remains is very tenacious and not disturbed by the further slight disturbances met with in ordinary work. This process is known as "artificial ageing."

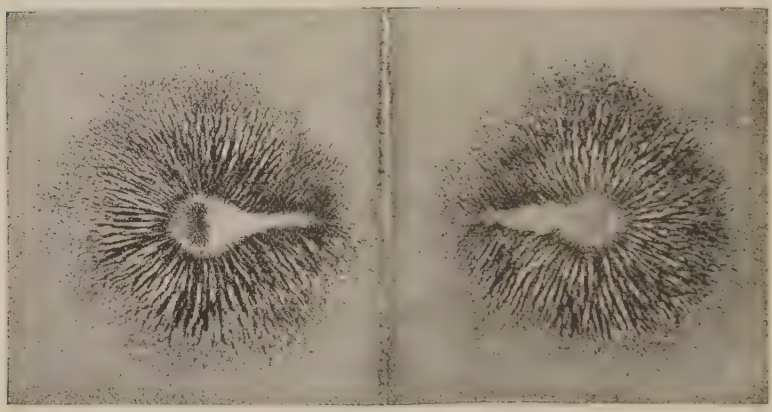


Fig. 10

Iron and steel magnets when heated to a bright red heat lose their magnetisation, which they do not regain on cooling.

A convenient way of ascertaining whether a body is a magnet or not is to dip it into iron filings. It is found that the filings do not adhere to all parts of the body, showing that the magnetic property is located in certain regions and not diffused throughout the body. If a long knitting-needle is carefully magnetised it will be found that filings will only adhere to it in the neighbourhood of the ends. The two points round which they cluster are called the *poles* of the magnet. The two poles have however essentially distinct properties, which may be examined as follows: Magnetise two knitting-needles AB, $A'B'$ similarly, so

that the end A corresponds to A' and B to B'. It is found that A repels A' and B repels B', while A attracts B' and B attracts A'. Like poles repel, unlike poles attract.

The ideal magnet is regarded as possessing two point-poles, approximately realised when a long steel bar is magnetised. As early as 1769 Robison showed that ball-ended magnets gave a better approximation, and such magnets have been reintroduced recently by Searle. If the ends of a steel rod are screwed into steel balls of about a centimetre diameter, and the whole magnetised, the magnetic property is found to reside almost entirely in the steel balls. This is shown on dipping such a magnet into iron filings, when hardly any of the filings adhere to the connecting steel rod. Moreover, the magnetic power is almost uniformly distributed over the spheres, as may be seen from Fig. 10, which represents the distribution of filings sprinkled on a card held over a ball-ended magnet. We thus acquire the notion of some influence emanating radially from the poles in all directions, which will shortly be made more precise.

A body magnetised at random may deviate considerably from the ideal of a body with two point-poles, but it is not desirable to reject the idea of poles. A body of this kind is regarded as an assemblage of like and unlike poles variously located inside it. We cannot isolate the opposite poles in any way: for if a bar-magnet is broken a new pole is formed at each of the fresh ends, and each half after separation behaves like a new bar-magnet.

9. Induced magnetisation. A magnet always attracts a piece of ordinary iron, although the like poles of two magnets repel one another. The explanation of this can be found by a simple experiment. Lay a steel magnet AB in the same straight line with a rod CD of wrought iron, previously unmagnetised. As long as AB is there the rod CD will attract iron filings, which

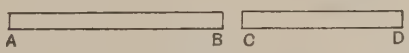


Fig. 11

will adhere to it mainly at the two ends, but the power of attracting such filings is almost entirely lost when AB is removed. This

shows that the rod CD has temporarily become a magnet, and the rod is said to be magnetised by induction. Further, we can easily test the nature of the induced poles by observing whether a particular end, say D, repels or attracts one end of a magnet similar to AB. In this way we can show that the pole induced at C in the wrought iron rod is similar to the pole A of the original magnet, while the pole at D is like the pole B. The process of attraction therefore consists primarily in the magnetisation of the rod by induction, and then B attracts the unlike pole C, the other poles having less effect on account of their being further off.

If a pole of a magnet is brought very near a weak pole of the same kind attraction may result, instead of the weak repulsion observed at moderate distances. Induced magnetism may thus overpower the ordinary or residual kind. In fact, magnetisation is always induced in a magnet to some extent when other magnets are near, although the induced effects are relatively much greater with wrought iron than with hardened steel. With steel magnets in ordinary experiments the induction effect is negligible in comparison with the permanent actions, and the magnetic properties of good magnets remain unaltered over considerable periods of time. For the present we need only consider magnets of invariable quality, which are called *permanent magnets*. We shall have occasion in a later chapter to form an estimate of the extent to which induction takes place in a magnet under the conditions of ordinary use.

10. Strength of a magnetic pole. The important question arises as to what number can be assigned to the pole of a permanent magnet which shall express its power of being attracted or repelled by other poles. The question is a theoretical one, since we cannot experiment, except approximately, on ideal poles: and moreover it is not very convenient to measure directly the forces between magnets. To obtain some notion of the strength of an ideal pole we may consider an ideal experimental arrangement and suppose the resulting forces to be known, merely for the sake of definition and without implying that it would be easy to realise the theoretical conditions in practice.

Suppose that we have any number of magnets distributed

in a region in any manner. Such a region, that is a region of magnetic attractions and repulsions, is called a *magnetic field*. If a new pole is introduced and placed at an assigned point P of the field, it will in general be acted on by a certain force due to the magnets in the field: and we shall make the natural assumption that the line of action of this force is the same whatever pole is placed at P. Suppose now that we place a number of poles in succession at P, and observe the magnitudes $F_1, F_2, F_3, \ldots$ of the forces acting on them, regard being had to the sign of the forces: then the *strengths* of the various poles are defined to be in the ratios $F_1 : F_2 : F_3$, etc. We may thus select any of the poles and call it the *unit pole*, or pole of unit strength, and the strength of any other pole is then known when the force on it is measured.

If the above definition is to be useful it must give to a particular pole of a permanent magnet the same strength under all conditions. We shall thus have to postulate further that the ratios of the forces F_1, F_2, F_3, ... are independent of the particular field used in their determination.

In order to determine pole-strengths in absolute measure, the strength $+1$ is assigned to the chosen unit pole, and other strengths determined by the above definition. We have only to imagine a field produced by a single magnetic pole to see that poles which are *like* the chosen unit pole have positive strengths, while the strengths of poles which are *unlike* it are negative. We shall for this reason speak in future of *positive* and *negative* magnetic poles, and in saying that the strength of a particular pole is m we understand that m has sign as well as magnitude.

11. Magnetic force and potential. Common observation shows that magnets exert more force on one another when they are near together than when far apart, so that a magnetic pole in general experiences different forces when placed at different points of a magnetic field. Taking a set of co-ordinate axes, let a unit pole be placed at the point $P(x, y, z)$. The field is completely specified by the mechanical force (H_x, H_y, H_z) with which it acts on this unit pole, and the vector H whose components are (H_x, H_y, H_z) is called the *magnetic force* at P. It follows that a pole of strength m placed at the same point would experience a force

(mH_x, mH_y, mH_z). The reader should bear in mind that the term *magnetic force* is always used in the above precise sense, and never loosely for the force acting on any magnetic pole.

If (H_x, H_y, H_z) are the same at all points, that is if H is constant in magnitude and direction, the field is said to be uniform.

The mutual forces between magnetic poles will be assumed to be of the type known in mechanics as *conservative*, and the potential energy of a unit pole when lying at a point P in a field is called the *magnetic potential* at P. Thus if the magnetic potential at P is Ω exactly that amount of work would be done by the forces of the field when a unit pole is taken from P to an assigned point. The assigned point is conveniently taken at an infinite distance from all the magnets in the field.

It is easy to show that a magnetic field is completely specified when Ω is given at all points of it. Let a unit pole be moved from the point (x, y, z) to the near point $(x + dx, y, z)$. The forces of the field do work $H_x dx$ on the pole, and this must be accounted for by the decrease $-d\Omega = -\dfrac{\partial \Omega}{\partial x} dx$ of potential energy. Hence $H_x dx = -\dfrac{\partial \Omega}{\partial x} dx$, and similar results hold for small displacements parallel to the axes of y and z.

Thus $$H_x = -\frac{\partial \Omega}{\partial x}, \quad H_y = -\frac{\partial \Omega}{\partial y}, \quad H_z = -\frac{\partial \Omega}{\partial z} \ldots\ldots (1),$$

so that H is known when Ω is given as a function of x, y, and z.

It is clear from the definition that magnetic forces are compounded according to the parallelogram law. Thus if two fields of strengths H and H' are superposed at right angles to one another the resultant field is of magnitude $(H^2 + H'^2)^{\frac{1}{2}}$ and makes an angle $\theta = \tan^{-1}(H'/H)$ with the direction of H. In a field specified by components (H_x, H_y, H_z) the magnitude of the resultant is given by

$$H^2 = H_x^2 + H_y^2 + H_z^2 = \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 + \left(\frac{\partial \Omega}{\partial z}\right)^2.$$

12. A magnet in a uniform magnetic field. The tendency of a compass-needle to set in a fixed direction shows that there is

a natural magnetic field at points on the earth's surface. This field, in the absence of special disturbing circumstances, may be regarded as uniform over the small space occupied by an ordinary room. Theoretically, we might examine the effect of the earth's field on a magnet by supporting the latter at its centre of gravity and thus eliminating the effects of gravitation. This is however not very convenient, and in practice it is better to examine separately the horizontal and vertical forces. The ordinary pivoted compass-needle is supported somewhat above its centre of gravity, and is thus not very sensitive to vertical actions. On the other hand, it turns freely round in a horizontal plane, so that we may begin by considering the horizontal component only of the earth's magnetic field, that is the magnetic force in a plane parallel to the earth's surface.

Regarding a magnet as an assemblage of ideal poles, let m be the strength of one of them. When the magnet is placed in a uniform field H the pole is acted on by a force mH in a fixed direction, and the resultant mechanical force on the whole magnet is $H\Sigma m$. The above case is experimentally realised when a magnet is floated on a cork in water, H representing the horizontal component of the earth's magnetic field, and vertical forces being practically inoperative for reasons similar to those advanced for the compass-needle. Such a floating magnet tends to point approximately north and south, but it has no tendency to move bodily in any direction. The resultant force on the whole magnet in the uniform field H must therefore vanish, i.e. $\Sigma m = 0$. Hence we have the important result that the total strength of the poles of any magnet is zero, or the magnet is compounded of positive and negative poles to equal amounts.

Let A, B be the poles of a simple bar-magnet, of strengths m and $-m$ respectively, and let the distance between the poles be δ. When the magnet lies in a uniform field H with BA making an angle θ with the direction of H, the forces on the magnet are mH at each end as shown in Fig. 12, and these forces clearly reduce to a couple of magnitude $mH\delta \sin\theta$ tending to decrease θ. The product $m\delta$ of the positive pole and the distance between the two poles is called the *moment* of a bar-magnet and denoted by M. Hence the couple on a bar-magnet in a uniform field is

$MH \sin\theta$, and in a given field the couple will be the same for two very different magnets if only they have the same moment M.

In defining the strength of magnetic poles it was supposed that any pole might be taken as a unit and given the strength $+1$. This indeterminacy will be removed later when we have considered the law of action of one pole on another at various distances apart: but in any case it is convenient to call the north end of a compass-needle a *positive* pole. The line joining the poles of a bar-magnet is called its axis, and the positive direction of the axis is that of a line proceeding from the negative to the positive pole. Hence when a bar-magnet is used as a compass the axis coincides with the direction of the earth's horizontal component, since the final position of rest of the magnet in Fig. 12 is clearly given by $\theta = 0$. This direction is known as *magnetic north*.

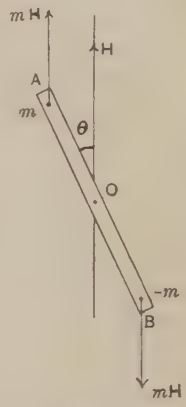


Fig. 12

In a non-uniform magnetic field a magnet will of course tend to set in a certain direction, but will in general be acted on by a resultant force in addition.

13. The magnetometer. Inverse square law of magnetic poles. The magnetometer is essentially an instrument for the comparison of magnetic fields. One form consists of a large compass-box let into a board about a metre long, the box being moveable about its centre and containing a short bar-magnet. A long

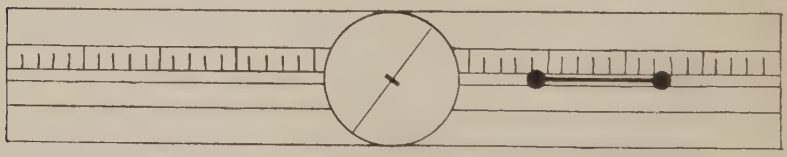


Fig. 13

double pointer of non-magnetic material is attached to the magnet and projects at right angles to the magnet so as to move over a graduated scale round the circumference of the box. A graduated groove for holding magnets is cut in the board and points towards the centre of the compass-box. To adjust the magnetometer the

compass-box is first turned till the zero line of its scale points along the groove, and the whole instrument can then be turned round until the pointers read zero. It is clear that in this position, the normal position for the magnetometer, the groove points magnetically east and west.

With the aid of the magnetometer and the definition of the strength of a magnetic pole we can demonstrate the law of action of magnetic poles, as follows:

(1) *The magnetic force due to a single pole is directed radially outwards from it.* Turn the magnetometer through 90° from its normal position, so that the groove points magnetically north and south. The needle will now point to 90° on the scale. The reading is unaltered when a bar-magnet is laid in the groove, and this holds good even if one end of the magnet is raised from the groove, provided that it remains magnetically north of the compass-needle. Hence neither pole exerts any east and west component of magnetic force on the needle. It may indeed happen that the needle suddenly swings through 180° and comes to rest pointing in the opposite direction, since the magnetic force due to the bar-magnet may overpower the earth's field; but the directions of the two fields are accurately parallel.

(2) *The magnetic force at an assigned distance from a pole is proportional to the strength of the pole.* Let P be the pole to be examined, its strength being m, and imagine a unit pole A to be placed in the assigned position. The pole A exerts on P a force proportional to m by definition, and this force is equal and opposite to the force exerted by P on A. The latter is however the magnetic force due to the pole P.

(3) *The magnetic force at distance r from a given pole is proportional to* $1/r^2$. Place the magnetometer in its normal position and lay a ball-ended magnet in the groove (Fig. 13). If the poles are distant r and $r+l$ from the compass-needle, they would, on the above law, produce a magnetic field H' proportional to $\frac{1}{r^2} - \frac{1}{(r+l)^2}$, and in an east and west direction. In any case, however, we have $H' = H \tan\theta$, where H is the horizontal component of the earth's field and θ the observed deflexion of the needle. Thus for a given magnet placed in various positions

along the groove we should have $\frac{1}{r^2} - \frac{1}{(r+l)^2}$ proportional to $\tan\theta$, i.e. $\frac{r^2(r+l)^2}{(r+2l)\,l}\tan\theta = \text{constant}$. In an ordinary experiment the constancy is found to hold good to about 2 per cent., which is about the limit of accuracy to be expected.

A more direct proof is to arrange a vertical groove directly above the compass-needle, and get rid of the effect of one pole by resting it in this groove, while the other pole lies in the original groove at varying distances r from the centre. With this arrangement it is easy to verify that $\tan\theta$ is inversely proportional to r^2.

It follows from what has been said that the magnetic force at distance r from a pole of strength m is $\lambda m/r^2$, where λ is some number. Hence two poles of strengths m, m' distant r apart repel each other with a force of F dynes, where $F = \lambda mm'/r^2$. By suitable choice of the *unit pole*, hitherto left indefinite, we can make the constant λ equal to unity. For if the unit pole is taken to be that (north-seeking) pole which repels an exactly similar pole distant one centimetre away with a force of one dyne, we have $F = 1$ when $m = m' = r = 1$. Hence $\lambda = 1$.

With these units the magnetic force due to a single pole m at distance r is m/r^2, and field-strengths are now measured in terms of a perfectly definite unit sometimes called the *gauss*, namely the field which would exist one centimetre away from a unit pole. The magnetic potential at distance r from a pole m, or the work done by the field when a unit pole is taken away to infinity, is $\int_r^\infty \frac{m}{r^2}\,dr = \frac{m}{r}$.

The most general magnetic field may be regarded as produced by a number of poles distributed in a certain way. The magnetic potential Ω at a point P is then given by the equation

$$\Omega = \Sigma\,(m/r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where r is the distance of P from a particular pole of strength m and Σ indicates summation over all the poles present.

Instead of being concentrated in points, magnetic poles may be distributed continuously through space in such a way that an element of volume $d\tau$ contains a total strength of pole $\rho\,d\tau$. Then ρ is called the volume-density of magnetism, and the

magnetic potential of such a distribution at a point P is $\Omega = \int \frac{\rho d\tau}{r}$, where r is the distance from P to $d\tau$ and the integration extends over the whole volume containing poles. Similarly we may have poles spread in layers over surfaces. The strength of pole per unit area near a point is called the surface-density of magnetism at that point and denoted by σ. The resulting magnetic potential is then $\Omega = \int \frac{\sigma dS}{r}$, where the surface-integral extends over the surface or surfaces carrying the magnetic layers.

14. Measurement of the horizontal magnetic force of the earth, and the moment of a magnet.

PART I. *Measurement of M/H.* The magnetometer is set up in its normal position with the groove pointing east and west, and a magnet laid in the groove as shown in Fig. 13. Let d be the distance of the centre of the magnet from the small compass-needle, $d \pm \delta$ the distances of the poles of the magnet. Then if $\pm m$ are the pole-strengths, the magnet exerts a magnetic force in the east and west direction of magnitude

$$H' = \frac{m}{(d-\delta)^2} - \frac{m}{(d+\delta)^2} = \frac{2Md}{(d^2-\delta^2)^2},$$

where M is the moment of the magnet. If θ is the deflexion, $H' = H \tan\theta$, so that

$$\frac{M}{H} = \frac{(d^2-\delta^2)^2}{2d} \tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

With ball-ended magnets 2δ is the distance between the centres of the spherical ends. The distribution of magnetism in cylindrical magnets is not so simple, but good results are obtained with *short* cylindrical magnets by assuming the empirical law that the distance between the poles is five-sixths the length of the magnet. It is advisable to read both ends of the double pointer on the compass-needle and also to take readings with the magnet turned over. Further, to correct for a possible error in fixing the compass-box of the magnetometer in the middle of the scale, the readings are repeated with the magnet on the opposite side of the compass-needle. The value of θ is thus a mean derived from eight observations.

PART II. *Measurement of MH.* The magnetometer is removed and the magnet M suspended horizontally from a torsionless silk fibre, so as to oscillate round a vertical axis through its centre O (Fig. 12). When the needle is swinging the earth's field exerts a couple of magnitude $MH \sin\theta$ on it in the direction tending to decrease θ, so that $I\frac{d^2\theta}{dt^2} = -MH\sin\theta$,

where I is the moment of inertia of the magnet about the axis of rotation. For small oscillations $\sin\theta$ may be replaced by θ, and we have $Id^2\theta/dt^2 + MH\theta = 0$. The motion is then simple harmonic, and of periodic time $T = 2\pi(I/MH)^{\frac{1}{2}}$. Hence if T is observed the value of MH is given by

$$MH = \frac{4\pi^2 I}{T^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The moment of inertia I can be calculated at once for a homogeneous magnet of simple form. Thus if the magnet is a cylinder of mass m, length $2l$ and radius a, $I = m(\frac{1}{3}l^2 + \frac{1}{4}a^2)$. Similarly for a ball-ended magnet consisting of two spheres of radius b with centres 2δ apart, joined by a rod whose radius a is small compared with its length, $$I = m\,\frac{a^2(\delta - b)^3 + 4b^3(\delta^2 + \frac{2}{5}b^2)}{3a^2(\delta - b) + 4b^3}.$$

From (3) and (4) we obtain M and H separately.

15. Magnetic field of a small bar-magnet. Let us have a small bar-magnet OA, with poles $-m$, m at O, A respectively. Take O as origin and OA, the positive direction of the axis of the magnet, as axis of z. If P is the point (x, y, z), we have $OP = r$, where $r^2 = x^2 + y^2 + z^2$. If δ is the distance between the poles OA, AP is obtained from OP by writing $z - \delta$ for δ, i.e. $\frac{1}{AP} = \frac{1}{r} - \delta\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ correct to the first order in δ. The magnetic potential at P due to the magnet is thus

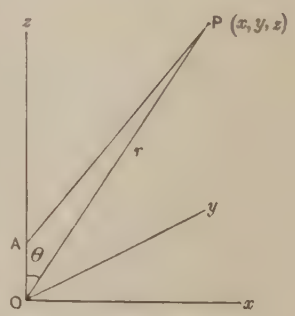


Fig. 14

$$\Omega = \frac{m}{AP} - \frac{m}{OP} = m\left\{\frac{1}{r} - \delta\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\} - \frac{m}{r}.$$

Hence $$\Omega = -M\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{Mz}{r^3} = \frac{M\cos\theta}{r^2} \quad \ldots\ldots\ldots(5),$$

where $M = m\delta$ is the moment of the magnet. Differentiating with respect to x, y and z we have the components of the field at P, namely

$$\left.\begin{aligned} H_x &= M\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right) = \frac{3Mxz}{r^5} \\ H_y &= M\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right) = \frac{3Myz}{r^5} \\ H_z &= M\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = M\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots(6).$$

The following method, due to Gauss, enables us to determine the law of action of magnetic poles *ab initio* by measurements with a small bar-magnet on the magnetometer.

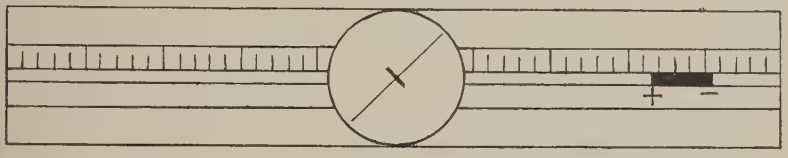


Fig. 15

The magnet is first placed magnetically east of the compass-box as shown in Fig. 15, with the magnetometer in its normal position. Let the strengths of the poles of the magnet be $\pm m$, and let $r \pm \delta$ be their distances from the compass-needle. If the magnetic force at distance r from a pole m is $mf(r)$, the small bar-magnet will produce at the centre of the compass-needle a magnetic force $mf(r-\delta) - mf(r+\delta)$ in a westerly direction. Expanding to the first order in δ, the magnitude of the force becomes $-2m\delta f'(r) = -Mf'(r)$, where M is the moment. Hence the compass-needle will be deflected to the west through an angle θ_1 given by

$$-Mf'(r) = H\tan\theta_1.$$

Now let the magnetometer be turned through a right angle and the magnet laid *across* the groove at the same distance r from the needle as before, as shown in Fig. 16. The positive pole of the magnet now produces at the centre a magnetic force practically

equal to $mf(r)$, and it is easy to see that the two poles neutralise each other except as regards the easterly component, which is $2mf(r)\frac{\delta}{r} = \frac{Mf(r)}{r}$. Hence in this position the needle is deflected to the east through an angle θ_2 given by

$$Mf(r)/r = H \tan \theta_2.$$

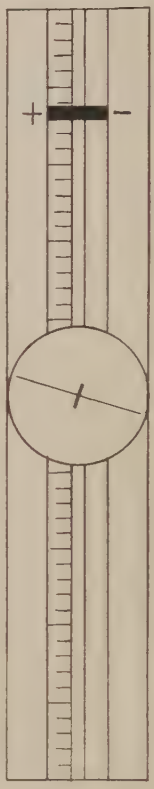


Fig. 16

It is found that $\tan\theta_1 = 2\tan\theta_2$ within the limits of experimental error. Hence

$$f'(r) = -2f(r)/r,$$

or

$$\frac{df}{f} + 2\frac{dr}{r} = 0.$$

Integrating, we have $r^2 f =$ constant, so that $f(r)$ varies as $1/r^2$.

16. Terrestrial magnetism. The quantities determining the earth's magnetic field at any place are

(1) the *horizontal component* H,

(2) the *declination*, or acute angle in a horizontal plane between magnetic and true north,

(3) the *dip*, or angle θ at which the axis of a compass-needle supported at its centre of gravity dips downwards.

In the latter case the direction of the needle is that of the *resultant* field of the earth. In addition to the horizontal component H we have a component V vertically downwards, and V is clearly given by the equation

$$V = H \tan \theta.$$

To measure the *declination* at any place we require a magnetic and an astronomical observation, the former to fix the position of the magnetic meridian and the latter that of true north. An accurate determination can be made with an instrument called the declination compass. Thanks to the exhaustive magnetic surveys carried out in recent years by Rücker and Thorpe, the declination at any given place can be estimated fairly accurately without measurement.

The fact that the compass-needle points in a nearly fixed direction has of course long been utilised in navigation: for rough surveying work a useful instrument depending on this principle is the prismatic compass (Fig. 17). In this instrument the compass-needle carries a graduated card or a light ring of non-magnetic metal divided into degrees. The observer looks through the vertical slit shown on the left of the figure, and holds the instrument so that the object is exactly behind the vertical thread shown on the right. By means of a right-angled prism covering the lower half of the slit he can also see the graduations on the divided scale, and it is easy to observe the position of the vertical thread among the scale divisions. The graduations are conveniently arranged so that the reading gives at once the angle between the magnetic meridian and the direction of the object from the observer. In this way angles can easily be measured to a fraction of a degree, and if absolute azimuths are required we only require to know the local declination, which can be found with fair accuracy from data given on most ordnance maps.

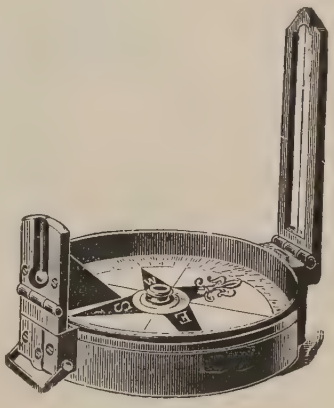


Fig. 17

It should be mentioned that a compass based on the principle of the gyroscope has recently been introduced for use in large vessels, the determination of azimuth being thus rendered independent of magnetic observations.

The *dip* is determined by means of the dip-circle, whose mode of action will be understood from Fig. 18. It consists essentially of a magnet capable of moving round a horizontal axis through its centre of gravity: such a magnet will make an angle θ with the magnetic north when the plane of the dip-circle is in the magnetic meridian. It is easy to see that the needle will dip vertically downwards when the plane is turned through 90° from this position—for the horizontal component in this case will only tend to strain the bearings, while the vertical

component will exert its full effect. This gives a convenient way of setting up the dip-circle in the magnetic meridian.

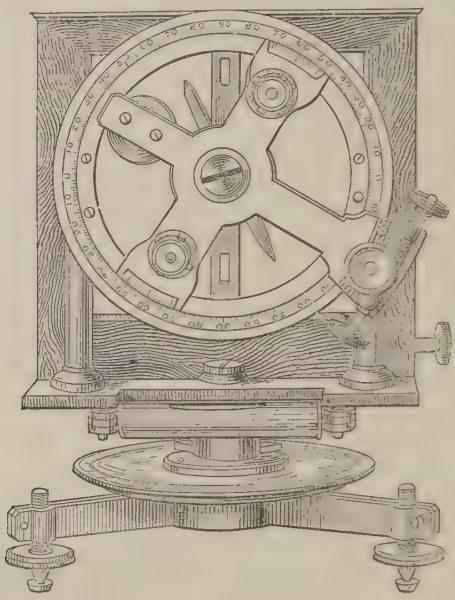


Fig. 18

The following are the approximate values of the magnetic elements in London at the present time:

Horizontal component, 0·185.
Declination, 16° W.
Dip, 67°.
Vertical component, 0·44.

All these elements are subject to large changes in time. It would appear that the changes are more or less periodic, going through a complete cycle in about 1000 years, though exact observations have not been made over this range. In addition to these *secular* changes there are smaller *diurnal* and other variations, indicating a direct or indirect dependence on the position of the heavenly bodies. Lastly, irregular variations frequently occur, particularly in arctic and antarctic regions, which have been called *magnetic storms**.

* For a fuller account of terrestrial magnetism, and of the precautions required for accurate measurements of the magnetic elements, the reader may consult Watson's *Physics*, pp. 602–617; *Practical Physics*, pp. 396–417.

The existence of magnetic elements of course shows that the earth itself is a magnet: measurements of these elements made in an ordinary laboratory do not however give the correct field due to the earth's magnetism, since the presence of iron causes great local disturbance and often entirely falsifies the measurement. The heavy electric currents now so frequently employed also have very appreciable magnetic effect. For example, the horizontal component in parts of the Electrical Laboratory, Oxford, is as low as 0·1, so that standard magnetic observations would here be out of the question. A magnetic observatory must be made entirely of non-magnetic material, and situated at a considerable distance from railway lines, generating stations, and other sources of disturbance. No modern electrical instruments however depend on the strength of the earth's field at the place where they are situated, but on fields artificially produced and of known strength.

17. Gauss' theorem. If dS is an element of a closed surface S and H_n the component of magnetic force along the outward-drawn normal to dS, then the value of $\int H_n dS$, taken over the

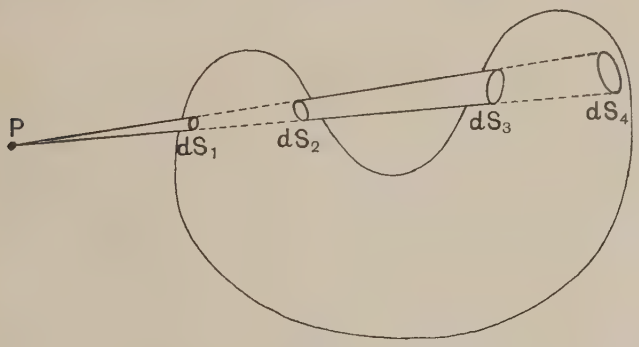


Fig. 19

whole surface, is 4π times the sum of the strengths of all the magnetic poles inside S.

Consider first of all the case of a single pole m placed at an external point P (Fig. 19). Describe a small cone of solid angle $d\omega$ round P, cutting the surface in an even number of surface elements (in the figure dS_1, dS_2, dS_3, dS_4). Let θ_1, θ_2, θ_3, θ_4 be

the angles between the ray and the normal at dS_1, dS_2, dS_3, dS_4 respectively. At points of entry like dS_1, dS_3, θ is obtuse and the projection of the element of surface perpendicular to the ray is $-dS\cos\theta$. Since this is equal to $r^2 d\omega$, we have

$$dS = -r^2 d\omega/\cos\theta.$$

But $H = m/r^2$, so that $H_n = m\cos\theta/r^2$. Hence at points of entry $H_n dS = -m d\omega$, and at points of exit $H_n dS = m d\omega$. If P is an external point the points of entry and exit occur in pairs, and the contribution of the small cone to $\int H_n dS$ is zero. Hence the integral is itself zero.

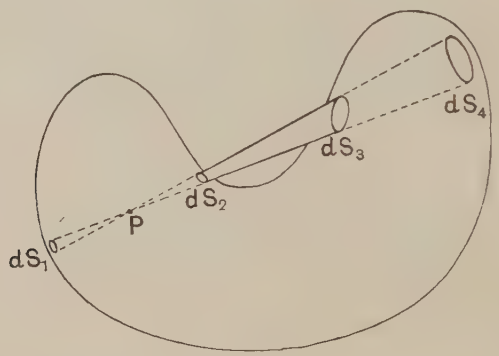


Fig. 20

If P is an internal point (Fig. 20), there are always two more points of exit than entry, and the cone contributes $2m d\omega$ to the integral. Now in tracing out the complete solid angle 4π each elementary cone is counted in twice. Hence $\int d\omega$ must be taken equal to 2π only, and

$$\int H_n dS = 4\pi m \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The result can easily be generalised to cover any distribution of magnetic poles. Each internal pole contributes $4\pi m$ to the integral, and each external pole nothing. Hence we have the complete theorem as enunciated above.

18. Lines and tubes of force. An *equipotential surface* is one at every point of which the potential is the same. In a given magnetic field Ω is a given function of x, y and z, and the equipotential surfaces are given by $\Omega = C$, where C is some constant.

By giving different values to C we get a set of equipotential surfaces.

Starting from a point P in a magnetic field imagine a very short line PQ drawn in the direction of the resultant magnetic force at P. From Q a similar line QR is drawn, and so on. In the limit the curved line $PQR\ldots$, which marks at every point the direction of the resultant magnetic force, is called a *line of force*. For a single pole the lines of force are straight, pointing from or to the pole according as its strength is positive or negative, and the same thing occurs in the immediate neighbourhood of an infinitely small pole in any magnetic field.

The lines of force have the important property of cutting the equipotential surfaces at right angles everywhere. To prove this, consider a short line AB of length ds drawn in an equipotential surface. If T is the component of magnetic force in the direction ds, the work done when a unit pole is moved from A to B is Tds. This expression must however vanish since A and B are at the same potential. Hence $T = 0$, i.e. there is no tangential component of magnetic force at any point of an equipotential surface. The lines of force must therefore run out normally.

The equations of equipotential surfaces and lines of force can be obtained in simple cases. Consider for example the field produced by two equal and opposite poles, which we may take to lie at the points $(-a, 0, 0)$ and $(a, 0, 0)$ respectively. If the pole-strengths are m and $-m$, the potential at the point (x, y, z) is

$$\Omega = m\{(x+a)^2 + y^2 + z^2\}^{-\frac{1}{2}} - m\{(x-a)^2 + y^2 + z^2\}^{-\frac{1}{2}}.$$

Confining our attention to the sections of equipotential surfaces by a plane through the poles, for example the plane $z = 0$, the equipotential surfaces are given by

$$m\{(x+a)^2 + y^2\}^{-\frac{1}{2}} - m\{(x-a)^2 + y^2\}^{-\frac{1}{2}} = C.$$

The lines of force are found by the condition that they cut the equipotential curves at right angles. Eliminating the constant C from the last equation by differentiation we have for the differential equation of all the equipotential curves

$$\{(x+a)^2 + y^2\}^{-\frac{3}{2}}\left\{x + a + y\frac{dy}{dx}\right\} - \{(x-a)^2 + y^2\}^{-\frac{3}{2}}\left\{x - a + y\frac{dy}{dx}\right\} = 0.$$

Replacing dy/dx by $-dx/dy$ gives the proper direction of the tangent to the line of force through the point (x, y): hence we have the differential equation of the lines of force in the form

$$\frac{(x+a)\dfrac{dy}{dx} - y}{\{(x+a)^2 + y^2\}^{\frac{3}{2}}} = \frac{(x-a)\dfrac{dy}{dx} - y}{\{(x-a)^2 + y^2\}^{\frac{3}{2}}},$$

which may be written

$$\frac{d\left(\dfrac{x+a}{y}\right)}{\left\{1 + \left(\dfrac{x+a}{y}\right)^2\right\}^{\frac{3}{2}}} = \frac{d\left(\dfrac{x-a}{y}\right)}{\left\{1 + \left(\dfrac{x-a}{y}\right)^2\right\}^{\frac{3}{2}}}.$$

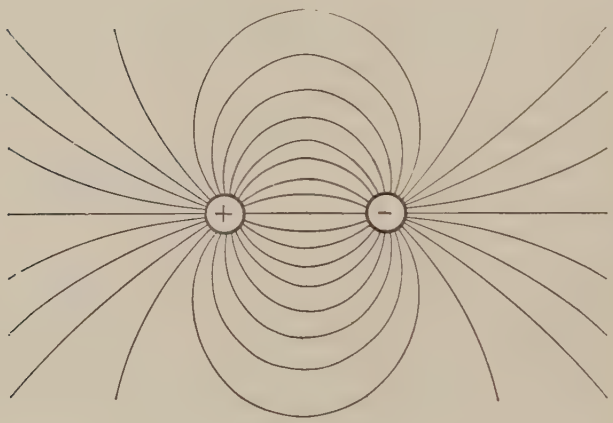


Fig. 21

Integrating, we have the equation of the lines of force, namely

$$\frac{x+a}{y}\left\{1 + \left(\frac{x+a}{y}\right)^2\right\}^{-\frac{1}{2}} - \frac{x-a}{y}\left\{1 + \left(\frac{x-a}{y}\right)^2\right\}^{-\frac{1}{2}} = \text{const.},$$

or $(x+a)\{(x+a)^2 + y^2\}^{-\frac{1}{2}} - (x-a)\{(x-a)^2 + y^2\}^{-\frac{1}{2}} = \text{const.}$

The lines of force are shown in Fig. 21. Similarly Fig. 22 shows the lines of force due to two equal magnetic poles of the same sign.

Diagrams of this kind are very instructive, as they show at a glance the state of a field at every point. Further, we shall see in the next article that the *strength* of the field is also indicated by the closeness with which the lines crowd together at different points of their length.

Lines of force can be exhibited experimentally by means of iron filings, which when sprinkled on paper or glass in a fairly strong magnetic field tend to orientate themselves along the lines

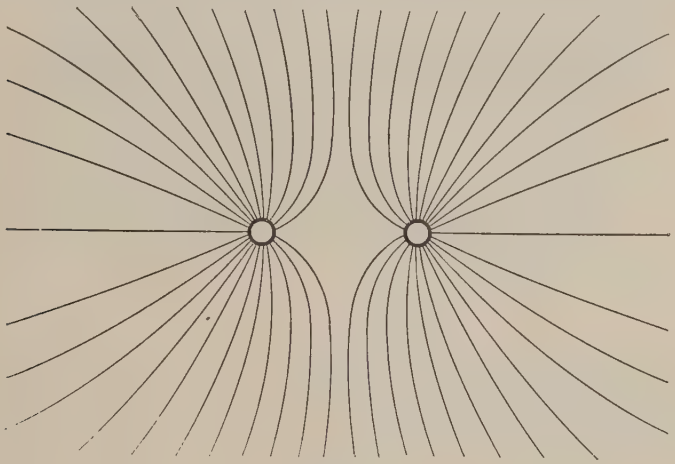


Fig. 22

of the field. Permanent diagrams can be made by using stiff paraffined paper, which is tapped to make the filings settle into position, and gently warmed so as to melt the wax and fix the filings.

Fig. 23 is made by using opposite poles of two similar magnets, and may be compared with the theoretical diagram Fig. 21. Similarly Fig. 24 corresponds to Fig. 22.

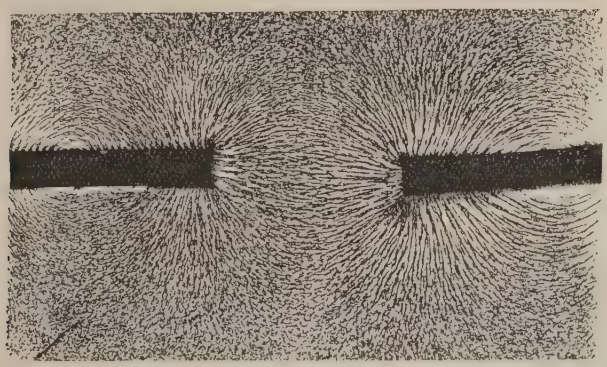


Fig. 23

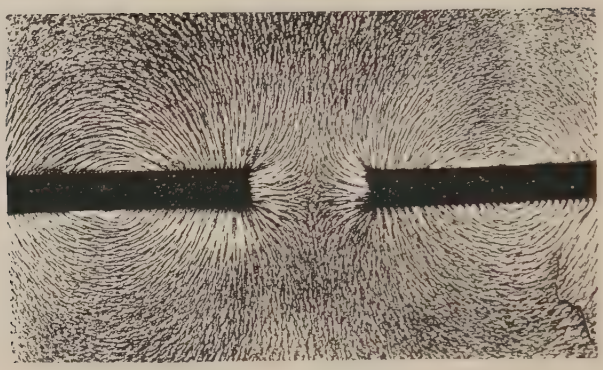


Fig. 24

The lines of force which start out from points on a closed curve form a tubular surface called a *tube of force*. Fig. 25 shows the tube formed by lines of force starting out normally from the perimeter of a small element of surface dS near P. Let dS' be the cross-section of the tube near another point P', and let H, H' be the resultant magnetic forces at P, P' respectively. The immediate neighbourhood will be supposed to be devoid of magnetic poles, so that we can apply Gauss' theorem to the truncated tube in the figure. The tubular sides contribute nothing to the surface-integral of normal component, since the resultant field is perpendicular to the normal to the surface at all points

Fig. 25

on those sides. The ends contribute $-HdS$ and $H'dS'$ respectively, and since the total integral vanishes we have $HdS = H'dS'$.

Thus for a small tube of force the product of the resultant field and the cross-section of the tube is constant. Lines of force therefore crowd together in the strongest parts of the field.

If we consider a closed circuit C in a region of space in which there are no magnetic poles, then by the type of argument used on p. 9 we see that the *flux of magnetic force* through C can be specified in terms of C alone, i.e. it is independent of the particular surface S used in its estimation. The reader can easily satisfy himself that a given tube of force will contribute nothing to the flux of force through C if it does not pass through C, but that if it does interlace C it gives a definite contribution to the flux irrespective of the surface S used to calculate it. For this reason the phrase "number of tubes of force passing through a circuit" is often used instead of "flux of force through a circuit": but we shall adhere to the latter phrase as expressing more precisely what is meant.

19. Magnetic effect of any small magnetised body. However irregularly a small body is magnetised, its magnetic effect is that of a certain small bar-magnet. It is of interest to find the moment and axis of this equivalent magnet in terms of the positions of the poles in the small body. Take a set of axes of x, y, z with the origin somewhere in the small body, and let a specimen pole in the body be one of strength m at the point $Q(x, y, z)$. Let Ω be the magnetic potential at the point $P(\xi, \eta, \zeta)$. Since x, y, z are small we have

$$\frac{1}{QP} = \{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{-\frac{1}{2}} = \frac{1}{r} + \frac{x\xi + y\eta + z\zeta}{r^3}$$

approximately, where $OP = r$. Hence if Σ denotes summation over all the poles in the small magnetised body,

$$\Omega = \Sigma \frac{m}{QP} = \frac{\Sigma m}{r} + \frac{\xi \Sigma mx + \eta \Sigma my + \zeta \Sigma mz}{r^3}.$$

The first term vanishes since $\Sigma m = 0$. If we write

$$M_x = \Sigma mx, \quad M_y = \Sigma my, \quad M_z = \Sigma mz,$$

we have

$$\Omega = \frac{\xi M_x + \eta M_y + \zeta M_z}{r^3}.$$

The quantities M_x, M_y, M_z are called the *components of magnetic moment* of the small magnetised body. If

$$M = (M_x^2 + M_y^2 + M_z^2)^{\frac{1}{2}},$$

M is called its *moment*, and the line whose direction-cosines are $(M_x/M,\ M_y/M,\ M_z/M)$ its *magnetic axis*. It is easy to see that these definitions agree, in the case of a simple bar-magnet, with those already adopted.

If the small magnet, instead of being at the origin, was at the point (x, y, z), the magnetic potential at (ξ, η, ζ) would become

$$\Omega = \frac{(\xi - x)\, M_x + (\eta - y)\, M_y + (\zeta - z)\, M_z}{r^3},$$

or
$$\Omega = M_x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + M_y \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + M_z \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \ldots\ldots (8),$$

where $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$.

20. Poisson's analysis of magnetism. Examination by the method of iron filings shows that a magnet continues to show

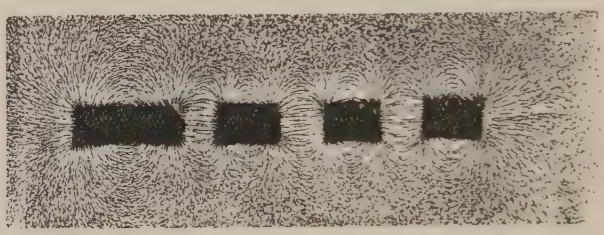


Fig. 26

magnetic properties at both its ends, however often it is broken. It is thus natural to suppose that the smallest parts of a magnetised body are themselves magnets, a medium of this kind being sometimes said to be *polarised*. Evidently the magnetic state of a volume-element $d\tau$ of a magnetised body is specified when its magnetic moment is given in magnitude and direction. If the components of its magnetic moment are given by

$$M_x = I_x d\tau, \quad M_y = I_y d\tau, \quad M_z = I_z d\tau,$$

the quantities (I_x, I_y, I_z) are called the components of the *intensity of magnetisation* I at the element $d\tau$, which we suppose situated near the point (x, y, z). The value of I depends on the magnetic force in the neighbourhood in a very complicated manner which has been determined by experiment and will be discussed fully in Ch. VII.

Let us first consider the case of a rod magnetised uniformly in the direction of its length. Since each element is polarised the positive and negative poles in the interior cancel one another, and there is no free magnetism. At the ends, however, the compensation ceases, the positive poles being left at one end and the negative poles at the other. Thus we have, in Fig. 27, a distribution of positive magnetism over the right-hand face and of negative magnetism over the left-hand face. Let $\pm \sigma$ be the pole-strength per unit area, which we may call the surface-density of magnetism. If l is the length and A the cross-section of the rod, the pole-strengths at the ends are $\pm \sigma A$, and the magnetic moment σlA. Since the volume is lA the magnetic moment is by definition equal to IlA, so that $\sigma = I$. Hence in a uniformly magnetised rod the surface-density of magnetism on the flat ends is equal to I.

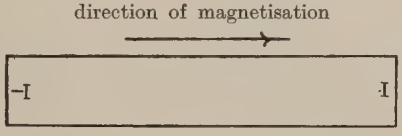


Fig. 27

In the general case in which the body is magnetised in any manner the compensation in the interior is not complete, but there is both volume- and surface-density of magnetism. Poisson found expressions for both of these in general and for a body of any shape, as follows.

Consider the magnetic effect of the body at a point P outside it. Remembering equation (8) we see that the magnetic potential at P is given by

$$\Omega = \int_\tau \left\{ I_x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + I_y \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + I_z \frac{\partial}{\partial z}\left(\frac{1}{r}\right) \right\} d\tau,$$

where the integration extends throughout the volume τ of the body. Since

$$I_x \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + I_y \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + I_z \frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{I_x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{I_y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{I_z}{r}\right) - \frac{1}{r}\left(\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z}\right),$$

Gauss' transformation shows that

$$\Omega = -\int_\tau \left(\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z}\right)\frac{d\tau}{r} + \int_S \frac{lI_x + mI_y + nI_z}{r}\, dS.$$

Writing

$$\rho = -\left(\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z}\right), \quad \sigma = lI_x + mI_y + nI_z \ldots (9),$$

it is evident that the same magnetic potential would be produced by a distribution of poles throughout the volume with volume-density ρ, and over the surface with surface-density σ. Since magnetic poles are known only by their effect it follows that the polarised distribution is actually equivalent to these distributions. In the simple case of a bar-magnet uniformly magnetised we have $\rho = 0$ always. On the flat ends σ, which in general is equal to the normal component of I, becomes identical with I, while on the cylindrical surface σ vanishes. This is in entire agreement with the result previously found.

Uniform magnetisation cannot be accurately realised in practice, as the magnetism is thrown back more or less from the ends. Further, the magnetic effect of the magnetism on the ends is such as to tend to remove the magnetism in the rod.

CHAPTER III

ELECTROSTATICS

21. Electrical attractions and repulsions. It has long been known that substances such as amber, glass, resin when rubbed are capable of attracting light bodies. These attractions are called *electrical*, and the rubbed body is said to be *electrified*, or to carry a *charge of electricity*, while a body in its normal state is called *neutral* by way of distinction. To study these effects, suspend a light metallic or other carrier from a silk fibre so that it can hold a glass rod in a horizontal position. If the rod is rubbed vigorously with a silk handkerchief and placed in the carrier, it will be seen to be attracted by the handkerchief when the latter is brought near. Two similarly prepared glass rods, on the other hand, repel one another. The glass rod is in this case said to be *positively* and the silk handkerchief *negatively* electrified, the terms positive and negative being used at present only in a descriptive sense. Now prepare an ebonite rod by rubbing it with flannel, and suspend it as before. Two similarly prepared rods repel one another and either attracts the flannel, as in the previous case. However, the ebonite rod will *attract* a glass rod, and be *repelled* by the silk handkerchief with which the latter has been rubbed. We must thus suppose that an ebonite rod becomes charged negatively by friction with flannel, and that two negative charges repel one another. We can now see the significance of the terms positive and negative: for if we assign a + sign to the phenomenon of repulsion and a − sign to that of attraction we see that + acting on + gives +, − acting on − gives +, while + acting on − gives −. This is in formal analogy

with the rule of multiplication of signs in algebra; an analogy which will be fully developed later.

Other substances may be used instead of glass and ebonite, but we never obtain any evidence of the existence of other kinds of electrification than those already named, because a rubbed body which repels a positively charged body is always found to attract a negatively charged one, and *vice versâ*. A convenient way of testing electrification is to prepare a number of pith balls by coating them with aluminium foil and suspending them from silk fibres. Pith balls that touch an electrified glass rod acquire positive electrification: that is, they repel one another and the glass rod, and are attracted by the silk handkerchief, while they attract pith balls that have been in contact with the ebonite rods. The effects with well-dried materials are more marked than those obtained directly with the rods, and in addition we verify the important fact that the force between two small charged bodies acts along the line joining them.

A pith ball that has not been electrified is always *attracted* by an electrified body, whether positively or negatively charged. As in the analogous case of magnetism, however, this phenomenon is in reality more complex than the original one of the repulsions and attractions of charged bodies. It is considered in detail in Art. 26.

22. The electroscope. Conductors and insulators. Simple electrical experiments are more conveniently performed by means of the *electroscope* than by either of the preceding methods.

This instrument consists essentially of a brass rod with a knob at the upper end, the lower end dipping into a metal case with glass sides (Fig. 28), the rod and case being separated from each other by a stopper of ebonite. Two strips of thin gold or aluminium foil are attached to the lower end of the rod. When the disc is touched with an electrified glass or vulcanite rod the leaves diverge on account of the mutual repulsion of the electrification on them, and being very light they are very easily deflected. The amount of divergence may be indicated by a graduated scale attached to the central rod or case.

If the electroscope is charged and left to itself, the leaves

remain in the same position, at any rate for a considerable time. The same is true of two pith balls suspended from dry silk fibres and left repelling or attracting one another in air. Thus the air, ebonite, and silk have the property of being able to resist the loss of electrification by a body: such substances are called *insulators*. On the other hand, if the knob of a charged electroscope is touched the leaves collapse immediately, showing that electrification can escape through or into the human body. Substances allowing the passage of electricity are called *conductors*. This view of the removal of electricity by passage through a conductor is supported by a simple experiment. If an ebonite rod is held in the hand and placed in contact with the knob of an electroscope the leaves do not collapse, because the electrification cannot pass through the insulator to the hand, although it would pass freely through the body if it once got so far. If the ebonite rod is replaced by one of metal, however, the leaves fall at once, showing that metals are conductors of electricity.

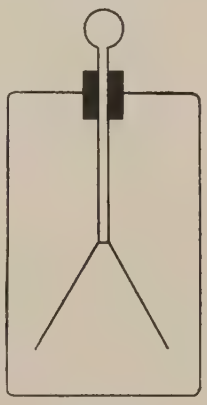


Fig. 28

The division of bodies into conductors and insulators is not absolutely strict, as there is a kind of continuous gradation between the two classes. Thus the leaves of an electroscope whose knob is touched by a wooden rod held in the hand collapse gradually, and at different rates depending on the dryness and nature of the wood. It must not, however, be thought that the distinction between conductors and insulators is an unreal one, for it is possible to find substances which approximate very closely to the ideal of both kinds. Thus, for example, a well-designed electroscope with ebonite, sulphur, or amber insulation will retain some charge for at least a week, while if metallic contact is made between the knob and the case the leaves collapse in the merest fraction of a second—in fact, the time taken for the electricity to disappear in the latter case is unmeasurably small.

In the following table we have endeavoured to place the

substances in order of insulating power, but it must be remembered that the exact order is somewhat uncertain and depends on the physical state of the substance.

Insulators	Air (best)
	Amber and amber preparations
	Sulphur
	Freshly cut ebonite
	Paraffin wax
	Quartz
	Dry glass
	Wood soaked in melted paraffin wax
Semi-Insulators	Wood
	Oils
Conductors	Water
	Salt solutions
	Carbon
	Mercury
	Metals (best)

Air is an admirable insulator under ordinary conditions, but it has its limits of resistance like all other insulators, and breaks down under high electric force with the passage of a *spark*. The conditions for a spark are favourably realised when conductors with sharp or pointed edges are present, a fact which is utilised in the construction of lightning-conductors, and also in influence machines (cf. Art. 27). The hot air in the neighbourhood of flames is also a deficient insulator.

When the insulation of a substance deteriorates with time, the trouble can usually be traced to the formation of a conducting layer on the surface. The moisture which deposits on glass is very harmful in this respect. Ebonite when exposed to strong light becomes oxidised and loses its insulating power, and it should not be fingered too much, or else a conducting layer of grease will be deposited on its surface. This can be obviated by making a narrow saw-cut round a piece of ebonite required for use in electrostatic experiments, leaving a fresh surface not likely to be contaminated.

The class of bodies described as semi-insulators cannot be

used in experiments with electroscopes: but we shall see later that for work with electric batteries, in which we deal with very large charges, they may be advantageously employed. Liquids also can never be classed as good insulators, because they *convect* electricity away by their motion, although they do not all *conduct* in the strict sense.

The conducting power of the human body and of metals explains a phenomenon which misled early investigators, namely that metals when rubbed do not appear to be electrified. That this is not the case is easily shown by rubbing a metal rod held in an insulating handle, when it is found to be electrified just like other bodies. When no handle is used the electricity is of course conducted away as soon as it is produced by friction.

23. Quantity or charge of electricity. Electrons. The use of the terms *electricity* and *charge of electricity* suggests an electric substance as distinct from a state of a body. It will be clearer, and in the end more satisfactory, to adopt this view; for it is now universally admitted that the process of electrification consists in the removal from the molecules of matter of something called electricity, which moreover is of atomic structure. Although the evidence for this latter statement cannot yet be fully stated, we shall adopt it here and give an account of the present state of speculation as to the nature of electrification.

It is supposed that *negative* electricity is fundamental in the universe, and consists of *electrons*, or indivisible atoms of electricity, which are identical in size and properties. The electron is, in size and mass, the smallest body at present known*, and each electron carries the same amount of electricity, which again is the smallest amount hitherto examined separately. Positive electricity behaves very differently. There is no evidence of the existence of carriers of positive electricity of less than atomic mass, and the nature of the carriers varies according to the substance with which we are experimenting, in sharp contrast with the electron, whose properties are independent of its origin.

These facts have led to the view that positive electricity is

* The negative electron has a mass of about 1/1800th of that of the hydrogen atom, while its diameter is probably of the order 10^{-13} centimetres, the diameter of a molecule being about 10^{-8} centimetres.

merely what occurs when electrons are removed from the molecules to which they are normally attached. That is, we regard a neutral body as one containing a certain normal number of electrons, so many on the average to each molecule. If more electrons are present the body is negatively electrified; if less, positively. Apart from the atomic notion, this theory is by no means novel, being identical with Franklin's "one fluid theory" of electricity. Franklin supposed that neutral bodies were capable of holding a definite amount of "electric fluid." When the fluid is present in excess there is positive electrification, and when present in defect negative electrification. If we interchange the words positive and negative in this and regard the electric fluid as consisting of a very great number of discrete particles, the theory becomes the modern electron theory.

On this theory insulators are substances whose molecules do not readily give up their electrons, although they may be stimulated to do so by external action such as friction. Thus when a glass rod is rubbed with a silk handkerchief the surface-molecules of the glass give up electrons to the silk, and the glass is positively electrified. This action, with insulators, is limited to those molecules which actually come into rubbing contact, but the behaviour of conductors is quite different. We must suppose that electrons move freely about in conductors, without being permanently attached to any particular molecule. If the conductor itself is *insulated*, i.e. separated by an insulator from other conductors, the electrons remain on it as they are unable to pass through the insulator and also cannot under normal conditions escape into the air.

The electron theory affords a simple and (theoretically) perfectly definite specification of *amounts* of electricity. If all electrons are supposed identical they must have the same amount of electricity, whatever that may be, and we may say that an electron carries a charge of q units of negative electricity (or carries a charge $-q$), where the precise unit need not yet be further specified. If a body contains n electrons more than in the neutral state it is said to carry a charge $-nq$; if n less, a charge $+nq$.

The mutual repulsion of negatively charged bodies is explained by supposing that all electrons repel one another. A neutral

molecule may be simply regarded as an electron attached to something which compensates its effect, which we shall call for brevity a residue. Suppose now that two electrons distant r apart repel one another with a force of $q^2f(r)$ dynes, and that a neutral molecule has no effect on either an electron or a residue. This latter condition can clearly only be fulfilled if a residue *attracts* an electron with a force of $q^2f(r)$ dynes, and *repels* another residue with the same force. The attractions and repulsions of charged bodies are very simply explained on these lines. Consider for example a small body carrying a charge $+e$ at distance r from a second small body carrying a charge $+e'$. The former body may be regarded as containing m residues and the latter n residues, where $e = mq$ and $e' = nq$. Each residue is supposed to act independently, and thus the force between the bodies is repulsive and of magnitude $mnq^2f(r) = ee'f(r)$. It is easy to see that this statement holds good when one or both of e, e' are negative, provided that an attraction is regarded as a negative repulsion. This statement gives the attractions and repulsions in precise quantitative form in terms of the charges on the bodies, being more far-reaching than the rule of signs in Art. 21.

Coulomb was able to show that the forces between two small charged bodies were inversely proportional to the square of the distance between them, i.e. that $f(r) \propto 1/r^2$. We shall not however make this assumption until we have indicated a more convenient and accurate method of verifying it.

24. The electric field. Electric force and potential. The description of the electric field now follows on the same lines as in the case of magnetism (Art. 11). The *electric force* at any point is the force which would act on a unit charge placed at the point. The precise unit is left at present unsettled, but it is perhaps not superfluous to emphasise the fact that a unit charge is always understood in the definition, and that electric force is *not* a mechanical force merely, as its name would suggest. If E, with components (E_x, E_y, E_z), is the electric force at a point (x, y, z) in an electric field, the force on a charge e placed there is (eE_x, eE_y, eE_z). Thus the electric force at distance r from a charge e is, with the notation of the last article, $ef(r)$.

The *electric potential* is the work that would be done by the forces in the field when a unit charge is moved to infinity from the point considered, i.e. the potential energy of a unit charge at the point. At distance r from a charge e the potential is

$$V = e\int_r^\infty f(r)\,dr.$$

The electrical potential in all cases vanishes at infinite distances from the charges in the field.

When a number of charges are present the potential is obtained by summation of the separate potentials due to each. A charged body contains a very large number of electrons which, though quite distinct, are very close together. We are not usually concerned with the precise location of each, but only require to know the average number per cubic centimetre or per square centimetre of a layer. We may thus consider charges extended through regions of space with *volume-density* ρ or spread over surfaces with *surface-density* σ, and then the potential of the distribution is obtained by integration (cf. Art. 29).

A difficulty arises in the definition of potential from the fact that electricity can move about in conducting bodies. Without entering here into details, we may point out that electricity settles down in conductors in a way which depends on the other charges present in the field. Thus when a unit charge is taken away from any point P to infinity the distribution of electricity in the field is changed, and the work actually done will differ from that which would have been performed if the charges had been fixed in position before the unit charge was moved. It is the latter work which defines the potential at P. In other words, in calculating potentials the disturbing effect of the unit charge is neglected. One way of avoiding this reservation would be to say that, if work W is actually done on a charge e moved from P to infinity, then the potential at P is the limit of the ratio W/e when e is indefinitely diminished.

If V is the potential and E the electric force at a point (x, y, z) in an electric field, we have the equations

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z} \quad \ldots\ldots(1),$$

analogous to those occurring in magnetism. More generally, the component E_s of electric force in any direction is equal to $-\partial V/\partial s$, where $\partial/\partial s$ denotes differentiation in the direction s.

The surfaces $V =$ constant are called *equipotential* or *level* surfaces.

We shall now prove a certain reciprocal theorem, which will be useful to us later. Let charges $e_1, e_2, \ldots e_n$ be placed at the points $P_1, P_2, \ldots P_n$, and let V_1 be the potential at P_1 of all the charges except e_1, V_2 the potential at P_2 of all the charges except e_2, and so on. Then

$$V_1 = \frac{e_2}{r_{12}} + \frac{e_3}{r_{13}} + \ldots + \frac{e_n}{r_{1n}},$$

$$V_2 = \frac{e_1}{r_{21}} + \frac{e_3}{r_{23}} + \ldots + \frac{e_n}{r_{2n}},$$

and so on. Let $e_1', e_2', \ldots e_n'$ be any other charges placed at the same points, $V_1', V_2', \ldots V_n'$ the corresponding potentials. If we form the sum $e_1'V_1 + e_2'V_2 + \ldots + e_n'V_n$, we see that it involves r_{12} in the form $\dfrac{e_1'e_2 + e_2'e_1}{r_{12}}$, and similarly for any other mutual distance. Hence

$$e_1'V_1 + e_2'V_2 + \ldots + e_n'V_n = \Sigma \frac{e_p'e_q + e_q'e_p}{r_{pq}}$$

extended over all possible pairs of numbers p, q. By symmetry, this is equal to $e_1V_1' + e_2V_2' + \ldots + e_nV_n'$. Hence

$$\Sigma eV' = \Sigma e'V.$$

This is known as Gauss' reciprocal theorem.

25. Potential of a conductor. All points on or inside a conductor on which electricity is at rest may be shown to be at the same potential. If two neighbouring points P, Q on or inside the conductor were at different potentials, the potential of P being the higher, electricity would tend to flow from P to Q because there would be an electric force from P towards Q, and in a conductor there is nothing to prevent the passage of charge under these circumstances. The common potential of all points of the conductor is called the *potential of the conductor*. It follows that the bounding surface of a conductor is an equipotential surface of the charges in the field. This conclusion is independent

of any particular assumption as to the law of force between electric charges.

If two charged conductors are joined by a wire held in an insulating handle, electricity flows from the one at higher to the one at lower potential. This process naturally ceases when the potentials have become equal, but if any means were provided to maintain the potentials at their original values it would go on indefinitely. In all cases in which electricity moves it is convenient to regard positive as well as negative electricity as moveable; and there is no objection to this way of speaking, since the movement of a negative charge from A to B is to all intents and purposes the same as the movement of an equal positive charge from B to A. In this chapter we shall however be chiefly concerned with electricity at rest.

26. Electric influence. Earthing a conductor. If an electrified ebonite rod is brought near to one end of an elongated

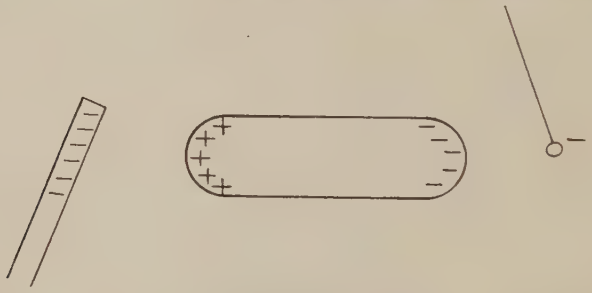


Fig. 29

conductor the other end repels a pith ball that has previously been touched by the rod, while the near end of the conductor attracts it. Thus negative electricity collects at the far end and positive electricity on the part near the rod, the middle parts being comparatively neutral. These effects are said to be due to *electric influence*.

An easy explanation can be given on the theory of the atomic constitution of electricity. The electrons of conductors are not permanently bound to any particular molecule, but even in the neutral state can move about in the metal. When a negatively charged rod is brought near, the random movement of the negative

electrons is partly prevented by the repulsion due to the charge on the rod. Thus on the whole electrons tend to congregate in the remote parts of the conductor, leaving positive charge on the parts near the rod.

Electrical influence effects are very easily shown with the electroscope. Thus if an electroscope is charged positively the leaves begin to diverge still further as a positively charged body approaches the knob, on account of the increase of the positive charge on the leaves. On the approach of a negative charge the leaves begin to collapse. If the knob is touched while the positive charge is near it the leaves at once collapse, as the positive charge originally on them tends to go even further away from the influencing charge, that is into the body of the experimenter. There remains only the negative charge attracted to the neighbourhood of the knob. Now suppose the hand to be taken away and the influencing charge subsequently removed. The negative charge will spread over the whole of the knob and leaves, and the latter will diverge, though with the opposite kind of electricity from that of the external charge.

We have thus a means of charging an electroscope with electricity of either sign by use of a charge of one sign only. If the rod had been removed before the hand was taken away there would of course have been no deflexion of the leaves. This illustrates the important fact that the order of operations in experiments depending on influence is not immaterial, but that quite different effects are produced by doing things in a different order. The human body in these experiments plays the part of a large conductor practically free from charge, and the same results would be obtained by connecting the knob to a large uncharged metal conductor, or to a gas-pipe having contact with the soil. A connexion of this kind is known as *earthing* a conductor, or putting it to earth.

Although in theoretical reasoning potential is defined to be zero at infinite distances, yet it is evident that we are really only concerned with differences of potential, so that we can take any convenient conductor to be at zero potential. It is conventional *to take an earthed body to be at zero potential.* An earthed conductor is also described as uninsulated.

We can now understand the attraction which a charged body exerts on neutral bodies. The first effect of a positive charge is to bring negative charge to the nearer part of the neutral body and repel positive charge to the remote parts, and since on the whole the negative charge is nearer the attraction exceeds the repulsion between the like charges.

27. Influence machines. Machines have been devised for the continuous production of electricity depending on the

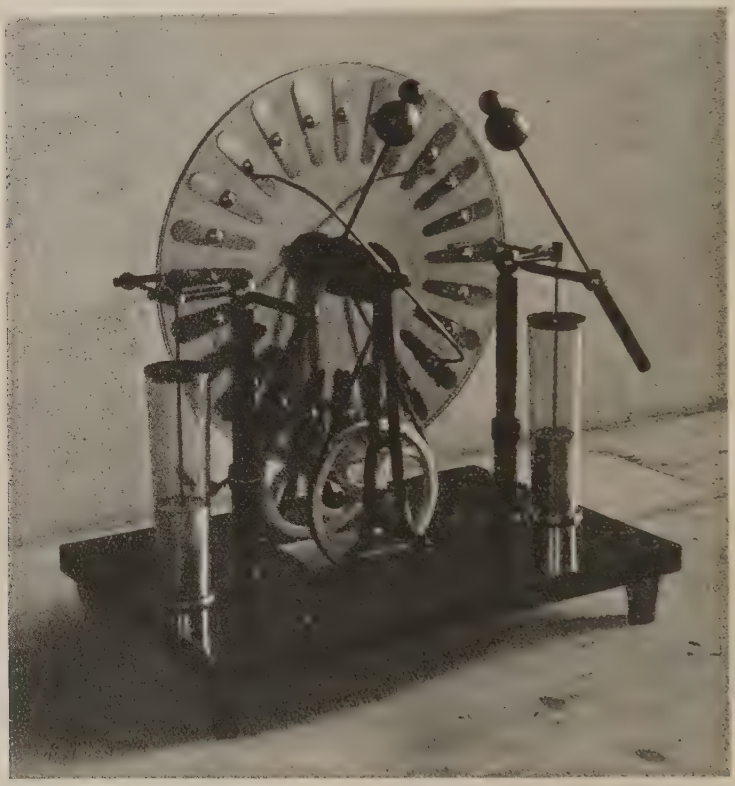


Fig. 30

fundamental principle of friction, but these are not altogether satisfactory, and they have been superseded by machines working by electric influence. Of the various types of influence machine the Wimshurst machine has generally been found the most

efficient. Two circular plates of glass or ebonite are placed parallel to one another on an axle, and arranged so that they rotate in opposite directions on turning a handle. The outer surfaces of the plates carry a number of thin sheets of metal cut into the form of sectors. At the level of the centre of the plates are fixed two double combs of metal enclosing the plates at the end without touching them, with the teeth pointing towards the sectors. The combs are affixed to insulating stands and also to a pair of brass knobs whose distance apart can be altered. On either side a brass rod is fixed to the axle, carrying at each end a metallic brush which touches each sector as it passes. These rods are inclined at angles of 45° to the horizontal as shown.

The mode of action of the machine is most easily explained by a method due to Prof. Silvanus Thompson, in which the two discs are imagined to be replaced by two coaxial cylinders revolving one inside the other. In Fig. 31 X, Y represent the combs with their knobs attached, and A, B, C, D the four brushes at the ends of the two brass rods.

Imagine to begin with that a particular sector P carries a negative charge, all the other sectors being uncharged. As the discs rotate P comes opposite to the brush A, and at the same time the sector originally at Q is in contact with the brush. The negative charge on P induces a positive charge on the sector and brush A, and this requires a negative charge to reside on the opposite brush B and its sector. The inductive effect of P is still appreciable when it is some distance from the brush A. Hence as the discs rotate further, other sectors passing A have positive charges induced on them, the opposite process occurring at the other extremity of the diameter. The figure shows the charges present just before the original sector comes in front of the comb Y. The sharp points of the teeth discharge this sector almost entirely: but the three upper sectors on the outer disc carry their positive charge with them until they come opposite the brush C, where they induce negative charges on the sectors passing C, causing positive charges to reside on the sectors passing D. Negative charges are subsequently induced on the sectors of the outer disc opposite B, thus continuing the supply of negative electricity to the sectors in this region. A little consideration shows that the

reinforcement of the charges present continually increases, and an ever-increasing number of sectors is brought into play. Ultimately all the sectors carry charges: on the inner disc the charge is negative in the part CYD and positive in the part DXC, while on the outer disc we have positive charge along AXB and negative along BYA. The charges on the sectors between Y and D, Y and A, X and C, and X and B are however comparatively small because these sectors have passed a discharging comb and

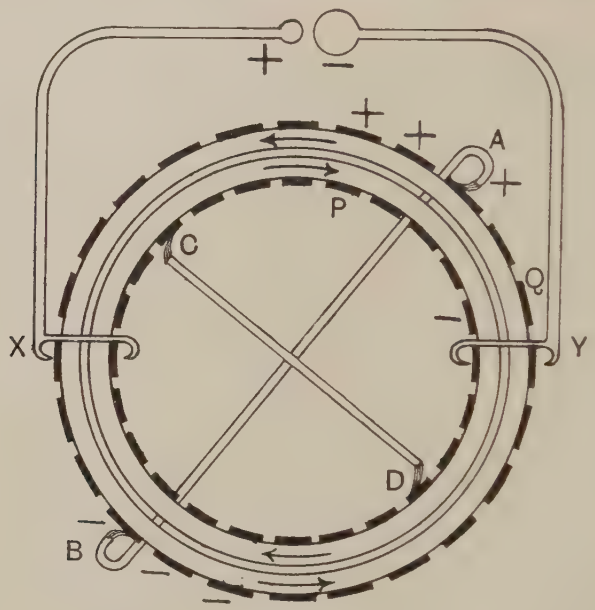


Fig. 31

have not yet arrived at a brush. As regards the collection of the charge on the knobs, consider the sectors passing the comb X. Both sets are positively charged, and induce negative electricity on the teeth of the comb, leaving positive electricity on the knob connected to X. The negative charge on the comb escapes almost entirely from the teeth and neutralises the positive charge on the sector, so that the conductor X is continually acquiring an excess of positive charge. The opposite charge is being accumulated on Y, and this may go on until a spark passes between the knobs themselves. The electrical energy of the

separation of the charges of course comes out of the increased work required to turn the handle against the mutual attraction of the charges on the sectors.

The slight initial charges required to start the action are generally present, and the machine is usually self starting. If this is not the case, an electrified rod may be held near the sectors opposite one brush. When driven at a constant speed with an auxiliary motor the influence machine gives a very constant source of high potential.

28. Cavendish's experimental proof of the law of force. A crucial test of the law of force between electric charges is afforded by an experiment first performed by Cavendish in 1773, and repeated with refinements and modifications under Maxwell's direction about a hundred years later. The experiment can be carried out in the laboratory as follows.

Take two large metallic conductors S_1, S_2 (such as two tin

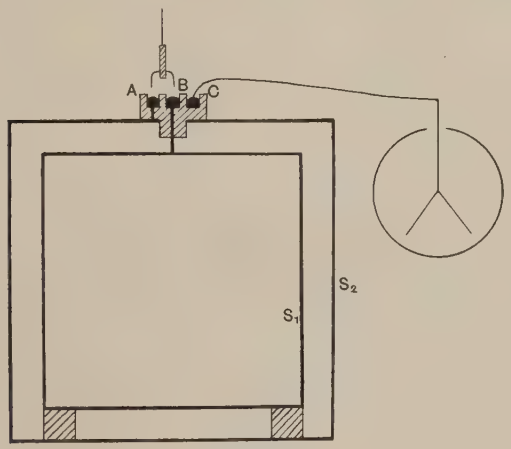


Fig. 32

boxes) and place S_1 inside S_2 and insulated from it by means of paraffin blocks. A small hole cut in the outer conductor is fitted with an ebonite plug in which are bored three small cups A, B, C to contain mercury. Wires soldered to S_1, S_2 and ending in the mercury-cups make contact with B, A respectively. A wire dipping in C leads to a sensitive electroscope, or better to one

terminal of a quadrant electrometer (Art. 43). We can make metallic connexion between B and A, or between B and C, by means of a "key" consisting of a short wire passing sideways through an ebonite rod and bent round at the ends as shown in the figure.

Initially B is joined to A and the electroscope discharged. In this state S_1 and S_2 form a single compound conductor of which one part lies completely within the other. The compound conductor is now charged to a high potential with a small influence machine, and the electricity settles down in its position of equilibrium on the two conductors. On removing the key from the cups the charges on S_1 and S_2 are insulated from one another. The outer conductor S_2 is now earthed and the key put into the cups B, C, thus connecting the inner conductor S_1 with the electroscope. It is found that no deflexion of the electroscope occurs, showing that there is no charge on S_1.

This was verified by Cavendish and Maxwell when S_1 and S_2 are spheres, and by Cavendish in the case of one cube inside another. We conclude that whenever one conductor lies entirely inside another, and the two are joined and charged with electricity, none of the charge resides on the inner conductor. When the two shells are joined so as to form one conductor, all points of both are at the same potential. None of this potential is due to the inner conductor since it is entirely devoid of charge: and since the inner conductor may have any shape whatever it follows that the electricity on the outer shell is sufficient by itself to produce constant potential at *all* points inside it. In particular, an electrified spherical shell gives constant potential at all internal points. The only law of force consistent with this can be shown to be that of the inverse square as follows.

Let the potential at distance r from a charge e be written in the form $e\phi'(r)/r$, where ϕ is some function to be determined. Let R (Fig. 33) be a point distant x from the centre O of an electrified spherical shell of radius a, and let PP', QQ' represent two neighbouring planes perpendicular to ORA, such that the angle $AOP = \theta$ and $AOQ = \theta + d\theta$. The electricity must be uniformly spread over the shell, say to amount σ per unit area. If $RP = r$ the potential at R of the part of the shell intercepted between the two

planes is $2\pi\sigma a^2 \sin\theta\, d\theta\phi'(r)/r$, since the area of the intercept is $2\pi a^2 \sin\theta\, d\theta$ and every point of it is at distance r from R. Regarding R as a fixed point and θ as the variable, the equation $r^2 = a^2 + x^2 - 2ax\cos\theta$ when differentiated gives $r\,dr = ax\sin\theta\, d\theta$. Hence the potential of the small intercept is

$$\frac{2\pi\sigma a}{x}\phi'(r)\,dr.$$

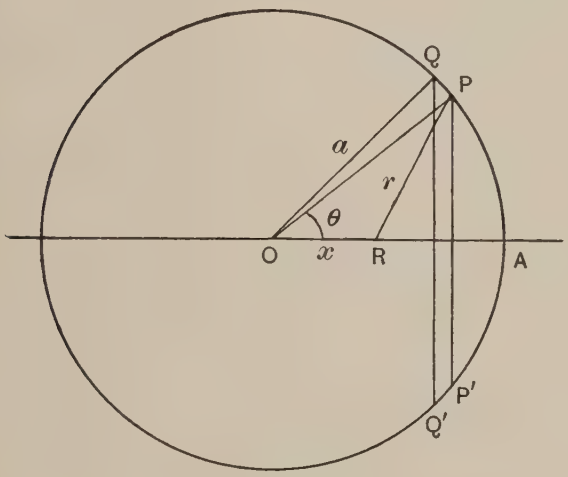


Fig. 33

The extreme limits of r are $a - x$ and $a + x$, so that the potential of the whole electrified shell at R is given by

$$V = \frac{2\pi\sigma a}{x}\int_{a-x}^{a+x}\phi'(r)\,dr = \frac{2\pi\sigma a}{x}\{\phi(a+x) - \phi(a-x)\}.$$

Since V is to be independent of x we have

$$\phi(a+x) - \phi(a-x) = \lambda x,$$

where λ is some constant. Differentiating twice, we get

$$\phi''(a+x) - \phi''(a-x) = 0,$$

showing that $\phi''(x) = B$, a constant. Hence $\phi'(x) = Bx + C$, and the potential at distance r from a charge e is of the form $e\left(B + \frac{C}{r}\right)$.

Now by hypothesis the potential vanishes when r is infinite, so that $B = 0$. Thus $V = eC/r$, and the electric force is

$$E = -dV/dr = eC/r^2,$$

which proves the result.

Maxwell showed that if the law of force was the inverse pth power, then p could not differ from 2 by more than 1/20,000 without giving rise to an observable charge on the inner conductor in his experiment.

29. Electrical units. We now see that two charges e, e' distant r apart in air repel one another with a force of F dynes, where $F = kee'/r^2$ and k is some constant. Evidently k cannot be determined until we have settled the unit in which e and e' are to be measured.

The unit charge of electricity is taken to be that positive charge which repels an equal positive charge distant one centimetre away with a force of one dyne. This amount of electricity is known as the *electrostatic unit of charge*. With these units $F = 1$ when $e = e' = r = 1$, so that $k = 1$. Hence the repulsion between two charges e, e' in general is ee'/r^2 dynes, an attraction being as usual regarded as a negative repulsion. The electric force at distance r from a charge e is

$$E = e/r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

radially outwards, and the potential

$$V = e/r \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

In the case of a volume-distribution the last formula is replaced by $V = \int \frac{\rho d\tau}{r}$, where r is the distance of the point where V is calculated from $d\tau$ and the integration extends over the volume containing charges. Similarly for surface-distributions we have $V = \int \frac{\sigma dS}{r}$.

It will appear that volume-distributions do not exist when electricity is at rest on conductors. We shall consider them, however, because such a distribution is the most general kind, and a surface-distribution can be regarded as a thin volume-distribution of great volume-density.

30. Electric force due to an infinite charged plane. Let RO be the perpendicular let fall from R on the plane, and let PP', QQ' be two circles in the plane with centre O and radii r, $r + dr$ respectively. If $OR = x$ and the angle $ORP = \theta$, then $r = x \tan \theta$, so that the area of the ring between the circles is

$$2\pi r\, dr = 2\pi x^2 \tan \theta \sec^2 \theta d\theta.$$

Each part of the ring is at a distance $x \sec \theta$ from R and acts radially along the line joining it to R. The resolved part along OR, which is evidently the direction of the resultant electric

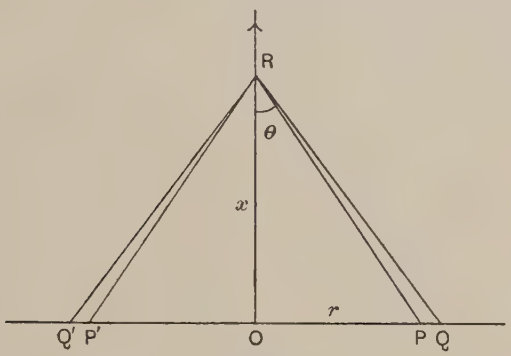


Fig. 34

force, is obtained by multiplying by $\cos \theta$. Hence the electric force at R due to the small ring is

$$2\pi\sigma x^2 \tan \theta \sec^2 \theta d\theta \,.\, \cos \theta / x^2 \sec^2 \theta = 2\pi\sigma \sin \theta d\theta,$$

where σ is the surface-density. Integrating from $\theta = 0$ to $\theta = \frac{\pi}{2}$, the electric force due to the whole disc becomes

$$E = 2\pi\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in the direction OR.

Thus the electric force is the same at all points on one side of the plane. On crossing the plane the direction of the force changes, so that the force undergoes a discontinuous change of magnitude $4\pi\sigma$. Since the force does not become infinite, however, there is no discontinuity in the potential as we cross the sheet.

31. Gauss' theorem. Distribution of charge on a conductor. Since the laws of magnetism and electrostatics are formally identical, the proof of Gauss' theorem (Art. 17)

may be transferred simply to the case of electric charges, and we have:

If dS is an element of a closed surface and E_n the component of electric force along the outward-drawn normal to dS, then the value of $\int E_n dS$, taken over the whole surface, is 4π times the total charge inside S.

Gauss' theorem may be used to show that the electricity at rest on a conductor is confined to a very thin layer near its surface, and is not diffused throughout the volume. If S is any surface described entirely on the body of the conductor, there is no electric force at any point of S, so that $\int E_n dS = 0$. Hence S contains no charge, and since S is quite arbitrary there can be no charge at any point definitely inside the conductor. This conclusion is strikingly exemplified by the experiment of Cavendish already mentioned.

It appears at first sight somewhat surprising that electricity can form up, as it were, in a thin superficial layer; but we must remember that electrons are much smaller than the molecules of matter and have plenty of room even in a very thin layer. The electric "fluid," in fact, must be regarded as almost indefinitely compressible on account of the smallness of the electrons.

A most interesting example of Gauss' theorem is to prove that *the electric force due to a charged sphere is the same, at all external points, as if the charge was collected at the centre of the sphere.* Let E be the electric force at an external point distant r from the centre, and describe a sphere S of radius r concentric with the conductor. It is obvious from reasons of symmetry that E is directed along the radius and coincides with its normal component at points on S. Thus applying Gauss' theorem to the surface S we have $4\pi r^2 E = 4\pi e$, or $E = e/r^2$. The electric force is therefore the same as that due to a charge e concentrated at the centre.

The following result, though not practically important in electrostatics, throws light on a certain theoretical point (see next article). A sphere of radius a contains electricity distributed uniformly throughout the volume with volume-density ρ. It is required to find the electric force at any point P. If P is an external point, then by applying Gauss' theorem as before we have $4\pi r^2 E = 4\pi \cdot \frac{4}{3}\pi\rho a^3$, or $E = 4\pi\rho a^3/3r^2$. For an internal point, the charge inside a sphere of radius r is not $\frac{4}{3}\pi\rho a^3$, but $\frac{4}{3}\pi\rho r^3$. Hence

$E = \frac{4}{3}\pi\rho r = er/a^3$. The electric force is therefore proportional to the distance from the centre of the sphere. The potential is $V = \text{const.} - er^2/2a^3$, and at the surface it takes the value e/a. Hence at internal points

$$V = \frac{3e}{2a} - \frac{er^2}{2a^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

32. Laplace's equation and Poisson's equation. Consider a distribution of electricity with volume-density ρ at the point (x, y, z), ρ being a function of x, y and z. Expressed analytically, Gauss' theorem becomes

$$\int_S (lE_x + mE_y + nE_z)\, dS = 4\pi \int_\tau \rho\, d\tau,$$

where S is any closed surface and τ the space inside it. Transforming the left-hand side by Gauss' transformation, we have

$$\int_\tau \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} - 4\pi\rho \right) d\tau = 0,$$

and τ may be any region of space whatever. It follows that the quantity under the integral must vanish, or

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots (6).$$

Since $E_x = -\partial V/\partial x$, $E_y = -\partial V/\partial y$, $E_z = -\partial V/\partial z$, this becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho,$$

or

$$\Delta V = -4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

which is known as Poisson's equation. In particular, if there is no electricity in the neighbourhood of the point (x, y, z), $\rho = 0$ and

$$\Delta V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

This is known as Laplace's equation.

The subject may also be approached with advantage from another point of view. The potential at the point (x, y, z) due to a charge e at the point (ξ, η, ζ) is $V = e/r$, where

$$r^2 = (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2.$$

Since $\partial r/\partial x = (x - \xi)/r$, we have

$$\frac{\partial V}{\partial x} = -\frac{e\,(x - \xi)}{r^3},$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{e}{r^3} + \frac{3e\,(x - \xi)^2}{r^5}.$$

Similarly $$\frac{\partial^2 V}{\partial y^2} = -\frac{e}{r^3} + \frac{3e\,(y-\eta)^2}{r^5}$$

and $$\frac{\partial^2 V}{\partial z^2} = -\frac{e}{r^3} + \frac{3e\,(z-\zeta)^2}{r^5}.$$

Adding, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{3e}{r^3} + \frac{3e}{r^3} = 0.$$

The effect of superposing any number of charges at different points is to add the separate potentials, leading to Laplace's equation in general. But the proof would obviously not apply without further consideration to charges indefinitely near to the point (x, y, z). It is obvious, therefore, that the term $-4\pi\rho$ in Poisson's equation arises from the electricity in the immediate vicinity of the point considered. A special example will make this clearer.

Let S (Fig. 35) be a surface containing a distribution of electricity of uniform volume-density ρ, O a point inside S. Draw a sphere

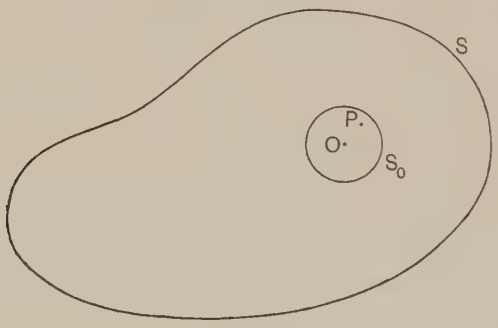


Fig. 35

S_0 of radius a with O as centre, and let V_0 be the potential at the point P of the electricity inside S_0, V_1 that of the electricity between S_0 and S. Then the total potential at the point P is $V = V_0 + V_1$. Since the electricity between S_0 and S is nowhere infinitely near to P, $\Delta V_1 = 0$. The electric force at P due to the charge inside S_0 has been shown to be $\frac{4}{3}\pi\rho r$ along OP: hence if (x, y, z) are the co-ordinates of P with respect to O,

$$\frac{\partial V_0}{\partial x} = -\tfrac{4}{3}\pi\rho x, \quad \frac{\partial V_0}{\partial y} = -\tfrac{4}{3}\pi\rho y, \quad \frac{\partial V_0}{\partial z} = -\tfrac{4}{3}\pi\rho z.$$

Hence $\dfrac{\partial^2 V_0}{\partial x^2} = \dfrac{\partial^2 V_0}{\partial y^2} = \dfrac{\partial^2 V_0}{\partial z^2} = -\frac{4}{3}\pi\rho$, and $\Delta V_0 = -4\pi\rho$

as we should expect. The difficulty therefore is that although S_0 may be made indefinitely small, the second differentials of its potential at internal points are finite, and have a sum distinct from zero. The potential itself is $V_0 = 2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2$, which, since r is less than a, can be made indefinitely small by sufficiently diminishing a.

It follows from Laplace's equation that the potential cannot be a maximum or a minimum at a point not occupied by electric charge. For if the potential has the value V at the point P (x, y, z), its value at a neighbouring point $(x + h, y, z)$ is

$$V + h\frac{\partial V}{\partial x} + \tfrac{1}{2}h^2\frac{\partial^2 V}{\partial x^2} + \ldots.$$

V will not be a true maximum, even if $\partial V/\partial x = \partial V/\partial y = \partial V/\partial z = 0$, unless $\partial^2 V/\partial x^2$ is negative. Similarly $\partial^2 V/\partial y^2$ and $\partial^2 V/\partial z^2$ must be negative, and this is inconsistent with Laplace's equation. In the same way V cannot be a true minimum. This result is known as Earnshaw's theorem.

Since eV is the potential energy of a charge e at P, it follows that a small charged body introduced into an electrostatic field cannot rest in stable equilibrium. Many electrostatic fields will permit of equilibrium positions, or points of no electric force, but it is always possible to find small displacements from those positions which lead to instability. This fact is interesting in showing that free electrons cannot come to rest permanently in separated positions, even though their repulsions are partly neutralised by positive charges scattered about.

33. General principles of electrostatics. The science of electrostatics, which seeks to determine theoretically the equilibrium of electricity on conductors, either isolated or under external influence, has been developed mathematically to a considerable extent. We shall explain the principles on which the development is based. To take a definite case, consider an insulated sphere under the influence of a positive charge e at a point outside it. We know, of course, that there will be positive charge induced on the

part of the sphere nearest the charge e, and negative charge on the remote portions. The effect of the induced charges is to produce an electric field opposed to that of the charge e, and the equilibrium distribution is *that which exactly neutralises the impressed field at all internal points*. That is, knowing only that the charges reside on the surface, we have to determine the surface-density at every point so that the total potential due to all causes is constant inside the conductor. A problem in electrostatics is thus not immediately soluble, since the potential depends on the distribution of the charge and the latter is not given beforehand. The process of solution is essentially tentative, although general methods have been devised for certain classes of problems. One of these methods is described in Art. 41.

34. Force just outside a conductor. Stress per unit area of a conductor. The electric force just outside a conductor is $4\pi\sigma$, where σ is the surface-density in the neighbourhood, and is directed along the normal to the conductor.

Take an element of area PQ on the conductor, its linear dimensions being of the first order of small quantities. Let R, S be two points whose distance from the element is of the second order of small quantities, R being inside the conductor and S in air. The calculation of the electric force at R, S is conveniently divided into two parts, namely that of the charge on PQ and that of the charge on the rest of the conductor and any other electricity present.

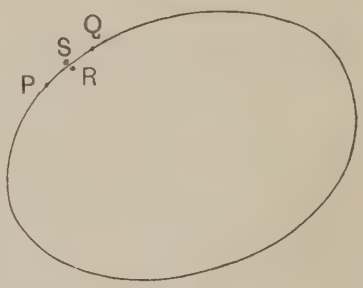


Fig. 36

Evidently the latter is nearly the same for the points R, S, while the former is reversed. As regards R and S the element PQ acts like an infinite electrified plane, so that the electric force at R due to PQ is $2\pi\sigma$ directed along the inward normal. The total electric force at R must vanish since R is inside the conductor: thus the electric force due to the remaining charges is $2\pi\sigma$ along the outward normal. At the point S the forces due to both causes are $2\pi\sigma$ along the outward normal,

altogether $4\pi\sigma$ in that direction. This result is known as Coulomb's law. If the electric force at a point just outside a conductor placed in any electric field is known, the surface-density in the vicinity can also be found.

If A is the small area of the element PQ, the charge on the element is σA. The electric force at points on PQ due to the charges in the field other than that on PQ itself is $2\pi\sigma$. Hence the charge on the element PQ is acted on by a force $2\pi\sigma^2 A$ due to the remaining charges; i.e. the surface of a charged conductor in any field experiences a stress of magnitude $2\pi\sigma^2$ per unit area, directed at every point along the outward normal to the surface. If the distribution of charge is known we can calculate from this formula the forces acting on the conductors in the field.

35. Lines and tubes of force. The theory of these follows on the same lines as the corresponding theory for magnetism (Art. 18). Recapitulating their properties in free space we have

(1) The direction of a line of force is that of the resultant electric force at every point. Thus lines of force run from high to low potential.

(2) The cross-section of a small tube of force at any point of its length varies inversely as the resultant electric force.

The following important property gives a relation between the tubes of force and the charges in the field:

(3) Lines of force start out from positive charges on conductors and end on negative charges on conductors of lower potential. The amounts of electricity at the ends of a tube of force are equal and opposite.

The first part of this theorem is evident from (1). To prove the second, let a small tube of force start from a place where the surface-density is σ and end where it is σ'. Cut off the tube by surfaces parallel to the surface of the conductor and just outside it (Fig. 37), and consider the truncated tube so formed. The normal component of electric force on dS is $-4\pi\sigma$ since the normal to the truncated tube is opposite to the normal to the conductor. Similarly at the end dS'. Thus, since the truncated tube encloses no charge Gauss' theorem gives

$$4\pi\sigma dS + 4\pi\sigma' dS' = 0, \quad \text{or} \quad \sigma dS = -\sigma' dS'.$$

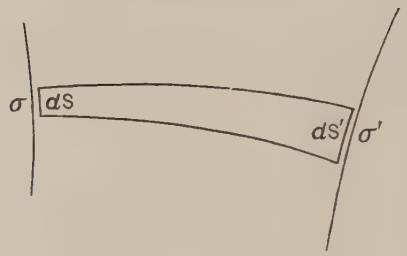


Fig. 37

Moreover, the constant product of the electric force and cross-section of the tube at any point is equal to 4π times the charge from which the tube started out.

The result (3) follows for any finite tube by addition. Another easy proof would have been to produce the tubes into the conductors at both ends and then close the ends and apply Gauss' theorem.

Just as lines of magnetic force can be exhibited to the eye by means of iron filings, so lines of electric force due to two conductors can be shown by the tendency of small elongated bodies to set along the lines of force. Several methods have been devised for making such diagrams, but it is not very easy to obtain good effects. The following simple method will be found fairly satisfactory.

A piece of stiff, smooth cardboard is stained black and well dried, and tinfoil is then pasted on one side to represent the conductors in the field. The conductors are joined to the terminals of a Wimshurst machine by spring clips, and the machine is rotated uniformly by means of an auxiliary motor. The place of the iron filings in magnetic charts is taken by small crystals of a substance which crystallises in prismatic form, for example oxalic acid. These are sprinkled uniformly on the cardboard out of a pepper castor. As the crystals fall they set along the lines of electric force provided that the electric field is strong enough, and the diagrams are then photographed. Fig. 38, showing the lines of force between a sphere and plane, has been obtained in this way. The appearance of the diagrams is improved by cutting out pieces of tinfoil of the right shape and pasting them

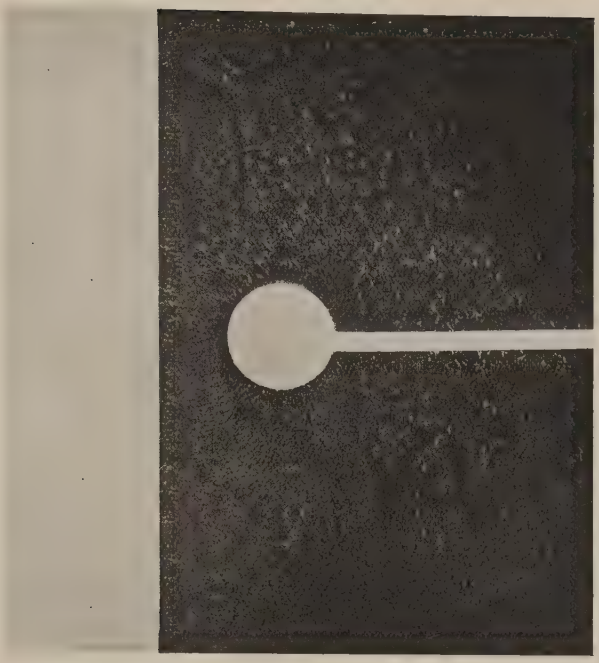


Fig. 38

over the negative so as to make the parts representing the conductors absolutely opaque. It is not necessary to tap the piece of cardboard during the experiment.

36. Capacity of a conductor. If a charge e is required to raise an isolated conductor to a potential V, a charge ne will raise it to a potential nV. Thus the ratio e/V depends only on the shape and size of the conductor, and not on its charge. Writing $e/V = C$, C is called the *capacity* of the conductor.

Since the charge on a conductor charged to potential V is CV, conductors of large capacity can receive considerable charges without having their potentials greatly raised. This is exactly analogous to a water-reservoir of large area, which can hold large quantities of water without having recourse to great depths.

For a sphere of radius a, $e/V = a$, so that the capacity of a sphere is equal to its radius. Thus if a concrete notion of the unit of capacity is required it is afforded by that of a sphere

of radius one centimetre. This is called the *electrostatic unit of capacity*.

As an example we may find the capacity of a circular cylinder whose length l is large compared with its radius a. The distribution of electricity on the cylinder will be nearly uniform in the middle, and only appreciably disturbed at distances from the ends comparable with the radius. Thus if O is the middle point of the axis the potential at O will be approximately that of a uniform layer of surface-density σ over the curved surface, the electricity on the ends being neglected. The charge on the strip between planes distant z and $z+dz$ from O is $2\pi\sigma a\,dz$, giving rise to potential $2\pi\sigma a\,dz\,(a^2+z^2)^{-\frac{1}{2}}$ at O. Hence the total potential at O, i.e. the potential of the cylinder, is given by

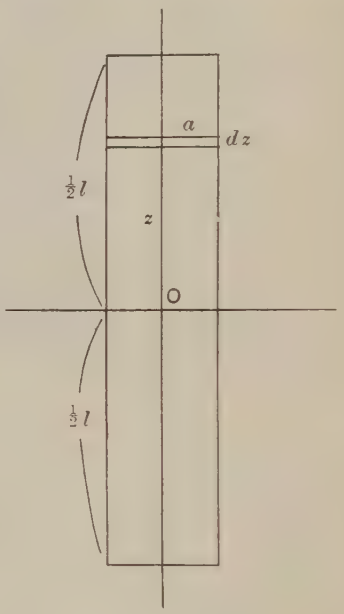


Fig. 39

$$V = \int_{-\frac{1}{2}l}^{\frac{1}{2}l} \frac{2\pi\sigma a\,dz}{(z^2+a^2)^{\frac{1}{2}}} = 4\pi\sigma a \sinh^{-1}\frac{l}{2a}$$

$$= 4\pi\sigma a \log_e \left\{\frac{l}{2a} + \left(1+\frac{l^2}{4a^2}\right)^{\frac{1}{2}}\right\}$$

$$= 4\pi\sigma a \log_e \frac{l}{a} \text{ approximately.}$$

Also the charge on the cylinder is approximately equal to $2\pi\sigma la$. Hence

$$C = \frac{e}{V} = \frac{l}{2\log_e (l/a)}$$

$$= \frac{0{\cdot}217\,l}{\log_{10}(l/a)} \quad \ldots\ldots\ldots (9).$$

For example, an isolated wire 1 metre long and 2 mm. in diameter has a capacity of 7·2 electrostatic units. When the wire is joined to other pieces of apparatus, however, the distribution of charge is altered and the total charge on the wire is not given by the above formula.

37. Electrical screening. If an electroscope is surrounded by a wire-gauze cage connected to earth, the two being insulated from each other, no movement of the leaves can be produced by

the presence of the most powerful electric charges outside the cage. If the cage is removed the leaves, of course, diverge under the influence. The screening effect depends on the fact that the cage is very nearly a closed conductor surrounding the electroscope. It can be shown that the screening from outside action is complete in the case of a conductor entirely surrounding another.

Of three conductors C_1, C, C_2, insulated from one another, let C completely enclose C_1 and be kept at a fixed potential V. The figure shows the conductors, with a sketch of the lines of force passing between them: S_1 is the inner surface of C and S_2

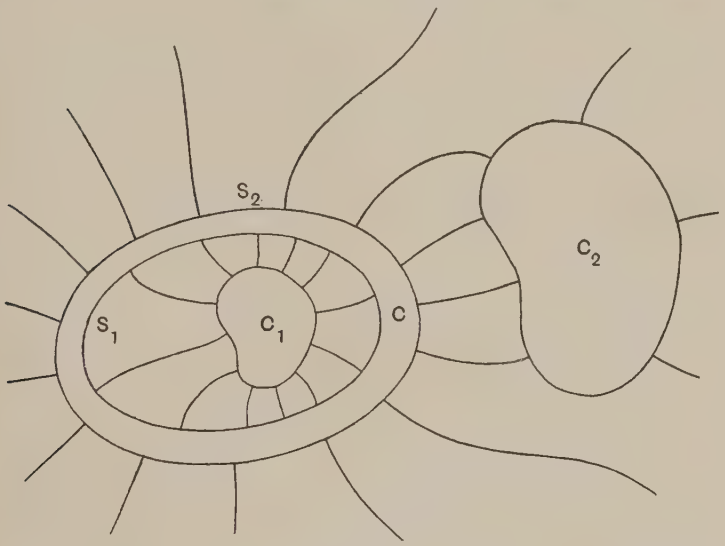


Fig. 40

the outer. First imagine that S_2 is removed to infinity, C_2 being absent, so that C becomes an infinite conductor with a cavity S_1. There will be a certain distribution of electricity on C_1 and S_1 corresponding to a given charge on C_1 when C is at zero potential. Secondly, suppose that the whole of the interior of C is filled up with metal. Then there will be a second distribution of electricity on C_2 and S_2 corresponding to a given charge on C_2 when C is maintained at potential V. If now the two distributions are superposed we get a distribution of charges on C_1, S_1, C_2, S_2 which satisfies all the conditions of the actual problem,

and is therefore the distribution when all the conductors are present. For the first distribution gives zero potential everywhere outside S_1, while the second gives potential V everywhere inside S_2. Hence in the combined distribution we have potential V in the space between S_1 and S_2, and also constant potential over C_1 and C_2.

The two distributions are evidently entirely independent of one another, so that the charge on C_1 does not move about when the position of C_2 is altered, nor is the potential at any point inside C changed in any way. Similarly C_1 has no effect on the electrostatic phenomena outside the conductor C.

This theory explains a celebrated experiment of Faraday, which consists in fixing a long metal cylinder to the knob of an electroscope and lowering small charged bodies into it by means of a silk fibre. On lowering a charged insulated body the leaves diverge, as the body induces an equal and opposite charge on the walls of the cylinder and repels a charge of its own sign on to the electroscope. It is found that the divergence of the leaves is independent of the position of the body inside, provided the body is fairly inside the cylinder, and that it does not alter even if the body touches the wall of the cylinder.

A second experiment of the same kind is to verify the fact that equal and opposite charges are produced by friction, required by electron theory. Let a silk handkerchief be tucked into the cylinder, and a glass rod then thrust inside and turned round vigorously. No divergence of the leaves of the electroscope is observed, so that the total charge generated is zero. The sensitiveness of the experiment may be tested by withdrawing the glass rod: the negative charge on the silk then causes a large divergence to take place.

The principle of screening is constantly used in the construction of electrical instruments, depending on the fact that, since a closed conductor screens perfectly, a nearly closed one is nearly perfect. Wires leading to electrometers are screened from outside influence by passing them down the centre of earthed brass tubes fitted with ebonite plugs, the instrument itself being screened by means of a brass case connected to earth. The electroscope in Fig. 28 is screened to some extent by its case.

38. Condensers. While it is easy to measure a difference of potential accurately, the direct measurement of electric charge presents great difficulties. It is thus advantageous to reduce the measurement of charge in some way to that of potential; and this might be done with the aid of conductors of known form and capacity, the charge on which would be known in terms of their potentials. There are however two difficulties in the way, namely that to obtain a large enough capacity large conductors would have to be used, and that isolated conductors are considerably affected by chance external influence. These difficulties are obviated by the proper use of *condensers*.

A condenser consists essentially of two conductors, all or nearly all the lines of force from one of which (called the *inner plate*) end on the other. Evidently one way of securing this is to have the latter conductor enclosing the former almost entirely, an arrangement which also protects the inside plate from outside influence. The *capacity* of the condenser is the ratio of the charge on the inner plate to the difference of potential between the two plates.

In addition to its great utility as a measuring instrument, the condenser is also used to store electricity temporarily, as a condenser of large capacity can store a great deal of electricity by the use of a low source of potential only. If the potential-differences are very large (of the order of 100 electrostatic units or 30,000 volts) the insulation of the air will give way: in this case it may be necessary to employ compressed air between the plates. At these potentials, however, insulation generally gives rise to some difficulty.

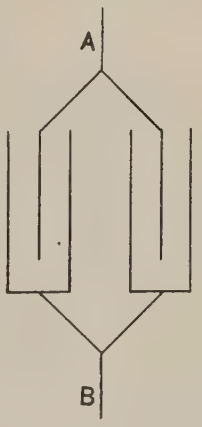


Fig. 41

We can obtain a condenser whose capacity is the sum of two given condensers by joining them *in parallel*. This is done by joining the outside plates together with a wire, and also the inside plates, as shown symbolically in the figure. Let C, C' be the capacities of the two condensers, V the difference of potential between the "terminals" A, B when the condensers are in parallel.

Then the charge on the inner plate of the first condenser is CV, and on the second $C'V$. In the compound condenser the total charge is $(C + C')V$, so that the capacity is $C + C'$.

A large capacity is conveniently built up by taking a number of smaller condensers and joining them in parallel, when we get a capacity which is the sum of those of the separate condensers. It is also possible to make a condenser whose capacity is less than that of any of the condensers by joining the inner plate of each condenser to the outer plate of the next, but this method of joining condensers is seldom used.

39. Parallel plate and cylindrical condensers. Guard-rings. Two parallel plates brought close together form a condenser, and then it is immaterial which plate we regard as the inner one. Suppose that we have two plates C_1, C_2, each of area A, so great in comparison with the distance d apart that the plates may be

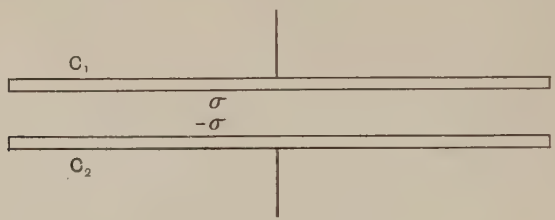


Fig. 42

considered as infinite. The lines of force then run practically straight across between the plates, and the surface-density is uniform except near the edges. Further, if the plates are joined at the back to other apparatus, nearly all the electricity lies on the opposing surfaces of the plates and there is very little at the back. If $\pm\sigma$ are the surface-densities, the charge on the positive plate is σA. Each plate gives rise to an electric force of magnitude $2\pi\sigma$, and it is easy to see that the separate forces assist one another between the plates and are in opposite directions everywhere else. Hence there is an electric field of strength $4\pi\sigma$ between the plates, and no force at the back. The difference of potential between the plates is $4\pi\sigma d$, so that the capacity is $\sigma A/4\pi\sigma d = A/4\pi d$.

Thus if d is made small the capacity can be made large without the use of large plates. Parallel plate condensers are sometimes

constructed with three plates, the two outer ones being joined together to a single terminal. The capacity is then $A/2\pi d$, and the arrangement is a better one for screening the inner plate.

Fig. 43 shows a diagram of the lines of force between the plates

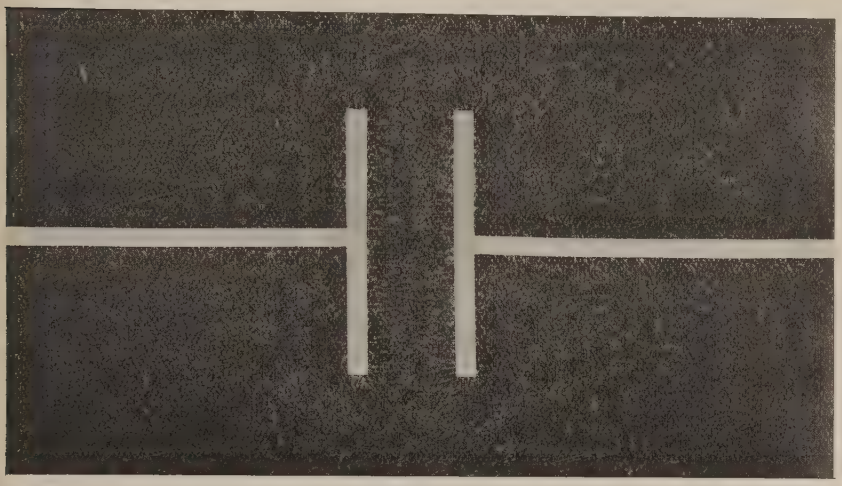


Fig. 43

of a condenser, prepared by the method previously described. It will be noticed that the lines of force pass straight across between the plates even when the latter are quite small, but that they spread out near the edges. The irregular arrangement of the crystals at the back of the plates shows that there is very little electric force there.

The spreading out of the lines of force near the edges makes the capacity of an actual parallel plate condenser somewhat greater than that calculated from the formula $A/4\pi d$. The

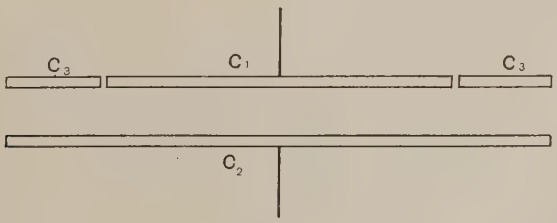


Fig. 44

guard-ring is a device due to Lord Kelvin for avoiding this uncertainty. The top plate is composite, consisting of a circular disc C_1 surrounded by a concentric ring C_3 separated from it by a narrow gap. When C_1 and C_3 are at the same potential the lines of force from C_2 pass very nearly straight across to C_1, and the charge on the inner plate C_1 is $AV/4\pi d$, where A is the area of C_1 and V the difference of potential between C_1 and C_2. In using such a condenser the plates C_1, C_3 must be at the same potential at the time of making the experiment, otherwise there will still be an error due to the edges.

Another arrangement is the cylindrical condenser, consisting of two coaxial cylinders C_1, C_2, whose length is large compared

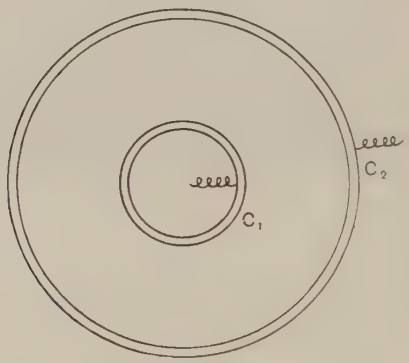


Fig. 45

with the radii. The "plates" C_1, C_2 are usually made of brass tubing. Let a be the *outside* radius of C_1, b the *inside* radius of C_2, l the length, and let E be the electric force at distance r from the centre, where $a < r < b$. Imagine a cylinder drawn with radius r and closed by two planes perpendicular to the axis and distant one centimetre apart. In applying Gauss' theorem to this cylinder we get no contribution to $\int E_n dS$ from the ends, since the electric force is everywhere radial. On the curved surface E_n is identical with E, so that if q is the charge per unit length of the inner tube, we have $2\pi rE = 4\pi q$, or $E = 2q/r$. The difference of potential between the plates is therefore $V = 2q \log (b/a)$, and

the charge on the inner plate approximately $e = lq$. Hence the capacity of the condenser is

$$C = \frac{l}{2 \log_e (b/a)} = \frac{0{\cdot}217l}{\log_{10} (b/a)} \ldots\ldots\ldots\ldots (10).$$

The reader will notice that the total charge on the outer cylinder is not necessarily equal to that on the inner, as there may be charges present on the outermost surface of all.

Fig. 46 shows a convenient form of cylindrical condenser for laboratory work. Here C_1 is the inner plate, C_2 the outer plate, and C_3 C_3 are short cylinders separated from C_1 by narrow air-gaps. When C_1 and C_3 are at the same potential the lines of force pass

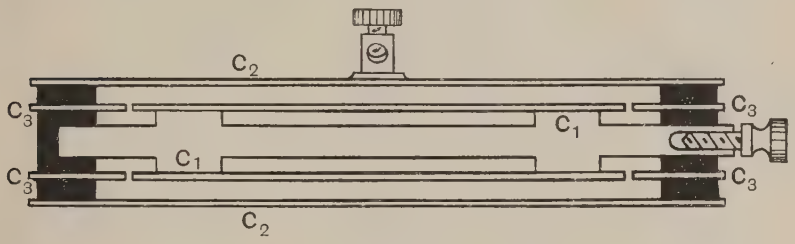


Fig. 46

nearly straight across from C_2 to C_1, and we can make use of equation (10) to calculate the charge on C_1. In use the end rings C_3 C_3 are usually earthed, and the condenser is valuable in experiments in which the inner plate is at zero potential at the time of measurement (cf. Art. 69).

40. Theory of a system of conductors. The condenser is a particular arrangement of two, or with a guard-ring three, conductors. We have now to consider the principles involved when there are n conductors at potentials V_1, V_2, ... V_n, the total charges being e_1, e_2, ... e_n. Since the potential at any point of a field only involves the charges to the first degree, there must be a set of linear equations connecting e_1, e_2, ... with V_1, V_2, ... Hence

$$\begin{aligned} e_1 &= q_{11}V_1 + q_{12}V_2 + \ldots + q_{1n}V_n, \\ e_2 &= q_{21}V_1 + q_{22}V_2 + \ldots + q_{2n}V_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_n &= q_{n1}V_1 + q_{n2}V_2 + \ldots + q_{nn}V_n. \end{aligned}$$

Here q_{11} is the total charge on the first conductor when its potential is unity and that of all the other conductors zero; and then q_{1r} is the charge on the rth conductor, and so on. The coefficients q_{rr} are called *coefficients of capacity*, and q_{rs}, with r and s unequal, *coefficients of influence*. The coefficients have several properties, the most important being the reciprocal relation $q_{rs} = q_{sr}$. A little reflexion will show that Gauss' reciprocal theorem $\Sigma eV' = \Sigma e'V$ (Art. 24) can be extended to the case in which e is the total charge on a conductor and V the corresponding potential. Taking as the two distributions those in which

(1) all the V's vanish except V_r,

(2) all the V''s vanish except V_s';

we have $e_s V_s' = e_r' V_r$. But by definition we have

$$e_s = q_{sr} V_r \text{ and } e_r' = q_{rs} V_s'.$$

Hence

$$q_{rs} = q_{sr}.$$

The coefficients q depend on the form and position of the conductors, and are only known in a few simple cases, since the electrostatic problem of the distribution of electricity over the surfaces has to be solved before they can be calculated. In particular, the coefficient q_{rr} is not the same as the capacity of the rth conductor when it alone is present in the field. We may illustrate this theory by considering the cylindrical guard-ring condenser depicted in Fig. 46. The general equations are

$$e_1 = q_{11}V_1 + q_{12}V_2 + q_{13}V_3,$$
$$e_2 = q_{21}V_1 + q_{22}V_2 + q_{23}V_3,$$
$$e_3 = q_{31}V_1 + q_{32}V_2 + q_{33}V_3.$$

Take $V_1 = V_3 = 0$, $V_2 = 1$. Then $e_1 = q_{12}$, $e_2 = q_{22}$, $e_3 = q_{32}$. Since C_3 acts as a guard-ring to C_1 we have, from the last article,

$$e_1 = -\frac{l}{2\log_e (b/a)}.$$

Hence

$$q_{12} = -\frac{l}{2\log_e (b/a)}.$$

The other coefficients q_{11}, q_{22}, q_{33}, q_{23}, q_{31} are difficult to calculate.

If C_3 is at zero potential, but C_1 at potential V_1, we should

have $e_1 = q_{11}V_1 + q_{12}V_2$. Thus the charge on C_1 would be incorrectly calculated from the formula $e_1 = \dfrac{l}{2\log_e (b/a)}(V_1 - V_2)$ unless C_1 and C_3 were at the same potential, and the error could not be estimated precisely, since q_{11} is unknown. This illustrates the improper use of guard-rings.

41. Two-dimensional problems in electrostatics. The distribution of electricity on infinitely long parallel cylindrical conductors can be found in many cases by the method of *conjugate functions*. Taking the axis of z parallel to the generators of the cylinders, the problem reduces to finding a solution of the equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ which shall make V constant over two given curves, namely the cross-sections of the cylinders, e.g. by the plane $z = 0$. Real quantities U and V such that $U + iV$ is a function of $x + iy$ only are said to be conjugate functions of x and y. It is well known that in this case

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y},$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Hence the curves $U = \text{const.}$, $V = \text{const.}$ cut at right angles everywhere, and V may be taken as the potential in some electrostatic problem. The curves $V = \text{const.}$ are the equipotential curves, or sections of the equipotential surfaces, and $U = \text{const.}$ the lines of force.

By making different assumptions as to the form of the function we obtain the solution of various problems. Suppose for example that $U + iV$ is connected with $x + iy$ by the inverse relation

$$x + iy = c \cot \tfrac{1}{2}(U + iV).$$

Writing $t = \tan \frac{1}{2}U$, $\tau = \tanh \frac{1}{2}V$, we have

$$x + iy = c\,\frac{1 - it\tau}{t + i\tau}.$$

On multiplying up and equating real and imaginary parts, this gives

$$xt - y\tau = c, \quad \frac{x}{t} + \frac{y}{\tau} = -c.$$

The equipotential curves are found by eliminating t from these equations.

Thus $$x^2 + y^2 + cy\left(\tau + \frac{1}{\tau}\right) + c^2 = 0,$$

or $$x^2 + (y + c \coth V)^2 = c^2 \operatorname{cosech}^2 V,$$

on reduction. Hence the equipotential curves are a family of coaxial circles. The circle corresponding to the potential V has centre $(0, -c \coth V)$ and radius $c \operatorname{cosech} V$. The lines of force are easily found to be given by the equation

$$(x - c \cot U)^2 + y^2 = c^2 \operatorname{cosec}^2 U \ldots\ldots\ldots\ldots (11),$$

which represents the orthogonal family of coaxial circles.

Let it be required to determine the distribution of electricity on two parallel conducting circular cylinders of radius a whose axes are at distance $2h$ apart and which carry equal and opposite charges. Taking as origin a point half-way between their axes, we can identify the cross-sections with the equipotentials $V = \pm V_0$ of the preceding transformation by writing

$$h = c \coth V_0, \quad a = c \operatorname{cosech} V_0.$$

Hence $$c = (h^2 - a^2)^{\frac{1}{2}}, \quad \cosh V_0 = h/a \ldots\ldots\ldots\ldots (12).$$

Since c is now known, the line of force for any value of U can be constructed from equation (11). Fig. 47 shows the lines of

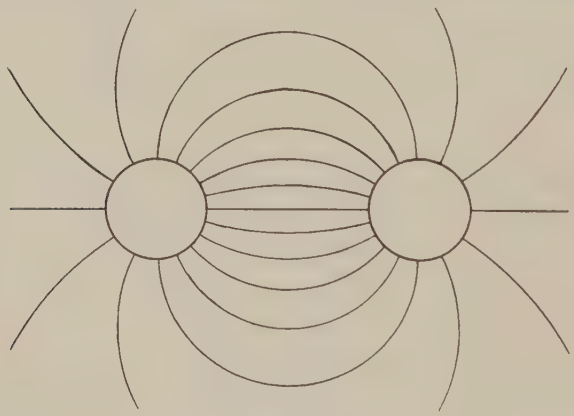


Fig. 47

force for equidistant values of U. It will be noticed that the system is really equivalent to a condenser when the charges are

equal and opposite, as all the lines of force from one cylinder end on the other. In order to find its capacity we have only to find the charge per unit length of either conductor. This is determined by the following general theorem.

Let ν (Fig. 48) be the normal to an equipotential, s the direction of the tangent, E the resultant electric force at P. The direction-cosines of s are easily seen to be $\left(-\frac{1}{E}\frac{\partial V}{\partial y}, \frac{1}{E}\frac{\partial V}{\partial x}\right)$, so that

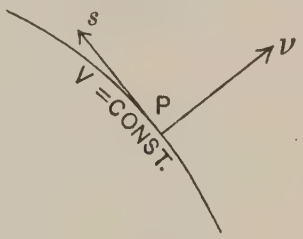


Fig. 48

$$\left(\frac{\partial U}{\partial s}\right)_{V \text{ const.}} = -\frac{1}{E}\frac{\partial V}{\partial y}\frac{\partial U}{\partial x} + \frac{1}{E}\frac{\partial V}{\partial x}\frac{\partial U}{\partial y}$$

$$= -\frac{1}{E}\left\{\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right\} = -\frac{E^2}{E} = -E.$$

But now if the equipotential considered is the surface of a conductor $E = 4\pi\sigma$. Hence, disregarding signs, we have

$$\sigma = \frac{1}{4\pi}\frac{\partial U}{\partial s}.$$

The charge per unit length of the conductor between the generators through the points P, Q is therefore given by

$$e = \int_P^Q \sigma ds = \frac{U_Q - U_P}{4\pi} \quad \ldots\ldots\ldots\ldots (13).$$

Hence if U is found in terms of the position of a point on the cross-section, the charge between any two points is known. Since the lines of force in Fig. 47 are drawn for equidistant values of U they divide the conductors into regions carrying equal charges, thus showing graphically the greater concentration of charge on the parts of the cylinders which are nearest together. In going once round one of the cross-sections U increases by 2π, so that the total charge per unit length of either cylinder in the present case is $\frac{1}{2}$ from equation (13). Since the difference of potential between the cylinders is $2V_0$, the capacity per unit length is

$$\frac{1}{4V_0} = \frac{1}{4\cosh^{-1}(h/a)}.$$

This formula will be practically applicable to two finite cylinders

whose length is great in comparison with the distance apart of their axes. Writing l for the former and $d = 2h$ for the latter, the capacity is

$$C = \frac{l}{4\cosh^{-1}\left(\frac{d}{2a}\right)} = \frac{0{\cdot}109l}{\log_{10}\left\{\frac{d}{2a} + \left(\frac{d^2}{4a^2} - 1\right)^{\frac{1}{2}}\right\}} \quad \ldots(14).$$

For example, if we take a length of wire of 2 mm. diameter, covered with cotton insulation $\frac{1}{2}$ mm. thick, cut it in two and lay the halves alongside one another so as to touch everywhere, we have a condenser whose capacity is 13 electrostatic units per metre of wire used. The capacity of long coils of wire wound back on themselves is often considerable*. In case the wires are at a great distance apart compared with their diameters, equation (14) becomes

$$C = \frac{0{\cdot}109l}{\log_{10}(d/a)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

42. Sensitive electroscopes. Electroscopes are very sensitive to charge on account of the small capacity of the gold leaf, by virtue of which a small charge produces considerable difference of potential between the leaves and the case. They are not, however, usually sensitive to potential itself. This sensitiveness can be obtained by improved design and by reading the deflexion of the leaves with a microscope. C. T. R. Wilson has designed a very delicate electroscope, an improved form of which is shown in Fig. 49. The "knob" is represented by a mercury cup C to which the external apparatus can be connected, leading down a quartz tube to the gold leaf L. An insulated plate P is arranged opposite L, so that its distance from L can be adjusted by means of a screw. The whole is surrounded by a metal case B connected to earth, a small window being left for microscopical observation of the gold leaf. The case is mounted on a stand so that it can be tilted at various angles to the horizon. In use the plate P is maintained at a constant potential of about 200 volts (i.e. about $\frac{2}{3}$ electrostatic unit). When the gold leaf is connected to earth it is attracted more or less to the plate, and on having its potential

* The capacity of a 500 ohm coil in a resistance box may amount to 100 electrostatic units, i.e. if cut in two the coil would form a condenser of this capacity.

raised by a small amount the leaf moves to the right or left according as its potential is of the same or opposite sign to that

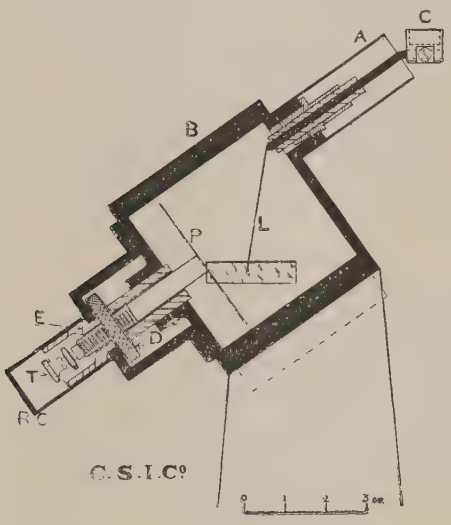


Fig. 49

of the plate P. The object of tilting is to diminish the effect of gravity on the gold leaf. If the angle made by the plate with the horizon is small, or if the potential of the plate is large, the mutual attraction of plate and leaf may be such as to make the equilibrium of the leaf unstable, but by seeking a position in which the leaf is just stable great sensitiveness can be obtained.

In another sensitive form of electroscope two small plates are set up parallel to one another and a fine quartz fibre is stretched between them on insulating supports. The fibre is silvered to make it conduct, and maintained at a high potential by means of a battery of small cells. If the two plates are at different potentials the fibre is pulled aside and the deflexion is read off with a microscope focused on the fibre. This instrument is very quick in action and suited to the observation of transient phenomena, since the fine fibre quickly takes up its position of equilibrium.

43. Electrometers. An electrometer is an instrument for the accurate measurement of differences of potential. Certain

electroscopes really fulfil this condition, but it is usual to reserve the name electrometer for instruments of rather different types, of which the following may be taken as examples.

Kelvin's *absolute electrometer* depends on measuring the pull between the plates of a parallel plate condenser. If A is the area of a plate fitted with a guard-ring, the thrust exerted on it is $2\pi\sigma^2 A$, where σ is the surface-density. If d is the distance between the plates and V_1, V_2 their potentials, then $V_1 - V_2 = 4\pi\sigma d$. Thus the thrust on the plate is

$$\frac{A\,(V_1 - V_2)^2}{8\pi d^2}.$$

It is clearly immaterial which plate is taken as positive.

Fabry and Perot have designed a very sensitive absolute electrometer, in which the plates are separated by three springs, variations of the distance being observed by an optical interference method. Their electrometer was capable of measuring accurately the difference of potential between the terminals of an ordinary electric battery, that is, differences of potential of the order of 1/200th of an electrostatic unit. The ordinary form is only suited for high potentials of the order of 10 electrostatic units.

Absolute electrometers are not suited for ordinary laboratory work, for which the best type is the *quadrant electrometer*, also due to Lord Kelvin. We shall describe a modification due to Dolezalek, now in general use (Fig. 50). The moveable part of the instrument is a so-called "needle" made of paper covered with tin-foil and cut out in order to decrease the weight. This is suspended from the central rod of the instrument by a fibre of quartz, which is very strong and true in its elastic properties. If the fibre is not silvered it can be made conducting by dipping it every now and then in a conducting liquid, for example a solution of calcium chloride. The needle can be charged up to a suitable potential by means of a battery of small Leclanché cells. The quadrants are four insulated pieces of brass so shaped that they fit close together and form a box nearly enclosing the needle. A light mirror is fixed just above the needle so as to show the angle turned through by reflexion of the light from a lamp on to a graduated scale. Fig. 51 shows a suitable arrangement. The

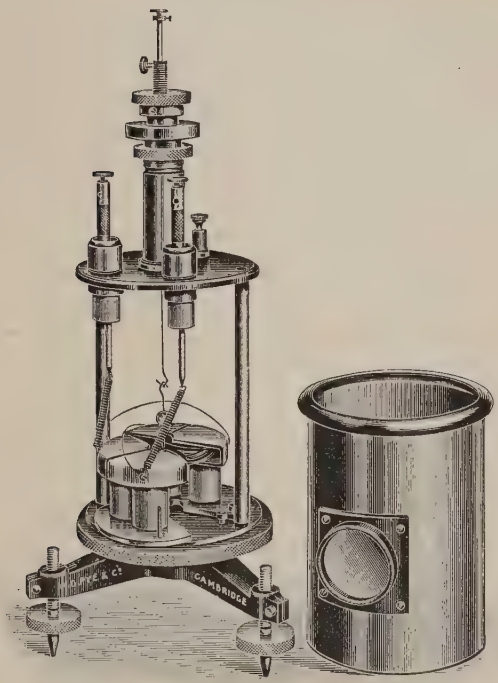


Fig. 50

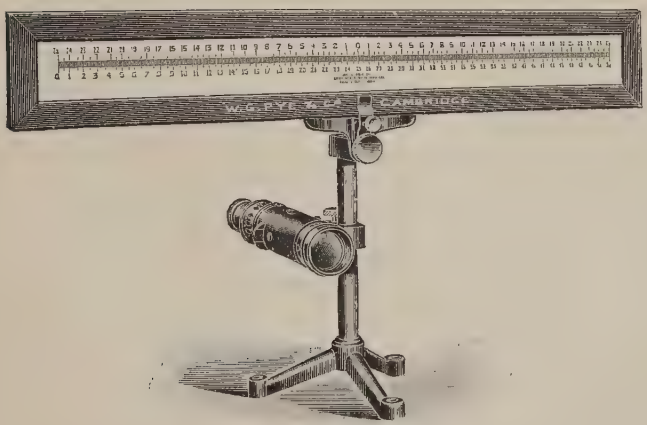


Fig. 51

electrometer is enclosed in a brass case connected to earth to guard against outside influence, and pierced only with small holes for the terminals and a small window for use with the lamp and scale.

Fig. 52 gives a top view of the needle, the upper half of the quadrants being removed in order to show it. The quadrants

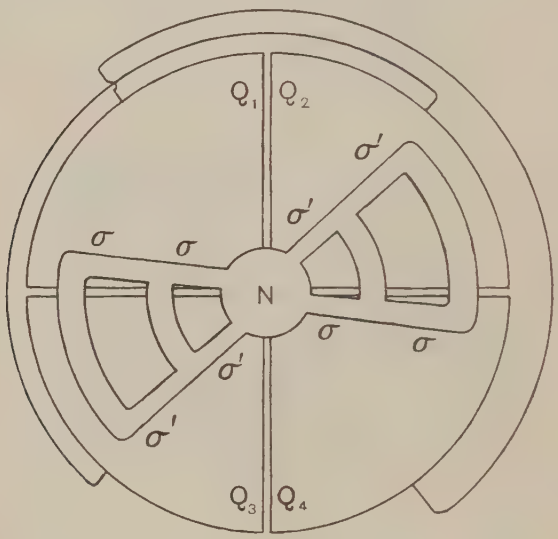


Fig. 52

Q_1Q_4, Q_2Q_3 are joined together, and each pair connected to a terminal of the instrument.

Suppose now that the pair of quadrants Q_1Q_4 is at potential V_1, the pair Q_2Q_3 at potential V_2, and the needle charged to potential V. We shall show that the deflexion of the needle is approximately proportional to $(V_1 - V_2)(2V - V_1 - V_2)$. At any point where the surface-density is σ the needle is acted on by a force $2\pi\sigma^2$ per unit area normal to the surface. Thus the only forces tending to produce a couple about the suspension are due to the charges on the straight outside edges of the needle, and if σ and σ' are the surface-densities at these edges the couple tending to turn the needle round is proportional to $\sigma'^2 - \sigma^2$. The angle of deflexion shown in Fig. 52 is considerably greater

than that occurring in practice (about 7°), so that the edges always remain well within the quadrants in which they are to begin with. The edge σ' under the quadrant Q_2 is thus practically affected by this quadrant alone, and the surface-density at the edge is proportional to the difference of potential between the needle and quadrant. Thus $\sigma' \propto V - V_2$ and $\sigma \propto V - V_1$, so that the couple on the needle due to all the charges is proportional to $(V - V_2)^2 - (V - V_1)^2$, i.e. to $(V_1 - V_2)(2V - V_1 - V_2)$. If θ is the deflexion, the torsional couple due to the fibre is proportional to θ. Hence in the position of equilibrium we have

$$(V_1 - V_2)(2V - V_1 - V_2) = \lambda\theta \quad \ldots\ldots\ldots (16),$$

where λ is some constant.

In the ordinary (or *heterostatic*) method of using the electrometer one pair of quadrants is earthed, and the potential of the needle is much higher than that of the other quadrants. Thus the deflexion is nearly proportional to the potential of the other pair of quadrants. Greater accuracy could be obtained if desired by reversing the potential of the needle and taking the mean of the two observations. Dolezalek has recently introduced an electrometer in which the needle is cut in two and the halves raised to equal and opposite potentials, the four quadrants being reduced to two "binants." The slight error due to the needle is thus eliminated and accurate readings can be taken with deflexions on one side only.

The quadrant electrometer is much more sensitive as a mere detector of electricity than the common electroscope. Ordinary patterns will detect a difference of potential of about 1/100th of a volt (1/30,000th of an electrostatic unit of potential), so that in spite of its considerable capacity (about 50 electrostatic units) very small charges can be measured. The sensitiveness depends of course on the use of a very fine fibre for the suspension. Of late years fine metal suspensions have come into use instead of quartz fibres, and are convenient when the greatest sensitiveness is not required, as they require no dipping to make them conduct.

44. Contact potential. Piezo- and pyro-electricity. Fig. 53 represents diagrammatically a parallel-plate condenser with one plate connected to an electrometer and the other to earth.

The other pair of quadrants is earthed as usual, while A is earthed to begin with and then insulated. If the plate B is moved about

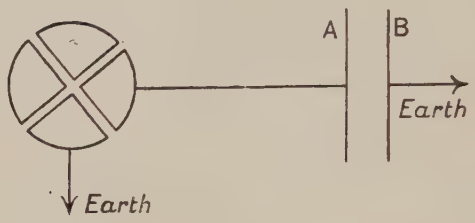


Fig. 53

in front of A the electrometer is unaffected as long as the plates are of the same metal. With *different* metals, however, if we start with the plates close together and then draw them quickly apart, the spot of light moves away, returning to its original position when the plates are moved back again. These influence effects are exactly what would happen if we moved one plate of a charged condenser about in front of the other, and we are thus led to suppose that two plates of different metals take up different potentials even when joined to a common conductor. It is easy to show that there is a definite difference of potential for every pair of metals, since it is found possible to prevent the motion of the spot of light altogether by raising or lowering B to a definite potential above or below A. The natural difference of potential of two metals is called their contact potential-difference.

We cannot determine contact potentials by connecting the two metals directly to a quadrant electrometer, because there would be no deflexion at all if this were done. This is due to the fact that contact potentials are additive, i.e. the contact potential of two metals A, C is the sum of the contact potentials of A, B and of B, C, where B is any other metal. Since the quadrants of an electrometer are both of brass no difference of potential arises between them on account of contact effects in connecting wires. Contact potentials can however be determined if the quadrants themselves are of different metals, as was originally done by Lord Kelvin.

The property of insulators which corresponds most nearly to the contact potential of metals is their behaviour during

friction. The electroscope, and for delicate tests the quadrant electrometer, enable us to examine very easily the charges produced by friction. It is found, for example, that while a silk handkerchief will acquire a negative charge by friction against glass, it is positively charged when rubbed against metals or ebonite. More generally, substances can be arranged in a series such that each when rubbed by the next acquires positive electricity, the following being the general order: Catskin, glass, wool, silk, metals, sealing-wax, ebonite, sulphur.

The effect is additive, i.e. any member of the series is positively electrified by friction with one later, and negatively electrified by friction with one earlier than itself.

In 1880 J. and P. Curie discovered that crystals devoid of a centre of symmetry, such as tourmaline and quartz, became electrified on the application of stress. This phenomenon is called piezo-electricity, and has been found to consist in the production of a polarisation analogous to that occurring in the inside of dielectrics, the direction depending on the external forces applied. The case of quartz has been carefully investigated. It is found that when a quartz crystal is compressed along its crystallographic axis of threefold symmetry (optic axis) no polarisation is produced, but polarisations do occur for other directions. If the crystal while under stress is passed through the flame of a Bunsen burner so as to remove the surface layers attendant on polarisation (Art. 47), and the stress then removed, the crystal is found to possess free charges at every point of its surface equal and opposite to those originally produced.

It has been known for at least 200 years that a crystal of tourmaline becomes electrically polar when heated, and the polarity persists in the parts into which a heated crystal is separated when broken. There is good reason to believe that these *pyro-electric* effects depend on alteration of form caused by rise of temperature, i.e. that they are really piezo-electric in nature, the temperature playing only a secondary *rôle*.

45. Dielectrics. We have hitherto regarded an insulator merely as something which prevents the passage of charge from one conductor to another: but the insulator plays a very distinct

part in electrical experiments, as is shown by the following. Take a parallel plate condenser and join one plate to one terminal of a quadrant electrometer in the ordinary way. Begin with the outer plate to earth, and discharge the electrometer and inner plate by connecting them temporarily to earth with a wire. If the outer plate is now raised to a definite positive potential there will be a negative charge induced on the inner plate, and a positive charge repelled to the electrometer, the latter showing a deflexion. If the experiment is repeated with a slab of glass or ebonite between the plates of the condenser the deflexion is much greater than before, showing that more charges reside on the plates for a given applied potential. In other words, the capacity of a condenser is increased when a solid insulator replaces the air between the plates.

This effect was first demonstrated by Cavendish about 1770, and rediscovered by Faraday. In view of the important part played by insulators Faraday gave them the name of *dielectrics*, and this is generally used when we wish to draw attention to the electrical phenomena occurring in the insulators themselves. Faraday showed that the capacity of a condenser was increased in a constant ratio when another dielectric was substituted for air; this ratio is called the *dielectric constant* (or specific inductive capacity) of the dielectric, and denoted by K. This increase of capacity is found very convenient in practice. For example, in order to separate the plates of a condenser by a finite small amount a solid insulator such as mica is used; and the capacity, already considerable, is increased in the ratio of about six to one by the presence of the mica.

Strictly speaking, air itself is a dielectric, though its properties differ but slightly from those of a perfect vacuum. However, it is convenient once for all to take a vacuum as the standard in electricity, and assign to it the dielectric constant unity. In other words, the electrostatic unit of charge is defined to be that which repels an equal charge placed one centimetre away *in vacuo* with a force of one dyne. This will be adopted henceforth.

The experimental methods of measuring dielectric constants are described in the next chapter: the following table gives their values for certain common insulators.

Dielectric constants (K), referred to vacuum.

Ebonite	2·7	Alcohol	26	
Glass	5 to 7	Paraffin	2·1	
Gutta-percha	2·5	Petroleum	2·1	
Mica	6	Water	81	
Paraffin wax	2·3	Air	1·00059	at 0° C. and 760 mm. pressure
Quartz	4·4 to 4·6	Hydrogen	1·00026	
Sulphur	3·8 to 4	Carbon dioxide	1·00096	

The results for gases show how little error is committed in taking $K = 1$ for them. The dielectric constant of air need not be taken into account in ordinary experiments.

46. Various forms of condenser. Many practical condensers are made with solid or liquid dielectrics between the plates, the size of a condenser of a given capacity being thereby considerably reduced. In addition, the insulation at high potentials is greatly improved by the choice of a suitable medium, the use of air being inadmissible when the plates are very close together, on account of the danger of a spark passing between the plates.

The *Leyden jar* (Fig. 54) is a convenient form of condenser to handle, and consists of a bottle with a wide neck covered outside and inside with tinfoil. Contact with the inside is made by means of a brass rod carrying a screw terminal, dipping into the bottle through a wooden or ebonite stopper. A hanging chain is often used to connect the rod with the inner tinfoil, but brass springs are much more reliable. As the Leyden jar is frequently used with high potentials good glass must be used, and the outside of the jar should be covered with shellac or other varnish to diminish the leakage of charge over the surface. The capacity of an ordinary jar is of the order of 1000 electrostatic units. Small Leyden jars are usually fitted to influence machines to store the electricity as it is formed and to strengthen the spark.

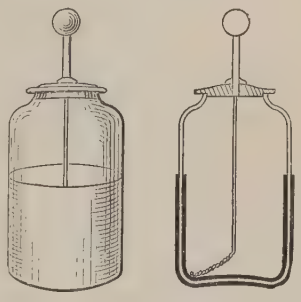


Fig. 54

Fig. 55 shows a condenser whose capacity can be varied continuously within wide limits. By turning the handle a greater or less area of the moveable vanes comes opposite the fixed vanes, and the capacity is changed accordingly. These condensers are extensively used in wireless telegraphic work, and are often

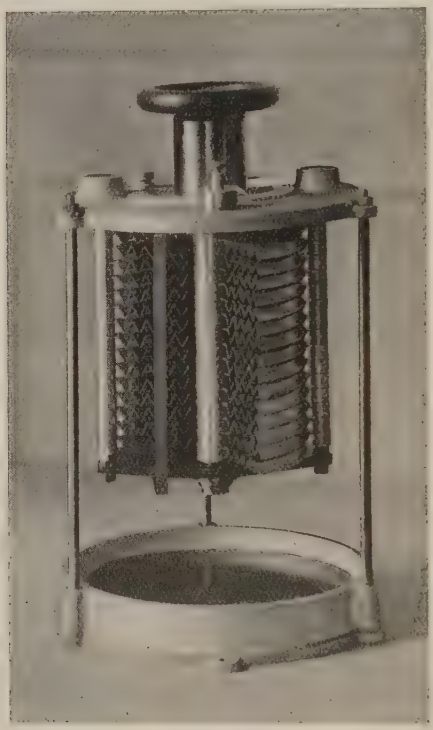


Fig. 55

immersed in oil to improve the insulation at high potentials. This also increases the capacity. For accurate work variable condensers must be calibrated, i.e. the capacity corresponding to any position of the pointer must be determined by comparison with a standard condenser.

Very large condensers, whose capacity is of the order of a million units, can be made of quite small dimensions by using mica or paraffined paper for the dielectric and tinfoil for the

plates, a large number being connected in parallel. For example, let us calculate the number of sheets of tinfoil 20 cm. square required in the construction of a condenser of a million units capacity, using mica insulation 0·2 mm. thick. As the sheets of tinfoil are laid alternately the capacity is $nKA/2\pi d$, where there are n sheets each of area A connected to each terminal. Taking $K = 6$, this gives $n = 52$, so that about a hundred sheets would be required altogether.

Fig. 56 shows the method of mounting large condensers for laboratory use. All the outside plates are joined together to one screw terminal T_1, while any number of the inner plates can

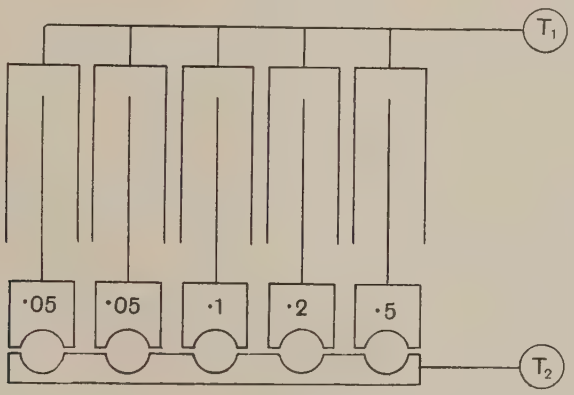


Fig. 56

be connected through a collecting bar to a terminal T_2 by means of metallic plugs, thus joining any number of the separate condensers in parallel. The values of the separate capacities are marked on the metal lugs supporting the plug-holes. Very large condensers are now often made by taking two long strips of tinfoil, covering one of them on one side with an insulating varnish, and then rolling the two up on top of each other. Such condensers are very portable and inexpensive.

Commercial condensers often show the phenomenon of the *residual discharge*; that is, for some time after their plates have been connected together they continue to discharge slightly. Well-made condensers with mica insulation, however, show little of this effect.

47. Theory of dielectrics. The point of theoretical importance in the discovery of Cavendish and Faraday is that the proportional increase of capacity is the same for all possible shapes of condenser employed. Thus for the same charges on the plates the difference of potential with a given dielectric is $1/K$ times its value for a vacuum, i.e. the electric force at any point is $1/K$ times its value *in vacuo*. Hence we have the fundamental rule that the electric force at distance r from a charge e immersed in a homogeneous dielectric is

$$E = \frac{e}{Kr^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

A satisfactory theory of dielectrics was first propounded by Kelvin on the lines of a corresponding theory of magnetisation due to Poisson. Translated into modern notation this theory is as follows. The electrons of insulators, except under special excitation such as friction, remain attached to their respective molecules. A certain freedom of motion however remains, and we may picture the electron as joined to the molecule by something like a spring. On applying an external electric force the electron and the residual positive charge move in opposite directions somewhat as shown in the figure. The actual state of affairs may not be as simple as this, but in any case we shall have positive electricity on the whole at one end of the molecule and negative electricity at the other. This state is exactly analogous to a small magnet if we imagine magnetic poles to be replaced by electric charges, and the molecule is said to be *polarised*.

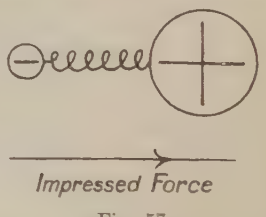


Fig. 57

By analogy with the simple magnet we can speak of an *electric doublet*, i.e. a positive charge e and a negative charge $-e$ at a small distance l apart, the *moment* of the doublet being le and the line joining the two charges being its *axis*. The electrical effect of a polarised dielectric is now calculated as in Articles 19 and 20 of the last chapter, since the laws of action of electric charges and magnetic poles are exactly similar. The total electric

moment of an element of volume $d\tau$ of a dielectric near the point (x, y, z) has components $P_x d\tau$, $P_y d\tau$, $P_z d\tau$, where the vector $P(P_x, P_y, P_z)$ is called the *polarisation* at the point. Poisson's analysis shows that the polarised dielectric is equivalent to a volume-distribution ρ of electricity through the volume of the dielectric and a surface-distribution σ over its boundary, where

$$\rho = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right), \quad \sigma = lP_x + mP_y + nP_z \ldots (18).$$

The potential at an external point A is given by

$$V = \int \frac{\rho d\tau}{r} + \int \frac{\sigma dS}{r},$$

where r is the distance from A of the element of volume or surface considered.

When the point A is actually inside the dielectric there is some doubt as to the definition of electric force and the exact meaning of equation (17). We shall take the electric force to be that due to the surface-distribution σ over the external boundary of the dielectric and the volume-distribution ρ, which may of course extend right up to A without giving rise to difficulty. With this understanding there exists a potential V, and the components of electric force are $-\partial V/\partial x$, $-\partial V/\partial y$, $-\partial V/\partial z$. Further, we have by Poisson's theorem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho = 4\pi\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) \ldots (19).$$

We shall show that the behaviour of dielectrics is completely explained by the hypothesis that the polarisation at any point is proportional to the electric force. Let us write

$$P_x = -k\frac{\partial V}{\partial x}, \quad P_y = -k\frac{\partial V}{\partial y}, \quad P_z = -k\frac{\partial V}{\partial z} \ldots (20).$$

For the sake of generality we shall suppose that the factor of proportionality k is a function of x, y and z, i.e. we consider a dielectric whose properties vary from point to point. Substituting from (20) in (19) we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\left\{\frac{\partial}{\partial x}\left(k\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial V}{\partial z}\right)\right\}.$$

Writing $K = 1 + 4\pi k$, this may be written

$$\frac{\partial}{\partial x}\left(K\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial V}{\partial z}\right) = 0 \quad \ldots\ldots(21).$$

The symbol K has been used because it is really identical with the dielectric constant as previously defined. To verify this, and also to gain additional insight into the action of dielectrics, let us calculate anew the electric force at distance r from a charge e immersed in a homogeneous dielectric.

Draw a small sphere S_0 of radius r_0 with e as centre, and imagine the dielectric inside the sphere removed to make room for the charge e. Let V be the potential at a point in the dielectric

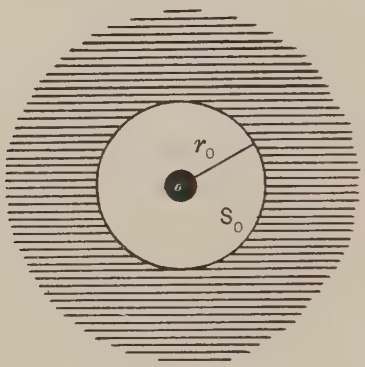


Fig. 58

at distance r from e. Since K is now constant, equation (21) reduces to Laplace's equation. Hence $V = \alpha/r$, where α is some constant; and the corresponding electric force at any point in the dielectric is α/r^2 radially outwards. It is easy to show that the volume-density ρ of Poisson's equivalent distribution vanishes, so that the electric force is that due to the charge e and the surface-distribution on S_0. The corresponding surface-density, which is k times the electric force along the normal pointing away from the dielectric, becomes $-k\alpha/r_0^2$. The total charge thus arising on S_0 is therefore $-4\pi k\alpha$, and this acts on external points as if concentrated at the centre of the sphere. The potential

V is that due to this charge together with the charge e, so that

$$\frac{a}{r} = V = \frac{e}{r} - \frac{4\pi k a}{r}.$$

Hence $$a = e - 4\pi k a, \quad \text{or} \quad a = \frac{e}{1 + 4\pi k} = \frac{e}{K}.$$

The electric force at distance r from e is thus e/Kr^2 as we should expect. It should be noticed that r_0 has disappeared from the equations, so that the dielectric can be taken to approach e as nearly as desired.

On this theory the reduction of the electric force from e/r^2 to e/Kr^2 is due to the formation of an induced charge on the contiguous surface of the dielectric, the magnitude of the charge being $-e(1 - 1/K)$. Since $K = 1 + 4\pi k$, we should expect the dielectric constants of all substances to exceed unity, and this is found to be the case. Gases, which contain but few molecules per unit volume, have small values of k, and K differs by very little from unity. Substances with large values of k are those which allow a large polarisation for weak electric forces, i.e. those whose electrons are but weakly bound to the molecules. We should thus expect conductors to behave like dielectrics of infinite dielectric constant, and we find in fact that those results in the theory of dielectrics which retain their meaning when $K = \infty$ hold for conductors. For example, putting $K = \infty$ in the dielectric just considered, we have an induced charge $-e$ on S_0, as we should expect. This charge of course screens the inner charge completely at external points, while the screening for dielectrics is not perfect.

48. Condenser with a partial layer of dielectric. Let d be the distance between the plates of a parallel-plate condenser (Fig. 59) and A the area of either plate, the linear dimensions being supposed large in comparison with d. Let there be a slab of dielectric of thickness h between the plates, the rest of the space being occupied by air, for which K may be taken as unity. The polarisation being uniform and of magnitude P, there is no volume-charge in the dielectric, but only two surface-charges of density $\pm P$. Let X be the electric force in the air-spaces and Y

that in the dielectric, $\pm\sigma$ the surface-densities on the plates. The electric force due to the surface-layers is easily found from Art. 30, and thus we have

$$X = 4\pi\sigma, \quad Y = 4\pi(\sigma - P).$$

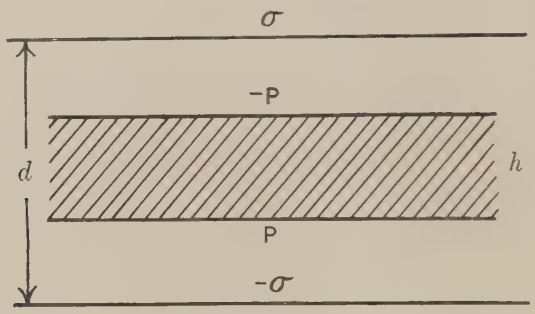


Fig. 59

But since $P = kY$ by definition, this gives $Y = 4\pi\sigma K$. The difference of potential between the plates of the condenser is $X(d-h) + Yh = 4\pi\sigma\left(d - h + \frac{h}{K}\right)$, and the charge on the top plate σA. Hence the capacity of the condenser is

$$\frac{KA}{4\pi\{Kd - (K-1)h\}}.$$

This reduces to $A/4\pi d$ when $h = 0$, and to $KA/4\pi d$ when $h = d$, as we should expect. The electric force inside the dielectric is $1/K$ times that in the air-gap. This is a particular case of a general theorem which will be given in the next article.

49. Boundary conditions at the surface of two dielectrics. Conductor immersed in a dielectric. Let C be the surface separating two dielectrics K_1, K_2, and let F_1, N_1

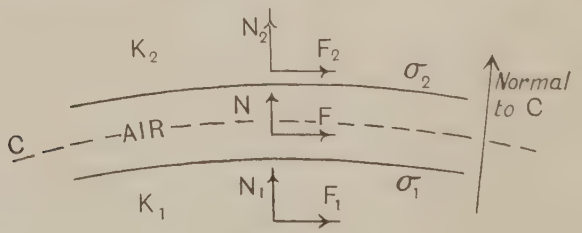


Fig. 60

be the tangential and normal components of electric force just inside K_1, F_2, N_2 the same quantities just inside K_2. In order to gain some insight into the effects where two dielectrics meet, imagine a thin air-gap to be left between them, its position coinciding very nearly with the surface of separation C. Let F, N be the components of electric force in the air-gap.

(1) *The tangential component of electric force is the same on either side of the surface of separation.* On crossing a surface-layer of surface-density σ there is a sudden change of $4\pi\sigma$ in the normal component of electric force (Art. 30), but no change in the tangential component, since the surface-layer acts practically as an infinite charged plane as regards points very near to it on either side. Thus $F_1 = F = F_2$.

(2) *The product of dielectric constant and normal component of electric force is continuous in crossing the surface of separation.* Let σ_1, σ_2 be the surface-densities of Poisson's distribution on the two dielectrics. Then from what has just been said we have

$$N = N_1 + 4\pi\sigma_1, \quad N_2 = N + 4\pi\sigma_2.$$

Thus $$N_1 + 4\pi\sigma_1 = N_2 - 4\pi\sigma_2,$$

and we also have

$$\sigma_1 = k_1 N_1, \quad \sigma_2 = -k_2 N_2.$$

Hence $$N_1(1 + 4\pi k_1) = N_2(1 + 4\pi k_2),$$

or $$K_1 N_1 = K_2 N_2.$$

(3) *The electric force just outside a conductor immersed in a dielectric is* $4\pi\sigma/K$. As before, imagine a thin air-gap to be removed between the conductor and the dielectric. The electric force in the air-gap is $E = 4\pi\sigma$ at right angles to the surface of the conductor; and the preceding theorems show that the electric force E' just inside the dielectric has no tangential component, and that $KE' = E$. Hence $E' = 4\pi\sigma/K$ normal to the conductor.

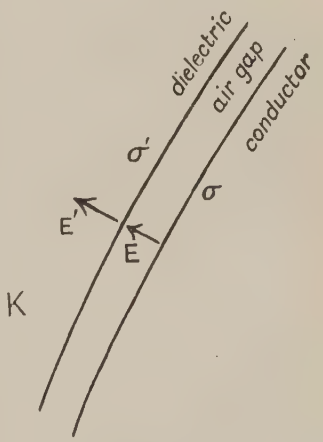


Fig. 61

50. Electric force and electric induction. It is of some importance, in view of later developments, to inquire what happens when the proportionality of polarisation to electric force does not hold. In this case we have at every point two vectors to determine the state of affairs, the electric force E and the polarisation P, independent of one another. Since $E_x = -\partial V/\partial x, \ldots$, equation (19) gives

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} + 4\pi\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = 0.$$

If we define a vector D as having the components

$$D_x = E_x + 4\pi P_x, \quad D_y = E_y + 4\pi P_y, \quad D_z = E_z + 4\pi P_z \ldots (22),$$

this becomes

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0 \ldots\ldots\ldots\ldots (23).$$

The vector D will be called the *electric induction*, and (22) may be written briefly $D = E + 4\pi P$. The vectors D and E enjoy characteristic properties. Thus from (23) it follows that D is always solenoidal, but in general is not derivable from a potential. On the contrary, E is derivable from a potential, but in general is not solenoidal. These characteristic properties are lost in particular cases: thus in a homogeneous dielectric D is a mere multiple (KE) of E, and the two vectors behave identically.

The boundary conditions of the last article are easily investigated in the present more general case. We thus find that the *tangential component of electric force* is continuous in crossing the boundary between two media, and so is the *normal component of electric induction*. The normal component of electric force and the tangential component of electric induction, on the other hand, undergo in general a sudden change.

CHAPTER IV

ELECTRIC CURRENTS

51. Electricity in motion. It has been seen in the last chapter that whenever two charged bodies of different potential are connected by any third conductor, such as a piece of wire, there is a redistribution of charge which brings the bodies to a common potential. This movement of electricity along the wire is called an electric current, and it will appear shortly that the effects are the same whether the charges move through the conductors, or whether the conductors themselves move with appropriate velocity carrying their charges with them.

In this chapter currents in solid conductors will be chiefly considered, and it is necessary at the outset to remember that whereas static charges are always confined to the surface of conductors, charges in motion, i.e. electric currents, in most cases move through the volume of the body. It is only in a single exceptional case that we can have currents confined to a thin skin on the surface of the conductor (Art. 203). The electric current produced by joining two conductors at different potentials is only momentary, and it is at this stage immaterial whether we consider the current to be due to the movement of positive charge alone, the negative remaining at rest, or *vice versâ*, or to both moving simultaneously. For the sake of generality the last assumption will often be made.

Imagine for simplicity the case in which the conductors have originally equal and opposite charges, so that after connexion neither will have any charge. Let attention be confined to some particular cross-section of the connecting wire: then, excluding the possibility of instantaneous neutralisation, a charge $+ de_1$ passes over the section in one direction in an interval of time

dt, and $-de_2$ in the other. This is regarded as a total transfer of charge in the first direction equal to de, where

$$de = de_1 - (-de_2) = de_1 + de_2.$$

The definition of *strength of current* is that it is the rate of transfer of charge in a given direction; current strength will be denoted by i, and if de is the charge transferred in time dt, $i = \frac{de}{dt}$. In this definition the (electrostatic) unit of current consists in the passage of one electrostatic unit of charge per second. In the case considered above i is not constant, all that is known at present being that it is zero at the beginning and end of the operation.

It should be observed that, provided there is no accumulation of charge anywhere along the wire, the current is at any moment the same at all points of the wire.

The passage of an electric current along a wire is accompanied by a number of effects which are not present in the case of charges at rest, but the above-mentioned method of producing a current is not suitable for their study, partly because of the smallness of the current and the short time for which it is acting, but chiefly because it is not constant in magnitude. But however currents are produced in practice, it is necessary to grasp clearly the idea that the "electric current" is merely the passage of electric charge.

52. The voltaic cell. So far the only methods described for producing differences of potential have depended upon friction or electric influence. Speaking generally, these methods produce small charges at high potentials. We shall now describe a method by which larger quantities of electricity may be generated at much lower potentials, the necessary energy being derived from chemical sources. If two different metals are taken and placed in a solution they will produce a deflexion when connected to the terminals of a sensitive electrometer, thus revealing a very essential difference between this arrangement and the case of metal plates separated by air, described in the last chapter under contact-potential. The liquid has a more extended property than that of merely allowing contact differences of potential to arise where it touches the plates.

The earliest observations in this subject are those of Galvani, published in 1791. He noticed that a frog's leg, attached to an iron hook so that the nerve touched the metal, contracted suddenly when touched with a piece of metal in contact with the hook. Since the same contractions could be produced with an electrical machine they were concluded to be electrical in nature. It was reserved for Volta to point out the essence of Galvani's discovery, namely that the production of electricity did not depend on the presence of animal preparations, but rather on that of certain liquids. Replacing the frog's leg by acidulated water, he devised in 1800 the arrangement already described, generally called the *voltaic cell*. If two simple cells are taken and the copper plate of one joined to the zinc of the other, they form a compound cell or *electric battery*, and if the two remaining plates are joined to the terminals of an electrometer the difference of potential is found to be twice as great as before. Any number of cells can be joined "in series," as it is called, so as to form a battery.

Other metals and liquids can be used in the construction of a voltaic cell, but certain combinations are advantageous in giving comparatively large differences of potential. The difference of potential between the plates of a simple cell, however, falls after it has been delivering current for any length of time. The cell is then said to be *polarised*.

One of the best forms of voltaic cell is the Daniell cell, shown in Fig. 62. A stout zinc rod or cylinder is placed in zinc sulphate solution contained in a porous pot of unglazed earthenware. Outside the pot is a larger vessel containing a solution of copper sulphate, into which dips a piece of sheet copper bent into the form of a cylinder. The action of the cell will be discussed in Ch. x: it will give current without becoming unduly polarised and is therefore useful for performing simple experiments, such as the following.

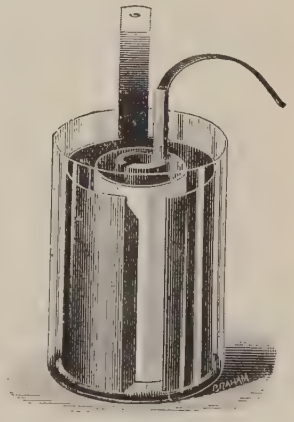


Fig. 62

Suppose that the zinc and copper, or as they are called, the poles, are joined respectively to the two plates of an uncharged condenser, and that then these connexions are broken and the plates connected to the terminals of an electrometer. Care must be taken by the use of insulating keys not to discharge the condenser in the process. The electrometer will show a deflexion, showing that the plates have been charged up by contact with the terminals of the cell. If now the condenser is discharged by joining the plates with a wire, and the cell again connected to them, it is found that they are charged up as before, and the process of charging and discharging can be continued almost indefinitely. It appears that the cell can go on supplying charge to the condenser, and it is natural to suppose that the energy of charging has come from certain chemical changes which the cell undergoes. Each time the condenser is discharged there is a momentary electric current, and if it was charged and discharged sufficiently rapidly an approximation to a continuous current might be obtained. An obvious modification is to do away with the condenser and connect the terminals directly by a wire; when several effects will be observed. Thus if the wire is thin and not too long it will become appreciably warmer. If the poles are joined to an electrometer it is found that the potential difference between them is somewhat less when they are connected by a wire than when they are not; thus the wire may be regarded as constantly tending to equalise the difference of potential while the chemical action in the cell strives to maintain it at its original value, with the result that an intermediate stage is reached.

The direction in which the spot of light of an electrometer is deflected by a positive charge, say, on one pair of quadrants may be found by an electrostatic experiment, and thus it is easy to determine which of the two poles of a cell is at the higher potential. In the Daniell cell the copper is found to be at a higher potential than the zinc, and it is called the positive pole of the cell. In the connecting wire, or external circuit as it is called, the current therefore flows from the copper to the zinc; and generally it is not difficult to find out beforehand the direction of a current flowing in a wire.

53. Magnetic action of currents. The tangent galvanometer. The most important fact discovered in experiments with voltaic cells is that a current in a wire produces a magnetic field in its vicinity, thus revealing an essential difference between the effects of an electric charge at rest, which produces no magnetic field whatever, and the same charge in motion which does. The discovery dates from 1820, when Oersted found that a conductor carrying a current would deflect a compass-needle near it. This affords a very convenient means of detecting electric currents, and it will appear shortly that the magnitude of a current is appropriately measured by its magnetic effect. The laws of the action of currents on magnets are in some ways very complicated, so that we shall only consider here as much as is necessary to understand the use of the magnetic effect in the measurement of currents.

If a straight wire is placed over a compass-needle pointing north and south, and a current passes along the wire from south to north (Fig. 63), the positive pole of the needle is deflected from north

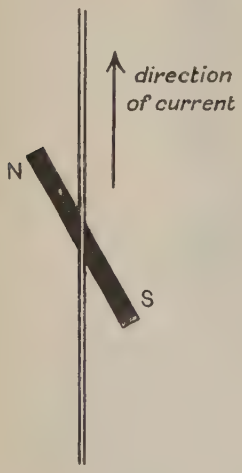


Fig. 63

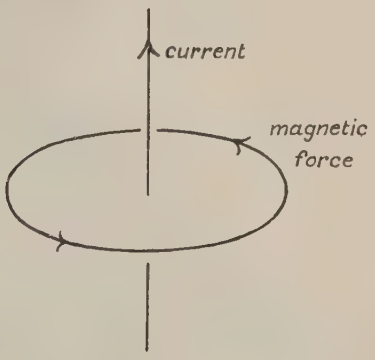


Fig. 64

towards the west. This effect may be described concisely by what is known as the *right-handed screw law* of electric currents; namely that the magnetic force is related to the direction of the current in the same way as the rotation to the direction of translation of an ordinary (right-handed) corkscrew, as shown in Fig. 64.

If a wire is doubled back on itself, the parts being insulated from one another, no magnetic effect is produced at points a large distance off compared with the distance between the two parts. This result is frequently applied in electrical experiments, as we can neutralise the magnetic effect of any wire by providing it with a return wire near to it. Further, if a straight wire carrying a given current produces a certain deflexion to the west when lying vertically over a compass-needle and pointing magnetically north, turning the wire through two right angles produces an equal easterly deflexion. Thus the magnetic field changes sign with that of the current.

A simple compass-needle may be used to detect electric currents; but there are other instruments designed for more

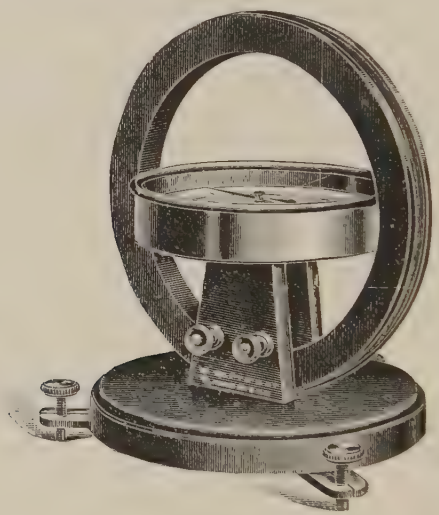


Fig. 65

accurate measurement which are called *galvanometers*. Of these the fundamental one is the tangent galvanometer (Fig. 65). It consists essentially of a circular coil of wire of one or more turns, moveable about a vertical axis, a small compass-needle being placed on a pivot at the centre of the circle. The compass-needle is fitted with two pointers and lies in a box with a graduated circle like that used in magnetometers. The graduations of the scale are most conveniently made so that the line of zeros is perpendicular

to the plane of the coil of wire. Thus when the plane of the coil is in the magnetic meridian and no current is flowing the pointers will read zero. If the coil is turned through a right angle, so that it lies magnetically east and west, and a current applied, either no deflexion is observed or the needle turns immediately through two right angles. This shows that the magnetic force at the centre of the circle is at right angles to the plane of the coil, since it either assists or directly opposes the earth's magnetic field. When the plane of the coil is in the magnetic meridian, however, the two fields are at right angles, and we have a deflexion, the direction of which is determined by the right-handed screw law.

The tangent galvanometer measures primarily the magnetic force F due to the current at the centre of the circular coil, and its normal position when in use is with the plane of the coil in the magnetic meridian. If H is the earth's magnetic field, a deflexion of θ to the west means that the force F acts to the west and is of magnitude $H \tan \theta$, since the compass-needle sets in the direction of the resultant of the two fields. If therefore we can find the connexion between F and the current we shall be in a position to measure currents with the instrument. This question is discussed in the following articles.

It is very convenient, and for many purposes absolutely

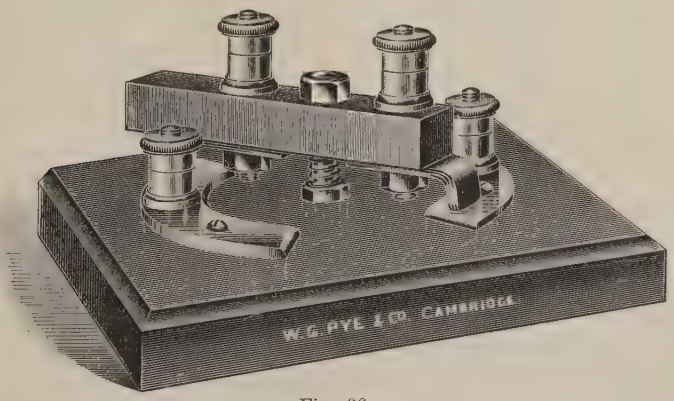


Fig. 66

necessary, to have special keys for putting currents on and off, and for reversing their directions, without taking the apparatus to

pieces. A simple form of key consists of two stout pieces of brass fastened on a board so as not quite to touch, and carrying screw terminals for attaching wires. A brass plug fitting into a conical hole makes contact with both pieces when in the hole. Fig. 66 shows a form of reversing key. Here the main circuit is connected to the two fixed terminals, while the part in which the current is to be reversed is joined to the other pair. The mode of action of the key will be readily understood from the figure.

54. Electromagnetic unit of current. We can prove experimentally that the magnetic force due to a current is proportional to the strength of the current. (These relations are not self-evident: thus the heating effect of a current is proportional to its square.) Two similar insulated coils of wire are wound close together round a wooden frame as shown in the figure, and a

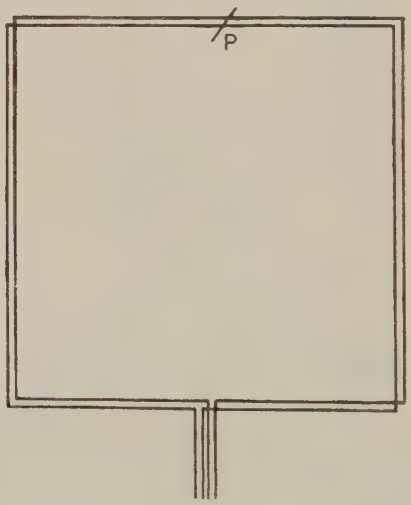


Fig. 67

compass-box is fixed with the centre of the needle in the plane of the coils. The same current is sent (1) through one coil only, (2) through both coils in the same direction, (3) through both coils in opposite directions, with the coils in the magnetic meridian. The equality of current in the three cases is tested with a tangent galvanometer at a considerable distance from the coils: evidently

the tangent galvanometer will on any theory always give the same deflexion for a given current passing through it. The following table gives the readings in an actual experiment:

Deflexion θ_1 with current in one coil only	Deflexion θ_2 with current in same direction in both coils	Deflexion θ_3 with current in the two coils opposed	$\frac{\tan\theta_2}{\tan\theta_1}$
16° 24′	30° 6′	0° 6′	1·97
20° 24′	36° 36′	0° 12′	1·99
23° 24′	41° 6′	0° 6′	2·01
27° 36′	46° 18′	0° 6′	2·00
31° 48′	51° 6′	−0° 12′	2·00
35° 36′	55° 12′	0° 6′	2·00
39° 24′	58° 42′	—	2·00
42° 54′	61° 30′	−0° 24′	1·98

The smallness of the numbers in the third column verifies the absence of magnetic effect of coils bent back on themselves. Now when the current flows in both coils in the same direction the amount of electricity passing a given point P of the composite wire is twice as much as when the current flows in one wire only; that is, the current by definition is twice as great as before. We should therefore have $\tan\theta_2 = 2\tan\theta_1$, and the numbers in the last column confirm this relation.

Returning to the tangent galvanometer, we have to find how the force F at the centre of the coils depends on the current i, the number of turns n, and the radius r of the coils. A current i circulating in n wires close together is equivalent to a current ni in a single wire, since the passage of charge is n times as great. Hence F can only be given by an expression of the form $F = nif(r)$. To determine f we require a separate experiment with a tangent galvanometer having two sets of coils, one of a single turn of radius r and the other of a double turn of radius $2r$, the windings being arranged so that the current is in opposite directions in the two sets. No deflexion will be observed, showing that F is unaltered when n, r are changed to $2n$, $2r$ respectively, and therefore depends on the ratio n/r only. Hence $F = kni/r$, where k is some quantity which is the same for all instruments. The constant k required to give F in gausses is now perfectly definite,

since i is defined as the number of electrostatic units of charge passing per second. Its value has been shown to be about 2×10^{-10}; but as it is not easy to make an accurate measurement we shall define a new unit of current, and incidentally a new unit of charge, by giving k an arbitrary value, leaving the question of the relation between the two units for later examination. It is most convenient to take k equal to 2π, and if this is done i is measured in what are called *electromagnetic units of current*. Thus the electromagnetic unit of current is such that it would give a field of 2π gausses at the centre of a circular loop of wire of radius 1 cm. With this understanding the deflexion θ of a tangent galvanometer used in the customary way is given by

$$\frac{2\pi ni}{r} = H \tan\theta,$$

or

$$i = \frac{rH}{2n\pi} \tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The quantity $\dfrac{rH}{2n\pi}$ is called the constant of the tangent galvanometer. Hence if H is known we have a practical method of determining the absolute magnitude of a current in electromagnetic units.

It appears at first sight a little artificial to put $k = 2\pi$ instead of putting $k = 1$. This is however done because it gets rid of an inconvenient factor in the subsequent theory. For in a tangent galvanometer of one turn the magnetic force at the centre is $2\pi i/r$ for a length $2\pi r$ of wire, that is i/r^2 per centimetre, so that the effect of a given length of wire is expressed in its simplest form.

Since the presence of electric charge gives rise mainly to electric repulsions, while the most important effect of a current is its magnetic effect, it is convenient to suppose charge measured in electrostatic units and current in electromagnetic units. This will be understood, in the absence of qualifications, throughout the book.

The ratio of the electromagnetic to the electrostatic unit of current will be denoted by the letter c, so that a current i in electrostatic units becomes i/c when measured in electromagnetic units. We shall describe later (Art. 140) experiments by which c has been shown to have the value 3×10^{10} very nearly,

and we shall adopt this value in all numerical calculations. The fact that we do not know c at present does not lead to any inconvenience, because we are here chiefly concerned with currents as defined by their magnetic effects.

55. The derived system of electromagnetic units.

(1) *The derived electromagnetic unit of current (ampere).* Both the electromagnetic and electrostatic systems of units have been fixed on purely theoretical grounds without reference to the commonly occurring values of the magnitudes in question. But in practice the chief requisite of a unit is that it should be more or less commensurate with the quantities that it is required to express. It is found that the electromagnetic units are of inconvenient size, and therefore a third system of units, called the "derived system of electromagnetic units," has been devised. On this system the unit of current is one-tenth of the electromagnetic unit; and like all the derived units it is given a special name, being called the *ampere*. Thus a current which has magnitude i on the true electromagnetic system is equal to $10i$ amperes or to ci electrostatic units, so that an ampere is equivalent to a current of 3×10^9 electrostatic units, very nearly. The reader should acquire early some knowledge of the magnitude of currents in common use. Thus the current passing through a 32 candle-power electric lamp under the usual conditions of electric supply in towns is of the order of $\frac{1}{3}$ ampere; but electrical effects must not be taken as depending solely on the magnitude of current, since other factors are equally important.

(2) *The electromagnetic unit of charge and the derived unit (coulomb).* The charge transferred by a constant current i in time t is $e = it$, or if the current is variable $\int i dt$. It is obvious, therefore, that the units of charge in the various systems are related to one another in exactly the same way as those of current. The charge conveyed by a steady current of one ampere flowing for one second is called a *coulomb*. Hence a charge e measured in electrostatic units $= e/c$ in electromagnetic units $= 10e/c$ coulombs, and 1 coulomb is 3×10^9 electrostatic units of charge.

(3) *The electromagnetic unit of potential and the derived unit*

(*volt*). Like the electrostatic unit, the electromagnetic unit of potential is defined in such a way that two points are at unit difference of potential when one erg of work is required to move a unit charge from one point to the other. Let P, Q be two points whose potential difference is one electrostatic unit. Since the electromagnetic unit of charge is c electrostatic units, the work done in moving a charge of one E.M. unit from P to Q is c ergs, and therefore the difference of potential between P and Q is c electromagnetic units. Hence the electrostatic unit of potential is c electromagnetic units, and a potential V measured in electrostatic units $= cV$ in electromagnetic units.

The derived unit of potential is taken as 10^8 times the absolute E.M. unit, and is called the *volt*. The electrostatic unit of potential is thus $c/10^8 = 300$ volts, a fact it is convenient to remember. It should be noticed that if we multiply charge in coulombs by potential difference in volts we do not obtain work in ergs, but to do so we must first convert to absolute E.M. units and then form the product.

The difference of potential between two points is often referred to as the electromotive force (E.M.F.) between them; a term in very general use, though objection can be taken to it on the ground that the word "force" is misleading. We shall use it generally for the difference of potential between the terminals of a cell or other source of electricity. The potential differences occurring in current electricity are usually small compared with those produced by electrostatic means. It is an easy matter to charge up a body by friction to 1000 volts: the E.M.F. of electric lighting circuits is from about 100 to 250 volts, while that of the Daniell cell is only 1·09 volts. There is, however, no difficulty in measuring small potentials by electrostatic means. Thus the quadrant electrometer when used heterostatically gives a deflexion of 80 scale divisions (mm.) or more for a potential difference of one volt between the quadrants, when the needle is charged to 50 volts.

(4) *The electromagnetic unit of capacity. Farads and microfarads.* The capacity of an isolated conductor in any system of units is the ratio of its charge to its potential, potential being defined so that it vanishes at infinity. Suppose that we have a

conductor of capacity 1 E.S.U. at a potential of 1 E.S.U. In electromagnetic units its charge would be $1/c$ and its potential c, so that its capacity would be $1/c^2$. Hence the electromagnetic unit of capacity is c^2 electrostatic units $= 9 \times 10^{20}$ electrostatic units.

The derived unit of capacity is called the *farad*, and is the capacity of a conductor which requires one coulomb to raise its potential by one volt. In practice we need only consider the capacity of condensers, and the above definition may be restated by saying that a condenser has a capacity of one farad if, when one plate is earthed, one coulomb given to the other raises its potential by one volt. It is easy to show as above that the farad is equal to 10^{-9} electromagnetic units; it is however an inconveniently large unit, and the microfarad*, which is one-millionth part of a farad, is used instead. We have

1 microfarad $= 10^{-15}$ E.M. units $= 900{,}000$ E.S. units of capacity.

The microfarad is itself a large unit: thus an ordinary Leyden jar has a capacity of about 1000 E.S. units or 1/900 microfarad. Really small capacities are best measured in electrostatic units.

The relation of the various units is exhibited in a form suitable for reference in the Appendix, together with a table of notation and other units to be introduced hereafter.

56. The magnetic effect of a moving charge. As the ordinary electric current is simply a passage of electric charge it is to be expected that a charged conductor when moved should affect a magnet in its neighbourhood. Such an effect was first detected by Rowland in 1876: the most perfect experiments from a quantitative point of view are those of Eichenwald, which we shall now describe.

A thin insulating disc A (Fig. 68) was mounted on an axle, and on it was pasted a circular strip of tinfoil BCD extending nearly round its circumference. One end B of the strip was permanently connected to an insulated brass slip ring on the axle, and could

* The prefix *micro-* when applied to the various units means one millionth part. Other prefixes are *milli-*, meaning one thousandth part, and in some cases *meg-*, meaning a million times.

be charged up by means of a brush E pressing against the slip ring. The disc was surrounded by a fixed earthed metallic case F nearly enclosing it, but with an aperture which enabled the end D of the tinfoil strip to be earthed at will. The magnet was suspended above the disc by a torsion fibre, and shielded

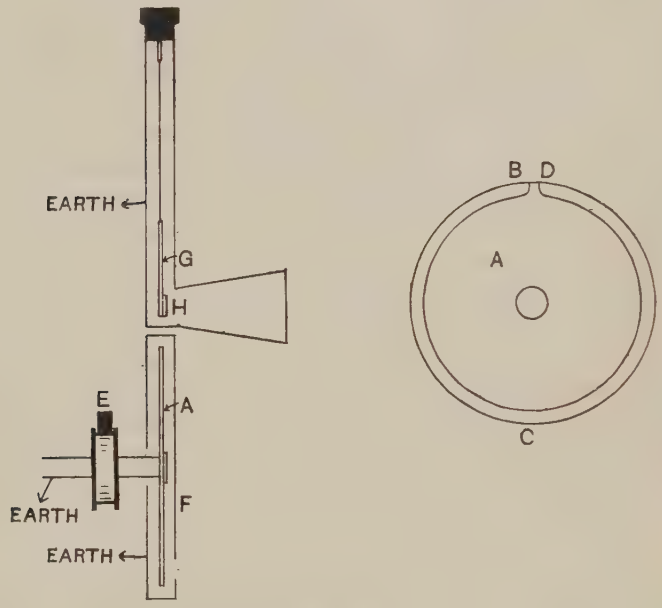


Fig. 68

from electrostatic influence. It consisted of a glass rod G carrying at each end a short horizontal magnet, the axes of the two magnets being turned in opposite directions so as to neutralise the effect of the earth's magnetic field. The position of the magnet was read off by reflexion with a lamp and scale, using the small mirror H attached to the magnet.

In the direct experiment the disc was charged to potential V by means of the brush and rotated n times per second, and the small deflexion θ read. The passage of the charge CV n times per second is equivalent to a current of nCV electrostatic units. Hence we should have $nCV = k\theta$, where k is a certain constant.

The constancy of nCV/θ in Eichenwald's experiments is shown in the following table:

n	C	V	θ	nCV/θ
77	39·0	1875	5·8	$\cdot 97 \times 10^6$
77	50·9	3750	14·1	1·04
77	50·9	5000	19·2	1·02
15	45·8	6250	4·1	1·05
30	45·5	6250	8·6	·99
77	45·5	6250	21·4	1·02
150	50·2	6250	45·5	1·03

The end D was next put to earth and a current of i electromagnetic units passed through the strip from a battery, the disc remaining at rest. This current is equivalent to ic electrostatic units, where c is the ratio of the units. Hence we should have $ic = k\phi$, where ϕ is the deflexion produced. The value of k is known from the preceding experiments, and thus c can be found. Eichenwald found values ranging from $2{\cdot}86 \times 10^{10}$ to $3{\cdot}16 \times 10^{10}$, mean $3{\cdot}01 \times 10^{10}$. This agrees very well with the accepted value 3×10^{10} (Art. 140), and thus the results are entirely conclusive as to the magnetic effect of moving charges.

57. Electrolysis. The currents considered so far have moved along metals, or been carried along by their motion as in the experiments of Eichenwald just mentioned. The passage of electricity through conducting liquids is somewhat different, in that chemical action is involved. This phenomenon is known by the name of *electrolysis*, and liquids that conduct in this way are called *electrolytes*. The current is conveyed to and removed from the liquid by wires or plates dipping into it and called *electrodes*, the one at higher potential being called the *anode* and the one at lower potential the *cathode*. The whole apparatus is called a *voltameter*. Pure water is a bad conductor, the best electrolytes being acids and salt solutions.

When the current passes constituents of the acid or salt appear at the electrodes, where they are set free in some cases. In other cases secondary actions take place depending on the material of the electrode. The following are two typical instances.

(1) A and B (Fig. 69) represent two platinum electrodes joined to a battery or other source of potential and dipping into pure hydrochloric acid. A current passes and bubbles of hydrogen appear at the cathode B, chlorine at the anode A. These bubbles rise and are set free, and it is found that the volumes of hydrogen and chlorine, if collected separately, are equal. Here the sole chemical action is the liberation of the constituents of the acid.

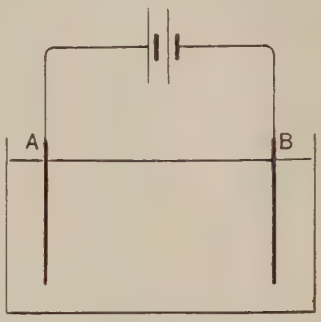


Fig. 69

(2) A slightly more complicated case occurs when A and B are copper plates dipping into a solution of copper sulphate. As a result of the passage of the current copper reaches the cathode and is deposited there, while the radicle SO_4 reaches the anode and combines with it to form copper sulphate, which enters the solution. The final effect is that one electrode gains copper and the other loses it, the solution remaining unaltered.

The quantitative laws governing these actions were first enunciated by Faraday in 1833. They are

(1) For any given substance the amount set free at an electrode is proportional to the quantity of electricity which passes, regardless of the time taken to do so.

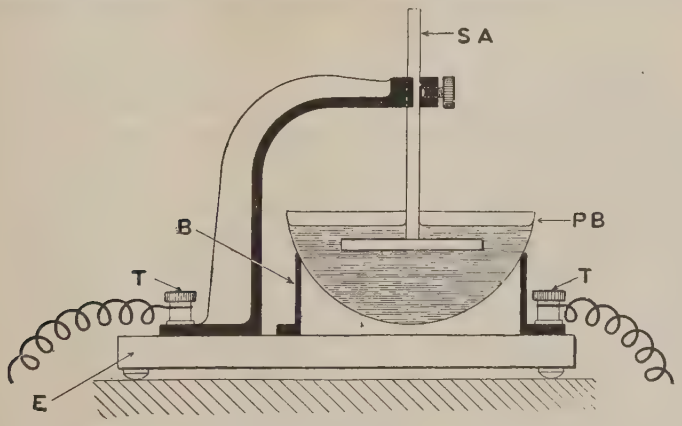
(2) The amount of a substance which is set free by a given quantity of electricity is proportional to its chemical equivalent weight. By chemical equivalent weight is meant the atomic or molecular weight divided by the valency.

We can easily verify Faraday's first law by means of three voltameters containing the same electrodes and electrolyte. If the current after passing through the first voltameter divides, part passing through the second voltameter and the rest through the third, the amount set free altogether from the second and third voltameters is the same as that set free from the first. Similarly if any number of voltameters are arranged in series so that the same current flows through each, the amounts of the

various substances liberated are easily ascertained, and Faraday's second law can be verified with fair accuracy (cf. Art. 163).

For our present purpose the first law is of importance because if the amount of a substance liberated by one coulomb has once been measured, we can apply this knowledge to determine how many coulombs have passed in a circuit in a given time, and thus to measure currents. For theoretical purposes currents are defined by their magnetic effects, as already explained, and it is probable that the *true ampere* will always be defined in this way. But the experiments necessary to determine a current accurately in absolute measure are elaborate and difficult, while great accuracy can easily be obtained by electrolysis. It is therefore better to take the results of certain standard experiments and re-define the ampere for practical purposes by its electrolytic effect. At an international conference held at London in 1908 the *international ampere* was defined as the unvarying current which, when passed through a solution of nitrate of silver in water, in accordance with an authorised specification, deposits silver at the rate of 1·11800 milligrams per second.

A good form of silver voltameter for determining the amount of silver deposited by the passage of one coulomb, or the electro-



C.S.I.Cº.

Fig. 70

chemical equivalent of silver as it is called, is shown in Fig. 70. The cathode is a platinum bowl *PB* which also serves to hold the

silver nitrate, a suitable strength for the solution being 15 per cent. The anode is a silver disc SA, which is covered with filter paper in order to prevent any particles which may become detached from the anode from falling into the bowl. The current enters and leaves at T, T, and after a known current has passed for a known time the amount of silver deposited on the bowl is determined.

Perhaps the most reliable of the many experiments that have been made on the electro-chemical equivalent of silver are those performed by T. Mather and F. E. Smith in 1906–7. The current was measured with a very accurate "current weigher" designed by W. E. Ayrton and J. V. Jones (see Art. 88), and great attention was paid to the purity of the electrolyte. The electro-chemical equivalent of silver was found to be 1·11827 milligrams per coulomb, estimated correct to about one part in 50,000. Incidentally, Faraday's first law was subjected to a very searching test, for it was found that the same number was obtained, within the limits of experimental error, for several forms of voltameter and for currents differing in the ratio of 16 to 1, while the concentration of the solution could vary from 1·5 to 50 parts of silver nitrate per 100 parts of solution without affecting the value appreciably. There seems little doubt that the international ampere is too small by about 1 part in 4000, but this is not enough to cause trouble in any ordinary laboratory experiments.

Currents produced by purely electrostatic methods can also be shown to cause electrolysis. For this purpose a small voltameter with electrodes of fine platinum wire is suitable, and we require a rather powerful influence machine. It is advisable to keep a large condenser connected across the terminals of the machine during the experiment. A sensitive tangent galvanometer included in the circuit will also be deflected, showing that currents taken from an influence machine produce magnetic effects. In fact, the experimental evidence for the identity of the voltaic current with a flow of ordinary electricity is overwhelming.

58. Polarisation. Cells and accumulators. When a circuit containing a voltameter is closed a comparatively large current may flow at first, but it will diminish to a steady value in time. This may be shown to be due to the formation of an

E.M.F. between the electrodes tending to oppose the applied E.M.F., for if the voltameter is taken out of circuit and the electrodes joined to the terminals of an electrometer a deflexion occurs which remains steady, under favourable conditions, for some time. The voltameter itself will now act like a cell, and if the cell in Fig. 69 is replaced by a galvanometer a current will flow in a direction opposite to that of the original current. The voltameter is then said to be polarised. The effect disappears more or less rapidly when the electrodes are joined by a thick wire, or short-circuited as it is called. Polarisation is due to some change in the surface of one or both electrodes caused by the passage of the current: thus it does not occur in the electrolysis of copper sulphate with copper electrodes, where the anode loses copper and the cathode gains it, without either electrode undergoing any change of composition.

When a voltaic cell is closed through an external circuit the current which circulates has an electrolytic effect inside the cell, and the polarisation of cells is of the same nature as that now described. The current in the interior of a cell flows from the negative to the positive pole; for example in the Daniell cell from the zinc to the copper. The effect on the electrodes is that copper is deposited on the positive pole and zinc dissolves into the zinc sulphate, so that the surfaces dipping into the liquids are not altered, at any rate for some time.

The *Leclanché cell* (Fig. 71) is convenient if weak currents are required at irregular intervals, as it requires very little attention and keeps well. The positive pole is a stout rod of hard gas-carbon closely packed with manganese dioxide and contained in a porous pot. Outside the pot is a saturated solution of ammonium chloride, in which dips a zinc rod forming the negative pole. When a current passes the layer of hydrogen which tends to form on the positive pole is removed by the oxidising action of the manganese dioxide, but as

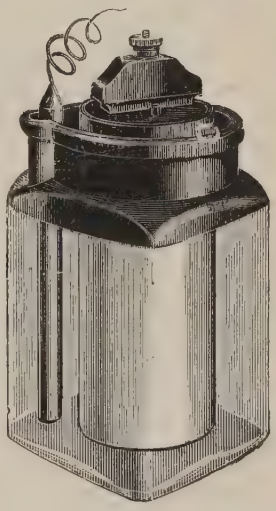


Fig. 71

this takes place slowly there is considerable polarisation for large currents. The E.M.F. on open circuit is about 1·4 volts, falling rapidly on the passage of a current and recovering when left to itself.

In the bichromate cell, which also has zinc and carbon poles, the electrolyte is a mixture of dilute sulphuric acid and potassium bichromate solution. The latter depolarises rapidly by oxidation, but the zinc pole is corroded even on open circuit. The E.M.F. is about 2·1 volts.

In many cases the effect of passing a current through a voltameter is reversible; that is, the chemical changes accompanying polarisation are reversed when the voltameter is removed from the circuit and its electrodes joined by a wire. Such a voltameter acts like a cell which has to be made (or charged, as it is called), and the energy expended in charging is stored up in the form of chemical energy which can be partially recovered on discharge. Thus the voltameter can be used to store electrical energy: this is the principle of the *accumulator* or *storage cell*, which is by far the most useful source of electricity for ordinary laboratory work.

The simplest form of accumulator, invented by Planté about 1879, consists of two lead plates immersed in dilute sulphuric acid and charged up for some time. The normal E.M.F. of the lead accumulator is from 2 to 2·1 volts, and it is not advisable to use it when the E.M.F. falls much below 2 volts. The amount of current obtainable depends very much on the size of the plates and the treatment they have received. The following simple experiment shows the behaviour of a very small cell whose plates were 10 sq. cm. in area. It was charged with a current of 2 amperes flowing for 5 minutes, and the E.M.F. was 2·2 volts at first, falling rapidly to 2·1 volts and then remaining steady. On short-circuiting for one minute the E.M.F. fell to 1 volt, but recovered to 1·5 volts in one minute more. Further short-circuiting reduced the E.M.F. still more.

Accumulators are made up in various forms giving E.M.F.'s from 2 volts upwards for laboratory purposes (Art. 168). A battery of small accumulators in series is a convenient source of high potential when but little current is required. Fig. 72 shows a battery of 10 cells yielding about 20 volts. Batteries of small

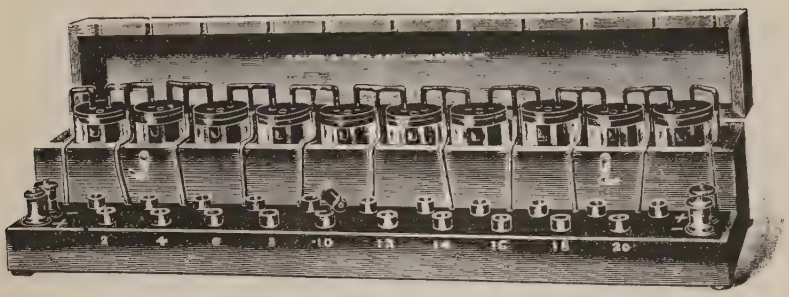


Fig. 72

Leclanché cells giving about 50 volts are also used for charging up the needles of electrometers.

59. Standard cells. It is very desirable to have a concrete realisation of potential difference in the shape of cells whose E.M.F. can be relied on to remain constant as long as the temperature remains the same. The Daniell cell satisfies this condition fairly well; but the Weston and Clark cells, which we shall now describe, are even more suitable, and the former is now almost universally used as a standard.

The *Weston normal cell* (Fig. 73) is very reliable when properly set up. The E.M.F.'s of well made cells differ by only one or two parts in 100,000, and remain constant for years. Cells which are

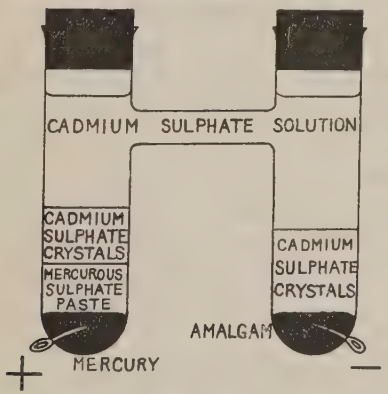


Fig. 73

accurate to about one part in a thousand may be constructed as follows.

(1) *The vessel.* The cell is prepared in an H-shaped glass vessel in which are fused two fine platinum wires. Rinse the vessel with dilute nitric acid and then with distilled water.

(2) *Preparation of the saturated solution of cadmium sulphate.* Take about 10 c.c. of commercially pure cadmium sulphate crystals and add 25 c.c. of distilled water. Prepare a saturated solution of cadmium sulphate by rubbing the ingredients together in a mortar for 10 minutes.

(3) *Preparation of the mercurous sulphate paste.* Take about 5 c.c. of commercially pure mercurous sulphate, add a little mercury and rinse it in a beaker with a little dilute sulphuric acid (1 part of acid to 6 parts of water by volume), afterwards rinsing twice with distilled water and once with a little of the cadmium sulphate solution for some time. Then pour off the solution.

Grind 1 c.c. of cadmium sulphate to a fine powder in a mortar together with 2 c.c. of the mercurous sulphate just prepared and $\frac{1}{4}$ c.c. of pure mercury. Add calcium sulphate solution if necessary till the whole forms a thin paste.

(4) *Preparation of the amalgam.* Weigh out about 6 gm. of commercially pure cadmium, and to each gram take $\frac{1}{2}$ c.c. of pure mercury. Heat the mixture to about 60° C. in a beaker over a water bath, and while still warm rinse with dilute sulphuric acid and then with distilled water.

(5) *Filling the cell.* Pour enough of the warm amalgam to cover the platinum wire into one limb of the H-vessel with a warmed pipette, and let it solidify. Into the other limb pour mercury to cover the platinum wire, and over it a layer of the mercurous sulphate paste about 1 cm. deep, using a pipette so that none of the paste adheres to the sides of the tube. Add the saturated solution of cadmium sulphate up to the level of the cross tube, and afterwards put some crystals of cadmium sulphate into each arm. The corks, previously soaked in distilled water, can now be inserted and the cell sealed.

The *Clark* cell is practically the same as the Weston cell, but with zinc everywhere substituted for cadmium. It has been extensively used in the past, but is now almost universally replaced by the Weston cell, on account of the much smaller variation of

the E.M.F. of the latter with temperature. Both cells show great recuperative power after being short-circuited. Thus a Weston cell short-circuited for one minute was found by F. E. Smith to have recovered its E.M.F. within one part in 10,000 one minute afterwards. Short-circuiting for five days affects the cell seriously for some days, but it recovers completely in six weeks. As actually used, standard cells are not called on to yield any but very small currents for short spaces of time.

E.M.F. of Weston and Clark cells (volts).

Temp.	Weston	Clark	Temp.	Weston	Clark	Temp.	Weston	Clark
10	1·0186	1·4386	17	1·0184	1·4304	24	1·0181	1·4215
11	1·0186	1·4374	18	1·0184	1·4291	25	1·0181	1·4202
12	1·0186	1·4363	19	1·0183	1·4279	26	1·0180	1·4188
13	1·0185	1·4351	20	1·0183	1·4267	27	1·0180	1·4175
14	1·0185	1·4340	21	1·0183	1·4254	28	1·0180	1·4161
15	1·0185	1·4328	22	1·0182	1·4241	29	1·0179	1·4147
16	1·0184	1·4316	23	1·0182	1·4228	30	1·0179	1·4134

For most purposes the E.M.F. of the Weston cell may be taken as 1·018 at all ordinary temperatures. The measurement of E.M.F. in absolute measure depends on principles not yet developed, so that we are not yet in a position to explain how the numbers in the table have been arrived at (cf. Art. 139).

60. Ohm's law. We shall now consider more closely the relation between the current and the E.M.F. in a circuit. The first step is to satisfy ourselves experimentally that the deflexion of a quadrant electrometer as ordinarily used, that is the difference between the actual reading on the scale and the reading when the quadrants are at the same potential, is proportional to the difference of potential between the quadrants. This can be done by making use of the fact that batteries keep up a constant difference of potential on open circuit. Let two different cells when applied separately to the terminals of an electrometer give deflexions θ_1 and θ_2; then if the two in series give a deflexion $\theta_1 + \theta_2$ the law of proportionality will hold. Testing an ordinary electrometer in this way with various cells we find that the readings may be

relied on to about one part in 200, provided that we take the mean of the deflexions with the cells joined in opposite ways to the quadrants. The direct (one-sided) readings are not so reliable, giving about 1 per cent. discrepancy.

Having now means for measuring current strength such as a tangent galvanometer, and having also the quadrant electrometer to give the difference of potential between any two points, it is possible to examine the way in which the current set up in a conducting wire depends on the potential difference between its two ends.

The principles involved in the passage of currents through conductors were not clearly grasped in the early days of electrical science. By 1827, however, G. S. Ohm had perceived that the ideas of current and electrical pressure, previously confused, were distinct though mutually dependent, and enunciated the law bearing his name. The idea of potential was not yet completely specified, so that the form in which Ohm's law was given differed somewhat from the form in which it can now be stated, which is as follows: If A, B are two points on a wire at constant temperature, the difference of potential between A and B is proportional to the current, other things being equal.

The wire AB to be tested is prolonged, and connexion made with a moveable contact to a cell and a tangent galvanometer. The ends of the wire are joined to the quadrants of an electrometer.

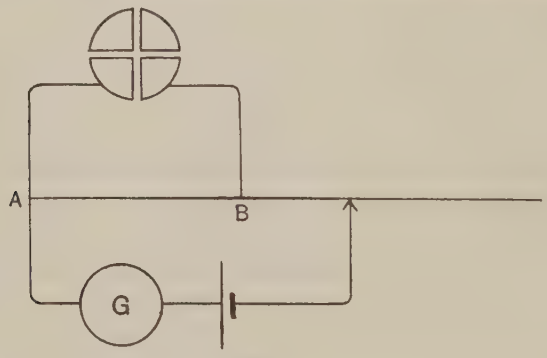


Fig. 74

it being understood that both instruments are provided with reversing keys. By moving this contact about we can get various

currents in AB and observe also the corresponding differences of potential between the ends of the wire. The following is the result of an actual experiment:

Deflexion θ of tangent galvanometer	Deflexion E of electrometer in arbitrary units	$\frac{E}{\tan\theta}$
15° 30′	24·0	86·8
23° 12′	37·8	88·2
33°	56·6	87·4
43°	78·3	87·0
50° 6′	105·0	87·8

The figures in the last column show no systematic variation, and therefore give a sufficient verification of Ohm's law.

A much more accurate verification can be made by indirect methods, that is by assuming the law and testing the deductions from it. Chrystal showed in 1876 that a deviation from Ohm's law of one part in ten million could have been easily detected if it had existed.

If the above experiment is repeated with a wire of different material, or of different length or cross-section, the ratio of the difference of potential to the current, i.e. V/i, will still be a constant though the constants will be different in each case. Thus the final result is reached that the magnitude i of a current flowing between two points differing in potential by V is given by $V/i = R$, where R is a constant depending upon the nature, length, and cross-section of the wire. This constant R is called the resistance of the wire; it also depends upon the temperature. But it is independent of both V and i except in so far as the current may cause a change in the temperature of the wire.

The derived electromagnetic unit of resistance is called the *ohm*, and a wire is said to have a resistance of 1 ohm if a current of 1 ampere flows when the difference of potential between its ends is 1 volt. The ampere being $\frac{1}{10}$ of the absolute unit and the volt 10^8 absolute units, it follows that the equation $i = V/R$ can be used unaltered in the absolute system if we take as the absolute unit of resistance 10^{-9} ohms. Large resistances are conveniently measured in megohms, a megohm being a million ohms.

Instances of resistances that may be mentioned are that of a 32 candle-power metallic filament lamp for a 100 volt circuit, which gives 300 ohms when hot; the resistance of a small electric bell is 3 or 4 ohms. Ordinary flexible copper wire, used in houses for electric light connexions, has a resistance of about 1 ohm per 40 metres, and that of solid copper wire of 1 mm. diameter is much the same.

It will appear later that electrolytes have real resistances, and that the same is true of the interior of a voltaic cell. The internal resistance of certain cells, for example accumulators, is low and is usually negligible in laboratory experiments with several hundred ohms in circuit. The behaviour of electrolytes is however not simple; Ohm's law is not satisfied in the sense of the current being proportional to the applied potential, but only when the disturbing effects of polarisation are eliminated. The conduction of electricity through gases affords a striking instance of deviation from Ohm's law (cf. Ch. XII), and another example is given by certain crystalline substances used in wireless telegraphy (Art. 179).

61. Resistances in series and parallel. Two resistances AB, BC having the end B of the first joined to the end B of the second are said to be *in series*, and if A and C are joined to

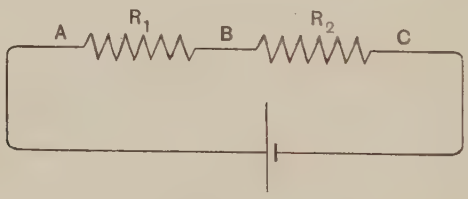


Fig. 75

a battery the current flows first through AB and then through BC. Let R_1 and R_2 be the resistances of the two coils, V_1, V_2 and V_3 the potentials of A, B, C respectively when a current is passing, and i the current. Then applying Ohm's law to the part AB we have $V_1 - V_2 = R_1 i$. Similarly $V_2 - V_3 = R_2 i$, and on addition we have $V_1 - V_3 = (R_1 + R_2)\, i$. But now if the whole coil acts like one of resistance R we shall have $V_1 - V_3 = Ri$ by

Ohm's law. Hence $R = R_1 + R_2$. Thus the resistance equivalent to two coils in series is obtained by adding the separate resistances, and so for any number.

Since a uniform wire of length l centimetres may be regarded as l pieces 1 cm. long in series, it follows that its resistance is proportional, *caeteris paribus*, to its length l.

Two resistances joined up as in Fig. 76 are said to be *in*

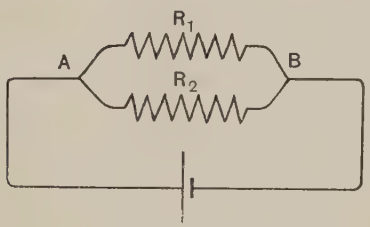


Fig. 76

parallel. The current i flowing from the battery divides at A, part i_1 going through R_1 and the rest i_2 through R_2. Since there is no accumulation of electricity at any point of the wire $i = i_1 + i_2$. If V_1, V_2 are the potentials of A and B respectively, Ohm's law applied to the resistance R_1 gives $V_1 - V_2 = R_1 i_1$, and similarly $V_1 - V_2 = R_2 i_2$. Hence

$$i = (V_1 - V_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

But $i = (V_1 - V_2)/R$, where R is the equivalent resistance of the branched circuit. Thus

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2},$$

or

$$R = \frac{R_1 R_2}{R_1 + R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It should be noticed that the equivalent resistance is less than either of the separate resistances R_1, R_2. It is often necessary to calculate the fraction of the main current i which passes through each branch, and it is obvious from the above that $R_1 i_1 = R_2 i_2$, i.e. the current divides in the inverse ratio of the resistances. The current through R_1 is $\dfrac{R_2 i}{R_1 + R_2}$ and that through R_2 is $\dfrac{R_1 i}{R_1 + R_2}$.

If there are any number of resistances in parallel we easily find instead of (2) the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots \ldots \ldots \ldots \ldots (3).$$

Although we cannot go inside a wire to see how the current is located in the interior, there is good reason to believe that steady currents are uniformly distributed across the cross-section: thus in electrolysis of copper sulphate the film of copper deposited on the cathode is appreciably uniform. Now suppose that a certain length of a wire, the area of whose cross-section is a, has resistance R. Then a wire of the same length and of cross-section na behaves like n of the preceding wires in parallel, and from (3) has a resistance R/n, so that the resistance varies inversely as the sectional area. Hence, to sum up, the resistance of a uniform wire is proportional to its length and inversely proportional to its sectional area, provided that the temperature remains constant. Thus the resistance is given by $R = kl/a$, where l is the length and a the area of the cross-section, and k is a constant.

The constant k at a given temperature is a characteristic of the metal; it is called its *specific resistance* and is the resistance of a cube of metal of side 1 cm. for currents normal to one face.

The best conductors are those whose resistances are lowest when drawn into a wire of given size, that is, those for which k is least. The quantity $1/k$, which is a measure of the conducting power of the material, is called the *conductivity* and denoted by σ. In speaking of the specific resistance and conductivity of substances we are dealing with their intrinsic properties independent of the dimensions of the wires into which they are drawn, and these are the appropriate quantities in terms of which the effect of influences such as temperature should be expressed. This we proceed to do in the next article.

62. Variation of resistance with temperature. Fig. 77 shows the variation of the specific resistance of certain metals with temperature, taken from the results of the best modern experiments (Dewar and Fleming, Jaeger and Diesselhorst, Lees, Kamerlingh Onnes and Clay). The metals were in all

cases carefully purified, but discrepancies always arise which cannot be explained as errors of experiment, being due to differences in the physical state of the metals used. Thus the resistance of a metal is increased by cold-drawing and diminished by annealing. Impurities generally cause a rise in the resistance, gold and platinum being very sensitive in this respect. The case of gold is particularly interesting as it has been carefully

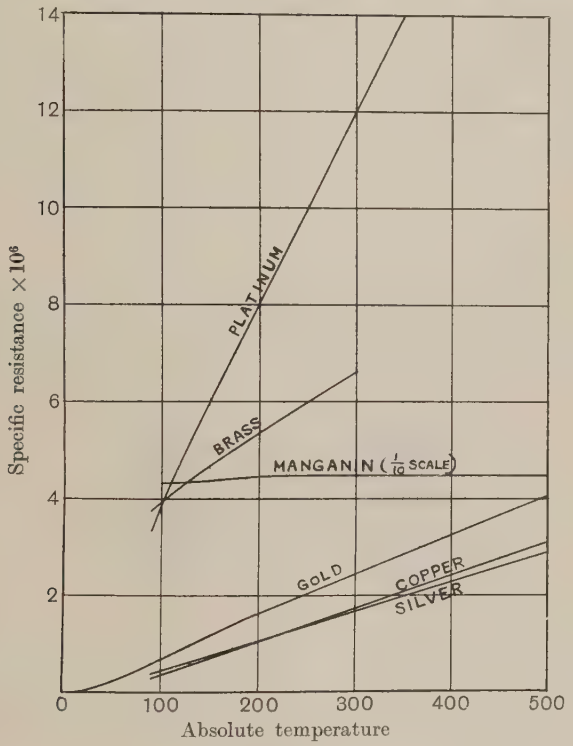


Fig. 77

studied down to very low temperatures (10° abs.). There is reason to believe that pure metals have very low resistances near the absolute zero, but for higher temperatures (100° to 500° abs.) the resistance in many cases is nearly proportional to the absolute temperature.

The *temperature coefficient*, or fractional rise of resistance per degree centigrade, affords a means of judging the behaviour of

a metal over a small range of temperature. In the following approximate table the numbers in the last column are the mean temperature coefficients between 0° C. and 100° C.

Metal	Specific resistance at 18° C. (ohms per cm. cube)	Conductivity at 18° C.	Temperature coefficient of resistance
Silver	$1{\cdot}65 \times 10^{-6}$	$6{\cdot}06 \times 10^{5}$	·00400
Copper (drawn) ...	$1{\cdot}78 \times 10^{-6}$	$5{\cdot}62 \times 10^{5}$	·00428
Gold	$2{\cdot}4 \times 10^{-6}$	$4{\cdot}2 \times 10^{5}$	·0038
Aluminium	$3{\cdot}0 \times 10^{-6}$	$3{\cdot}3 \times 10^{5}$	·0038
Zinc	$6{\cdot}1 \times 10^{-6}$	$1{\cdot}6 \times 10^{5}$	·0037
Platinum	$1{\cdot}16 \times 10^{-5}$	$8{\cdot}62 \times 10^{4}$	·00367
Nickel	$1{\cdot}2 \times 10^{-5}$	$8{\cdot}3 \times 10^{4}$	·0062
Iron	$1{\cdot}4 \times 10^{-5}$	$7{\cdot}1 \times 10^{4}$	·0062
Lead	$2{\cdot}1 \times 10^{-5}$	$4{\cdot}8 \times 10^{4}$	·0043
Mercury	$9{\cdot}41 \times 10^{-5}$	$1{\cdot}062 \times 10^{4}$	·0009
Bismuth	$1{\cdot}2 \times 10^{-4}$	$8{\cdot}3 \times 10^{3}$	·0042

Specific resistances are reduced to absolute electromagnetic units by multiplying by 10^9, conductivities by dividing by 10^9.

The curves for brass and manganin (Fig. 77) illustrate the behaviour of alloys, the temperature coefficient of which is generally smaller than for pure metals. Manganin (84 % copper, 12 % manganese, 4 % nickel) has a very small temperature coefficient (about ·00001) and is almost universally used for laboratory and standard resistances as the correction for change of temperature is usually negligible. German silver (temp. coeff. about ·0004) is useful if great constancy is not essential.

The specific resistances of indifferent conductors are given in the following table, but the numbers are only approximate, and for the best insulators include to some extent the effects of surface-leakage.

Substance	*Specific resistance*
Graphite	·003
10 % solution of sodium chloride	8
Tap water	1500
Ordinary distilled water ...	$1{\cdot}5 \times 10^{5}$
Pure water	$2{\cdot}5 \times 10^{8}$
Paraffin wax, quartz, ebonite, sulphur	Of the order 10^{15}.

It appears from these numbers that the criterion of good insulation is much less stringent in current electricity than in electrostatic work. All that is required for experiments in current electricity is that practically all the current in a circuit shall be carried by the metallic wire and that none leaks away through the insulating sheath. Now if we have a copper wire and a "wire" of ebonite of the same size in parallel, the current divides in the ratio of the specific resistances (Art. 61); that is, in the ratio 10^{21} to unity. Even if the available cross-section of ebonite was a billion times that of the copper no appreciable part of the current would be carried by the insulator. It is thus not surprising that substances like wood and oil can safely be regarded as insulators for these purposes.

The resistance of *selenium* is very much reduced when light falls on it. This fact has been applied to the telegraphic transmission of pictures.

63. Rheostats and other resistances. The *sliding rheostat* (Fig. 78) is a convenient laboratory instrument for adjusting

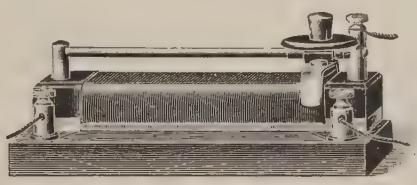


Fig. 78

the current in a circuit. It consists of a long coil of German silver wire wound in grooves on a slate cylinder, with its ends connected to two screw terminals. A third terminal is connected with a slider by means of which more or less of the coil may be inserted between it and either of the other terminals, and the adjustment of resistance is practically continuous. Fig. 79 shows a form of rheostat for high currents of the order of 10 amperes. Here one terminal is joined to one end of a continuous coil of thick wire interrupted at intervals by heavy brass studs, and the other terminal (not visible) is joined inside to the moveable handle, carrying a brass slider which moves over the studs. The adjustment proceeds in a number of stages. For some purposes

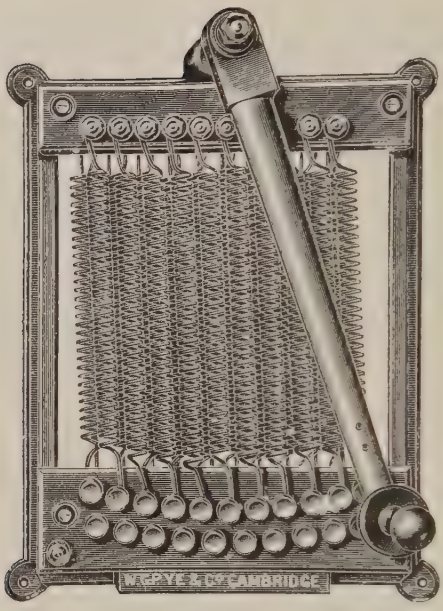


Fig. 79

a tray of electric glow-lamps (Fig. 80) is useful for controlling the current in a circuit. Here one or more lamps can be joined in parallel between two fixed terminals.

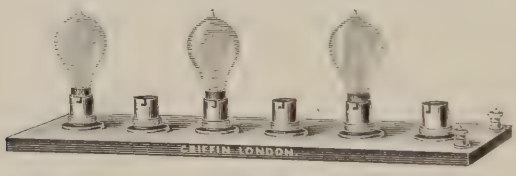


Fig. 80

An arrangement of coils known as a resistance box is generally used for accurate laboratory work. Fig. 81 shows the completed instrument, Fig. 82 the arrangement of the coils. Each coil consists of a manganin wire doubled back on itself and wound on an insulating spool, and its ends are fastened to the lower sides of the heavy brass strips which form the top of the box. When a brass plug is placed in a conical hole between two strips

the resistance of the corresponding coil is cut out since nearly all the current passes through the plug. The resistances of the coils

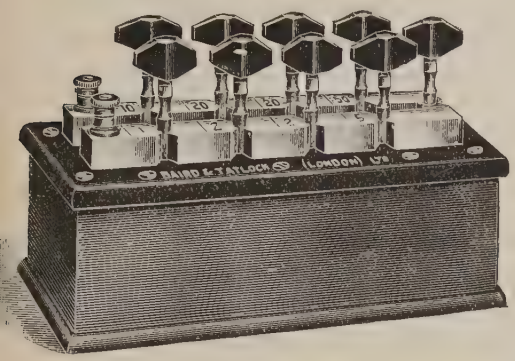


Fig. 81

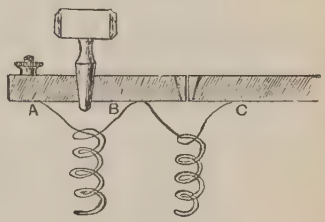


Fig. 82

are accurately adjusted to their marked values, and by combining them properly any required number of ohms can be put in the circuit (in Fig. 81 from 1 to 100). It is necessary in using resistance boxes to press the plugs in firmly, particularly with coils of high resistance, otherwise variations will occur in the current. Many troubles in electrical measurements can be traced to insufficient contact between parts of a circuit, and care should be taken to screw down all terminals tightly before beginning an experiment. The coils of a well-made resistance box should be correct to about one part in 2000.

Standard resistances are made as nearly as possible an exact number of ohms, and are used immersed in a water or paraffin bath at a known temperature. It will appear shortly that all that is done in ordinary laboratory experiments is to *compare* resistances: thus all the resistances in a laboratory may be supposed known in terms of a single coil, which we may call the laboratory standard. The relation of this standard to the true ohm as defined in Art. 60 is a matter requiring special investigation (see Art. 138).

64. Galvanometers, ammeters and voltmeters. Mention may here be made of certain laboratory instruments for measuring current and potential, but it is convenient to defer the detailed description until the principles of their action have been discussed

(see Arts. 80, 81). Sensitive *galvanometers* are generally read by the reflexion method with a lamp and scale: many of those in current use will give a deflexion for a current of one-millionth of an ampere. A modern galvanometer is independent of external magnetic fields such as that of the earth, and may therefore be fitted with a pointer moving over a scale graduated so as to give the current directly in absolute measure. Fixed-scale instruments of this kind for measuring large currents are called *ammeters*.

The accuracy of the scale-readings of an ammeter may be checked by the electrolytic deposition of copper as follows.

The electrolyte is a 25 % solution of copper sulphate to which 1 % of sulphuric acid has been added. The electrodes are copper plates, the cathode B being removeable and smoothed off at the edges. Set up a circuit containing the ammeter, a rheostat R and a key in addition to the cell and voltameter. If it is desired to test the accuracy of (say) the reading 50 on the ammeter scale, first take a trial observation to adjust the current roughly to its right value. Cut off the current and clean the cathode thoroughly with sand-paper, then dip it in dilute sulphuric acid and afterwards in distilled water and alcohol, drying it *very gently* over a Bunsen flame. The cathode is then weighed accurately and put in its place, and the current turned on. During the experiment the rheostat is used to keep the deflexion of the ammeter as near to 50 as possible, and the time the current is flowing is measured. On cutting off the current dip the cathode successively in dilute sulphuric acid, distilled water and alcohol, then dry gently and weigh as before. If m is the mass of copper deposited and t the time of the experiment in seconds, the true current i in amperes is given by

$$m = 3{\cdot}294 \times 10^{-4}\, it.$$

The reading of the ammeter is compared with this.

The *voltmeter* is an instrument which indicates directly the difference of potential between its terminals. A common type of voltmeter consists of a sensitive fixed-scale galvanometer in series with a high resistance. At present, however, we may confine our attention to the *electrostatic voltmeter*, which in principle is the same as an electrometer in which the needle is joined to one

pair of quadrants. An electrometer so used is said to be connected up *idiostatically*, as it works without extraneous charging of the needle. Referring to the theory of the quadrant electrometer (Art. 43) we have in general

$$(V_1 - V_2)(2V - V_1 - V_2) = \lambda\theta \ldots\ldots\ldots (4),$$

where θ is the deflexion and λ some constant. When $V = V_1$ this becomes

$$(V_1 - V_2)^2 = \lambda\theta \ldots\ldots\ldots\ldots\ldots (5).$$

Thus the deflexion of a quadrant electrometer used idiostatically is proportional to the square of the difference of potential between the quadrants, so that it increases very rapidly with increase of potential. This is a common characteristic of electrostatic voltmeters.

It is interesting to compare the sensitiveness of an electrometer when used heterostatically and idiostatically. In the first case, taking V as large compared with V_1 and V_2, we have

$$2V(V_1 - V_2) = \lambda\theta \ldots\ldots\ldots\ldots\ldots (6).$$

When $V = 50$ volts, $V_1 = 1$ volt and $V_2 = 0$, a fair deflexion is $\theta = 100$ (mm. on scale 1 metre away). This gives $\lambda = 1$. But now in order to produce the same deflexion idiostatically we must have, from equation (5),

$$V_1 - V_2 = 10 \text{ volts.}$$

A difference of potential of 1 volt between the quadrants, in fact, only produces a deflexion of 1 scale division instead of 100.

Commercial electrostatic voltmeters usually consist only of the "needle" and one set of quadrants, the other set being suppressed. Fig. 83 shows a pattern designed by Lord Kelvin for voltages up to 10,000 or so, in which the motion of the needle is controlled by gravity.

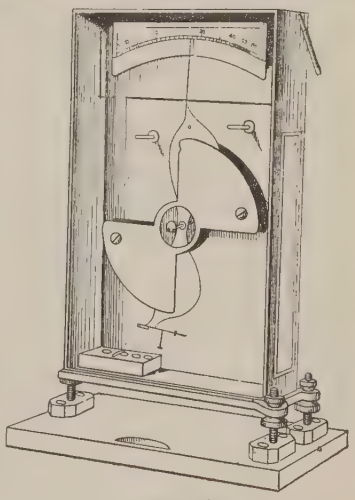


Fig. 83

65. Heat produced by currents. Let V be the difference of potential between two points A, B of a circuit, and in time dt let a charge de_1 of positive electricity pass from A to B and a charge $-de_2$ of negative electricity from B to A. The work done on the positive electricity is Vde_1 and on the negative $-V(-de_2)$; in all $V(de_1 + de_2) = Vidt$ by Art. 51. The work done in driving the current is therefore Vi ergs per second as far as the part AB of the circuit is concerned, and this work must reappear as heat developed in AB. For steady currents the heat developed in time t is therefore Vit measured in work units; but in case the current is not steady it is expressed by $\int Vidt$. True electromagnetic units must here be used.

For steady currents we have in addition $V = Ri$ by Ohm's law, and the heat developed is Ri^2/J calories per second, where $J = 4{\cdot}18 \times 10^7$ is the mechanical equivalent of heat. Thus the rate of heat-development is proportional to the square of the current.

A simple laboratory experiment is to find the mechanical equivalent of heat by electrical methods. To do this we require a coil of fine wire immersed in water in a calorimeter, with thick leads which do not become appreciably heated. First weigh the calorimeter and coil when dry, and also when containing enough water to cover the fine part of the wire. Take the air-temperature a and the initial temperature θ_0 of the water. Set up a circuit containing the calorimeter, an ammeter and a key, together with a cell of E.M.F. sufficient to cause a rise in the temperature of the calorimeter of at least half a degree per minute. Pass a current through the coil for about 20 minutes, reading the ammeter each minute and stirring the water in the calorimeter constantly. Note the final temperature θ_1 and the time t of the experiment in seconds. After stopping the current observe the falling temperature each minute for 10 minutes, stirring as before. In this way we find the average fall of temperature ω per minute on account of radiation at the temperature θ_1. The mean temperature during the rise is $\frac{1}{2}(\theta_0 + \theta_1)$, so that we may expect radiation to go on during the rise at an average rate

$$\omega \{\tfrac{1}{2}(\theta_0 + \theta_1) - a\}/(\theta_1 - a)$$

per minute, assuming that the rate of radiation is proportional

to the difference of temperature between the calorimeter and the air. The radiation correction to be applied to θ_1 is therefore approximately

$$\phi = \frac{\omega t}{60} \frac{\frac{1}{2}(\theta_0 + \theta_1) - a}{\theta_1 - a}.$$

If i is the mean current in amperes and R the (known) resistance of the coil in ohms, the heat developed in the coil is $10^7 Ri^2/J$ calories per second. Hence if W is the water value of the calorimeter and m the mass of the contained water, the mechanical equivalent J is given by

$$\frac{10^7 Ri^2}{J} t = (m + W)(\theta_1 - \theta_0 + \phi).$$

The values of J obtained electrically agree with those derived from mechanical experiments, thus justifying the application of the principle of energy to these phenomena. Further, the experiments afford a fresh test of Ohm's law which is in some ways more satisfactory than the one already described. For since the same value of J can be obtained with widely different currents, it follows that the rate of development of heat is proportional to the square of the current, and the proof of this result involves Ohm's law.

The development of heat by electric currents is utilised in many ways, the electric glow-lamp being a familiar example. Another is afforded by the use of *fuses*, or short wires of fusible metal inserted in a circuit, which become heated and melt when more than a certain current passes through them, thus protecting the ammeters or other electrical apparatus in the circuit from excessive current. That the action of fuses is due entirely to heat can be shown by immersing a fuse wire in water to keep it cool, when it will carry many times its normal current without yielding.

The unit of work in the derived system of electromagnetic units is the *joule*, named in honour of J. P. Joule, who first showed that the heating effect of currents was in accordance with the conservation of energy*. Its value is 10^7 ergs, and it is the work done on one coulomb moving through a difference of potential of one volt. The *watt* is the derived unit of power, and denotes

* The Ri^2/J law is often called Joule's law

the performance of work at the rate of one joule or 10^7 ergs per second. It follows that if a current of i amperes is flowing in a circuit, the rate at which energy is being converted into heat in a part AB of the circuit is measured by Vi watts, where V is the difference of potential between A and B in volts.

Since the mechanical equivalent of heat, in terms of the calorie at 15° C., is $4{\cdot}18 \times 10^7$, we have

$$1 \text{ calorie} = 4{\cdot}18 \text{ joules.}$$

The joule is however a small unit of energy, and a more usual unit of energy is the *Board of Trade unit*, or *kilowatt-hour*. This is the energy consumed when 1000 watts are used for one hour, so that

$$1 \text{ kilowatt-hour} = 3{\cdot}6 \times 10^6 \text{ joules.}$$

It may be mentioned that the cost of electrical energy for lighting purposes is generally from 4*d*. to 6*d*. per kilowatt-hour. This amount of energy will keep a 16 candle-power metal filament lamp alight at normal brilliancy for about 60 hours.

66. The Wheatstone's bridge. The arrangement of four resistances R_1, R_2, R_3, R_4 shown in Fig. 84 is due to Wheatstone

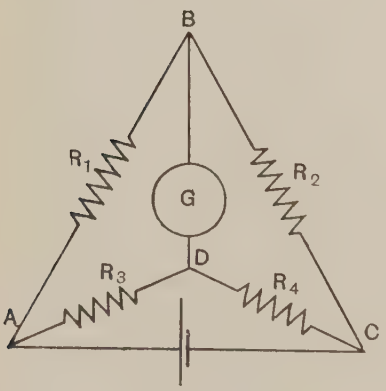


Fig. 84

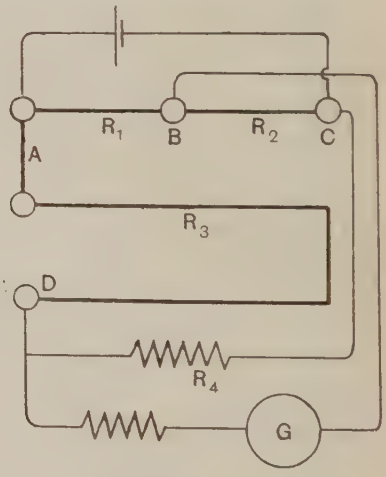


Fig. 85

(1843) and known by his name. A and C are joined to the poles of a battery, B and D to a galvanometer G.

The special advantage of the Wheatstone's bridge is that the current in BD is made zero, and therefore it is possible to compare resistances accurately without depending on the uniformity of the scale-readings of galvanometers. We shall show that, if the resistances R_1, R_2, R_3, R_4 in the arms of the bridge are adjusted so that no current flows in the galvanometer whether the battery in AC is on or off, they satisfy the condition

$$R_1/R_2 = R_3/R_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

For if there is no current in BD the current in AB has the same value i as that in BC. Similarly the currents in AD and DC have the same value j. Then applying Ohm's law to the four arms in succession we have $A - B = R_1 i$, $B - C = R_2 i$, $A - D = R_3 j$, $D - C = R_4 j$, where the letters A, B, C, D are used for the potentials at the corners of the bridge. But $B = D$ since no current passes in BD. Hence $R_1 i = R_3 j$ and $R_2 i = R_4 j$, from which the above result follows. Thus if we have three known resistances R_1, R_2, R_3, equation (7) enables us to measure any other resistance R_4 in terms of them, and great accuracy is obtainable by the use of a sensitive galvanometer. A resistance box should be inserted in the galvanometer circuit and a key in both this and the battery circuit.

The so-called Post Office box (Fig. 86) is a Wheatstone's bridge ready for the measurement of unknown resistances. The

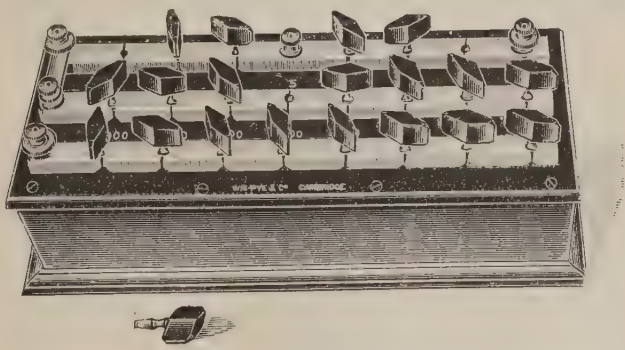


Fig. 86

method of using the box to measure a resistance R_4 will be evident from the diagram (Fig. 85) which shows its relation to the theoretical form of bridge.

The following example shows how the variation of the resistance of a coil of wire with temperature may be determined in the laboratory. The coil employed was of copper, wound on a hollow spool inside a thin-walled glass tube so that it could be placed in a water bath, and a thermometer was passed down the spool to indicate the temperature of the coil. A Post Office box was used in which $R_1 = 100$, $R_2 = 1$, and balance was obtained when the coil was cold with $R_3 = 108$. Thus the resistance of the coil when cold was 1·08 ohms. When R_3 was altered to 113 the balance of the bridge was destroyed and the galvanometer deflected, but the balance could be recovered by heating up the coil slowly until the galvanometer returned to zero. The thermometer was read when this occurred, and thus the temperature at which the resistance of the coil is 1·13 ohms was known. Proceeding similarly with $R_3 = 118$ and so on, we can find the variation up to the temperature of boiling water.

The principle of the Wheatstone's bridge is very important and is very extensively used. One application is to compare, with great accuracy, the resistances of two coils A, B known to be nearly equal, for example two one-ohm coils from a resistance box. Let C, D be two other resistances, also nearly equal. Then a Wheatstone's bridge made up with the four coils is very nearly balanced, and exact balance can be obtained by shunting A with a comparatively large resistance a and B with a resistance b. Then

$$\frac{\frac{1}{A}+\frac{1}{a}}{\frac{1}{B}+\frac{1}{b}} = \frac{D}{C}.$$

Next interchange A and B and balance again with shunt a' and b' respectively. Then

$$\frac{\frac{1}{A}+\frac{1}{a'}}{\frac{1}{B}+\frac{1}{b'}} = \frac{C}{D},$$

from which we have

$$\left(\frac{1}{A}+\frac{1}{a}\right)\left(\frac{1}{A}+\frac{1}{a'}\right) = \left(\frac{1}{B}+\frac{1}{b}\right)\left(\frac{1}{B}+\frac{1}{b'}\right).$$

With well-made coils A and B will not differ by more than one part in 1000, and A/a, B/b, etc. are of the same order of magnitude. Hence neglecting terms of the order 10^{-6} we have

$$\frac{1}{B^2} - \frac{1}{A^2} = \frac{1}{A}\left(\frac{1}{a} + \frac{1}{a'}\right) - \frac{1}{B}\left(\frac{1}{b} + \frac{1}{b'}\right).$$

If A and B are both nearly equal to A_0, this gives approximately

$$A - B = \frac{A_0^2}{2}\left(\frac{1}{a} + \frac{1}{a'} - \frac{1}{b} - \frac{1}{b'}\right),$$

so that the difference of the resistances of A, B is known. In order to obtain the ratio of the resistances correct to the order 10^{-6} it is only necessary that A_0 and the shunting resistances should be known to one part in 1000 of their value, for we have

$$\frac{A - B}{A_0} = \frac{A_0}{2}\left(\frac{1}{a} + \frac{1}{a'} - \frac{1}{b} - \frac{1}{b'}\right),$$

and the terms on the right are already small of the order 10^{-3}.

67. Comparison of electromotive forces. The potentiometer. A cell such as the accumulator will deliver a current for a considerable time without suffering any appreciable change in its electromotive force, but standard cells are not constructed to give more than the minutest current. It is thus advantageous to compare the electromotive forces of two cells without requiring both of them to deliver steady currents. This can be done by

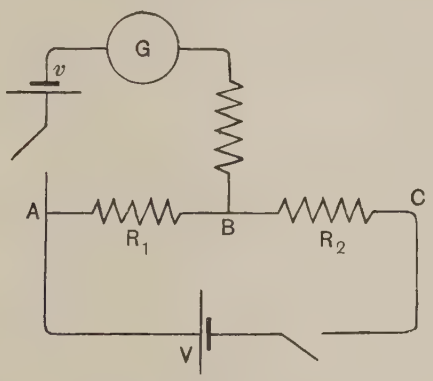


Fig. 87

the compensation method due in principle to Poggendorf and du Bois-Reymond.

Let R_1, R_2 (Fig. 87) represent two resistance boxes connected in series with an accumulator of electromotive force V. A "shunt" circuit containing another cell v, a resistance box and sensitive galvanometer, is connected to the ends of the resistance R_1 and can be put into operation by means of a key. If the adjustment of the resistances R_1, R_2 is such that no current passes in the galvanometer circuit, we have $v/V = R_1/(R_1 + R_2)$. To prove this let i be the current flowing in the main circuit, and let A, B, C stand for the potentials of the corresponding points in the diagram. Then $A - B = R_1 i$ and $V = A - C = (R_1 + R_2)\, i$, and since no current flows in the shunt circuit $v = R_1 i$. Hence

$$\frac{v}{V} = \frac{R_1}{R_1 + R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The E.M.F. of the cell V in the main circuit must exceed that in the shunt circuit if balance is to be obtained. In practice it is convenient to keep $R_1 + R_2$ at a fixed value by using two resistance boxes of the same pattern with only half the full number of plugs, and arranging so that the plugs in one box correspond to gaps in the other, and *vice versâ*.

If v is a standard cell, the E.M.F. V of the accumulator is accurately known from (8), and then the E.M.F. of any other cell can be found by substituting it for the standard cell in the shunt circuit. Moreover, a voltmeter can be attached to the poles of V during the experiment, and its reading can be compared with the true E.M.F. given by (8). We can thus calibrate a voltmeter at the points on its scale corresponding to 2, 4, 6, ... volts approximately.

The arrangement of the two resistances R_1, R_2 is also useful as a potential-divider or *potentiometer*. Thus the difference of potential between A and B is the fraction $R_1/(R_1 + R_2)$ of the E.M.F. of the cell V applied to the extremities of the compound coil, and AB can be applied to keep any two parts of an apparatus at an adjustable difference of potential. A sliding potentiometer giving adjustment in small stages is found useful in experiments in electrostatics and in the conduction of electricity through gases. A number of equal coils of wire are prepared and soldered to brass strips arranged alongside one another as shown in Fig. 88, the

two end strips A, B acting as terminals. A sliding brass piece P moves over the segments and also slides on a brass rod from which

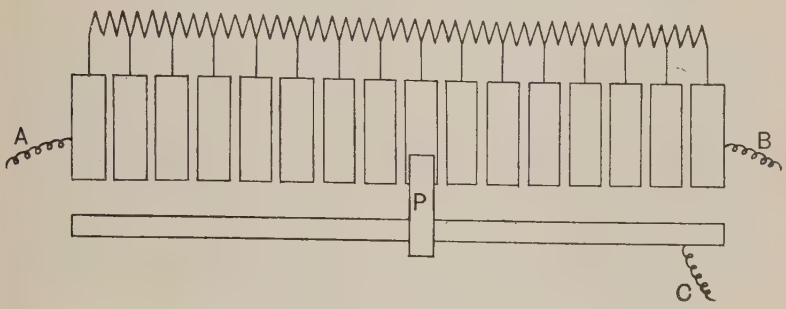


Fig. 88

connexion is taken off at C. If there are 50 coils of (say) 20 ohms each, and a 2 volt accumulator is applied to AB, the difference of potential between A and C may be quickly adjusted to 1/50 volt, 2/50 volt, etc., at will.

68. Other applications of Ohm's law.

(1) *Standardisation of a sensitive galvanometer by shunting.* It is often necessary to determine the current in a sensitive galvanometer which corresponds to a given deflexion in absolute measure; or the current per scale division if this is the same all over the scale.

An accumulator V is used, and a circuit set up as shown. Here R and S should be considerable (at least 100 ohms) and X is either 1 ohm or $\frac{1}{10}$ ohm. R or S is then adjusted until the right deflexion is obtained. Neglecting the internal resistance of the accumulator, it is not difficult to show that the current i through the galvanometer is given by

$$V = i\left\{R + S + G + \frac{R}{X}(S + G)\right\},$$

where G is the resistance of the galvanometer.

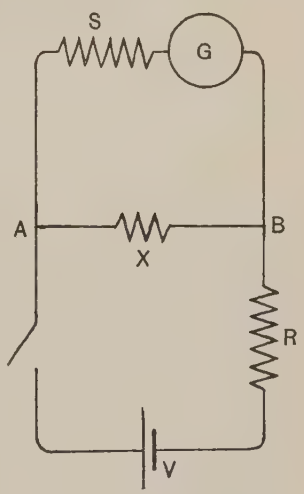


Fig. 89

In practice X is small compared with R and S, and we have

$$i = \frac{VX}{R(S+G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

This approximate formula is more easily arrived at as follows. Since $S + G$ is merely a large shunt to X, the equivalent resistance of the parallel circuits is nearly equal to X, so that the difference of potential between A and B is approximately VX/R. Applying Ohm's law to the galvanometer circuit then gives the formula (9). The E.M.F. of the accumulator may be read on a voltmeter if necessary, but it often happens that the same cell is used in another part of the experiment and that V cancels out.

(2) *Measurement of a small resistance.* Let 12 be the terminals of the small resistance X which it is desired to measure, 34 those of a standard small resistance Y, for example the $\frac{1}{10}$th ohm coil of a resistance box. Set up a circuit as shown with an additional

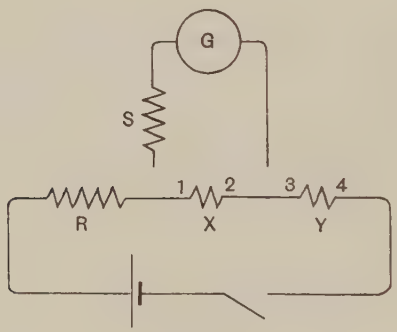


Fig. 90

resistance R sufficient to prevent the passage of an excessive current, leaving the galvanometer circuit free to connect to any points in the main circuit. First join the galvanometer circuit to 12 and adjust S for a certain deflexion θ of the galvanometer. Since $S + G$ is large in comparison with X the current in the galvanometer may be taken as $Xi/(S + G)$, where i is the current in the main circuit. Now disconnect the galvanometer circuit from 12 and connect it to 34, changing the resistance S to the value S' necessary to make the deflexion θ once more. Then

since the galvanometer currents are the same in the two cases we have

$$\frac{X}{S+G} = \frac{Y}{S'+G},$$

giving X.

69. Laboratory experiments with the quadrant electrometer.

(1) *Comparison of capacities.* With the key up insulate the electrometer. Then adjust the ratio of the resistances R_1, R_2 till there is no deflexion on pressing down the key. On pressing down the key the outside plates of the condensers, previously at zero potential, take up the potentials $R_1V/(R_1+R_2)$ and $-R_2V/(R_1+R_2)$ respectively. Since the electrometer is not deflected the inner plates are both at zero potential, and therefore the charges on the inner plates are

$$\frac{R_1C_1V}{R_1+R_2} \quad \text{and} \quad -\frac{R_2C_2V}{R_1+R_2}.$$

But there is no charge on the electrometer, and initially no charge on either of the plates. Hence the actual charges on the plates must be equal and opposite, i.e.

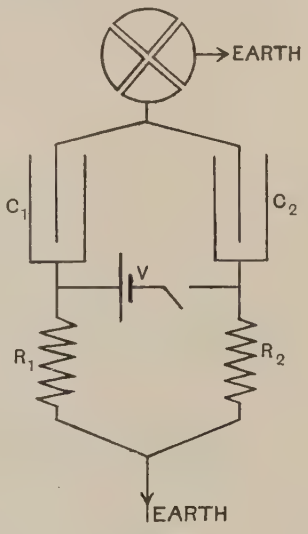


Fig. 91

$$R_1C_1 = R_2C_2 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

If a condenser of known capacity is taken as C_1 (cf. Art. 39), the capacity of another condenser C_2 is found from the last equation. It will sometimes happen that one of the condensers leaks slightly, so that on pressing down the key the electrometer moves off continuously. In this case what should be looked for on making the circuit is a sudden jump of the needle, and the adjustment made until this vanishes as nearly as can be judged.

(2) *Measurement of the charge required to produce a given deflexion of the quadrant electrometer.*

For this we require a condenser of known capacity C. First

insulate the electrometer with the battery circuit open, and then close the circuit, obtaining a steady deflexion of θ scale divisions. If V, v are the potentials of the outside and inside plates of the

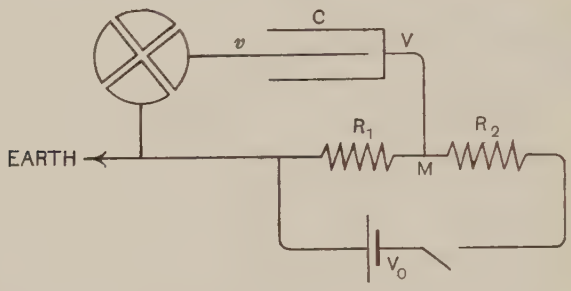


Fig. 92

condenser, the charge on the electrometer, being equal and opposite to the charge on the inner plate of the condenser, is $e = C(V - v)$. Here $V = R_1 V_0/(R_1 + R_2)$. To find v connect the electrometer directly to the point M and reproduce the deflexion θ with resistances S_1 and S_2 instead of R_1 and R_2. Then

$$v = S_1 V_0/(S_1 + S_2).$$

It is convenient to make $R_1 + R_2 = S_1 + S_2$ as suggested in Art. 67, in which case the charge on the electrometer corresponding to the deflexion θ is

$$e = \frac{C(R_1 - S_1) V_0}{R_1 + R_2} \quad \ldots\ldots\ldots\ldots (11).$$

V_0 is read on a voltmeter and reduced to electrostatic units by dividing by 300, and then e is obtained in electrostatic units. Or, if preferred, C is reduced to farads by dividing by 9×10^{11}, and e is then in coulombs, V_0 remaining in volts.

It is usually found that the charge is proportional to the deflexion if the latter is not too large, and then the charge per scale-division is $C(R_1 - S_1) V_0/(R_1 + R_2)\theta$. Again if c is the capacity of the electrometer at the given deflexion, we have $e = cv$.

Hence $$c = \frac{C(V - v)}{v} = \frac{C(R_1 - S_1)}{S_1} \quad \ldots\ldots\ldots\ldots (12).$$

The capacity of a quadrant electrometer is of the order of 50 electrostatic units.

(3) *Measurement of large resistance with a condenser and an electrometer or electrostatic voltmeter.* We shall describe a suitable arrangement for resistances of the order of 1 megohm. For larger resistances a smaller condenser is used.

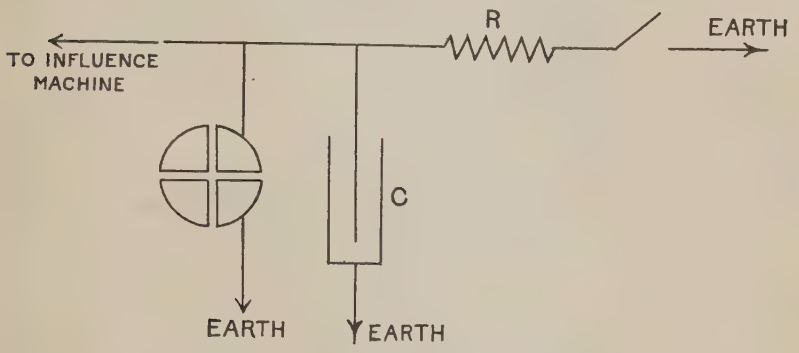


Fig. 93

Charge up the condenser C (about 10 microfarads) to (say) 150 volts by bringing a wire from an influence machine near to a wire extended from the condenser circuit, the other terminal of the influence machine being earthed. The potential of the inner plate of the condenser is read on an electrostatic voltmeter. Having observed this, press down the key for about 10 seconds and then insulate and wait for the deflexion to become steady again. If the potential drops from V_0 to V in t seconds, the resistance R is given by $t = RC \log_e (V_0/V)$, or

$$t = 2{\cdot}302\, RC \log_{10} \frac{V_0}{V} \ldots\ldots\ldots\ldots (13).$$

For if V is the potential at time t, the current is $-C\,dV/dt$, and Ohm's law gives $V = -RC\,dV/dt$, from which the above result follows on integration. The insulation in the experiment must be carefully looked after.

(4) *Measurement of contact potential.* The plates whose contact difference of potential it is required to measure are placed in a metal case so that they can slide in and out while remaining parallel to one another (Fig. 94), and the case is put to earth to protect the inner plate from electrostatic disturbances. It is possible to adjust the ratio R_1/R_2 so that the electrometer is not affected

by pulling the right-hand plate away from a short distance in front of the other (cf. Art. 44). The natural contact potential of the right-hand above the left-hand plate is then equal to

$$R_1V/(R_1 + R_2).$$

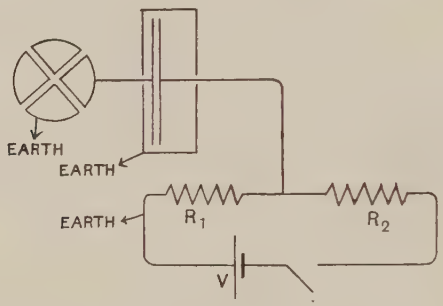


Fig. 94

The magnitude of the contact potential depends very much on the state of the opposing surfaces. The following table, due to Anderson and Bowen, gives the values for freshly-cleaned surfaces.

Contact Potential Differences (volts).

Aluminium – copper ...	+ ·83
Zinc – copper	+ ·76
Lead – copper	+ ·55
Brass – copper	+ ·15
Gold – copper	– ·11
Silver – copper	– ·13

The additive law is always found to hold within the limits of experimental error.

70. Measurement of dielectric constants. Dielectric constants of well-insulating solids may be measured by using two parallel-plate condensers of identical dimensions and comparing their capacities as in the last article. The two condensers may be combined into a single composite condenser with a common middle plate (with guard-ring), having air on one side and the solid dielectric on the other. As the induced charges will leak on to the dielectric in time, it is advisable to close the battery circuit only momentarily.

The use of the electrometer is not advisable with badly-insulating solids or with liquids, but the capacities can be compared in all cases by the following method, which does not differ in principle from the preceding one. The two condensers C_1, C_2 form a Wheatstone's bridge with two resistances R_1, R_2. The

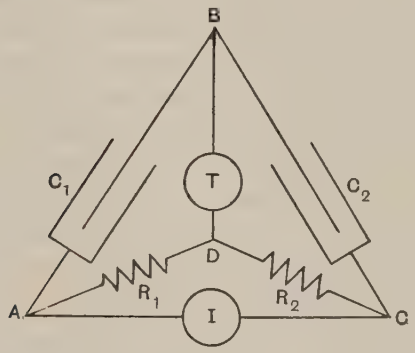


Fig. 95

battery is replaced by a small induction coil I (see Art. 135) which gives intermittent currents alternating in direction, and the galvanometer by a telephone receiver T, which is a very sensitive detector for these currents. The resistances R_1, R_2 are of the order of 10,000 ohms, and are preferably liquid resistances (for example electrolytic cells with platinised platinum electrodes, filled with distilled water). It is not very convenient to vary the resistances, so that one of the condensers must be adjustable. The adjustment is made till there is no sound in the telephone, i.e. till there is no current in the arm BD whatever changes occur in the arm AC. Then if i is the current in the arms AB, BC and j that in the arms AD, DC, we have

$$A - B = \frac{1}{C_1}\int i dt, \quad B - C = \frac{1}{C_2}\int i dt,$$

$$A - D = R_1 j, \qquad D - C = R_2 j,$$

where A, B, C, D stand for the potentials at the corresponding points. Since $B = D$, this gives

$$R_1 C_1 = R_2 C_2 \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Suppose that the condensers have plates of the same area A, and that one or both of the distances d_1, d_2 are adjustable with micrometer screws. Then if C_1 is the air-condenser,

$$C_1 = A/4\pi d_1; \quad C_2 = KA/4\pi d_2.$$

Hence (14) gives $R_1/d_1 = KR_2/d_2$, so that K is known.

This method is not applicable to gases, for which K differs by very little from unity. The dielectric constants of gases may be measured by the following method, due to Boltzmann.

A parallel-plate condenser is placed under a bell-jar connected to a mercury manometer and an air-pump, and the bell-jar is then exhausted. One plate of the condenser is joined to an electrometer by a wire leading out through the bell-jar, and the other to a source of high potential consisting of a battery of cells one pole of which is earthed. Suppose that the electrometer is initially earthed and the outside plate at potential V_0. On insulating there is then a charge CV_0 on the inner plate, where C is the capacity of the condenser. Air is now admitted to the bell-jar, and the electrometer shows a slight deflexion θ. The potential of the electrometer is $\lambda\theta$ where λ is some constant. There is thus a charge $KC(V_0 - \lambda\theta)$ on the inner plate of the condenser and $c\lambda\theta$ on the electrometer, where c is the capacity of the electrometer, making $KC(V_0 - \lambda\theta) + c\lambda\theta$ altogether on the insulated system. A second experiment, with exhausted bell-jar, consists in first insulating the electrometer and then changing the potential of the outside plate of the condenser from V_0 to $V_0 + V$, producing a deflexion ϕ. Since the charge on the insulated system is the same in all three cases, we have

$$CV_0 = KC(V_0 - \lambda\theta) + c\lambda\theta = C(V_0 + V - \lambda\phi) + c\lambda\phi.$$

It follows from these equations that

$$(K-1)CV_0 = (KC - c)\lambda\theta, \text{ and } CV = (C-c)\lambda\phi.$$

Hence

$$\frac{(K-1)V_0}{V} = \frac{KC - c}{C - c} \cdot \frac{\theta}{\phi}.$$

Since K differs by very little from unity, the last equation is practically equivalent to

$$K - 1 = V\theta/V_0\phi.$$

Boltzmann satisfied himself that the deflexions observed were

really due to the difference of the dielectric constants of air and vacuum, and not to charges produced by friction during the letting in (or out) of the air. In his experiments V_0 was of the order of 300 volts, V 1 volt, and θ/ϕ of the order 1/10.

71. Platinum resistance thermometry. The variation of the resistance of metals affords a ready means of ascertaining their temperature or that of any vessels in which they may be immersed. Platinum has been found most reliable, and platinum thermometers can be used between temperatures of $-200°$ C. and 1000° C. Suppose we have a coil of platinum immersed in a vessel, and that its resistance is R. If R_0 is the resistance when immersed in melting ice and R_{100} in boiling water, the "platinum temperature" t is given by the equation

$$\frac{t}{100} = \frac{R - R_0}{R_{100} - R_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Callendar has found that the relation of t to the corresponding true temperature θ (thermodynamic temperature with 0° and 100° fixed points) is accurately given by

$$\theta - t = \frac{\delta}{10000}\,\theta\,(\theta - 100) \quad \ldots\ldots\ldots\ldots (16),$$

where δ is a constant for a particular specimen of wire, and always nearly equal to 1·5. To determine δ it is only necessary to take an observation of the boiling-point of sulphur, which has been shown by Callendar and Griffiths to be at $\theta = 444{\cdot}53°$ C. For low-temperature work the boiling-point of oxygen ($-182{\cdot}5°$ C.) may be used; but platinum thermometers cannot be advantageously used at very low temperatures.

The resistance of the platinum coil is measured by a Wheatstone's bridge method, Fig. 96 showing the connexions. The thermometer itself, shown diagrammatically in the figure, consists of a platinum coil wound on mica in a glass or porcelain tube. In order to avoid the uncertainty as to the temperature of the leads in the upper part of the tube a pair of dummy leads is also used, identical with the true leads except that they are not attached to a coil. The leads are of copper or platinum and are kept apart by discs of mica inserted in the tube, which also serve

to prevent convection currents from circulating too freely. The difference of the resistances of the coil and dummy leads is measured by an arrangement known as the Carey Foster bridge, in which the two fixed arms of the bridge are kept equal. The point of contact C of the galvanometer can be moved along a stretched wire XY by means of a sliding contact, and adjustment is made till the galvanometer shows no deflexion when the battery is on. The resistances in the arms AD, DB are equal, so that if x and y are the resistances of the two segments of the wire, the condition of balance is

$$\rho + R_1 + x = \rho + R + R_2 + y,$$

where ρ is the resistance of the dummy leads and $\rho + R$ of the thermometer coil with its leads. Hence

$$R = (R_1 - R_2) + (x - y) \ldots (17),$$

which gives R, and then θ is known from equations (15) and (16). Several values can be given to $R_1 - R_2$ on the same bridge by means of plug contacts, so that the range of variation of R may greatly exceed that obtainable from the bridge-wire alone. As it is difficult to obtain a wire of any prescribed resistance, the bridge-wire is often shunted so that the range afforded by the wire corresponds to some particular rise of temperature, for example a rise of 10 platinum degrees. For very accurate work it may be necessary to take account of the heating of the platinum coil by the battery current.

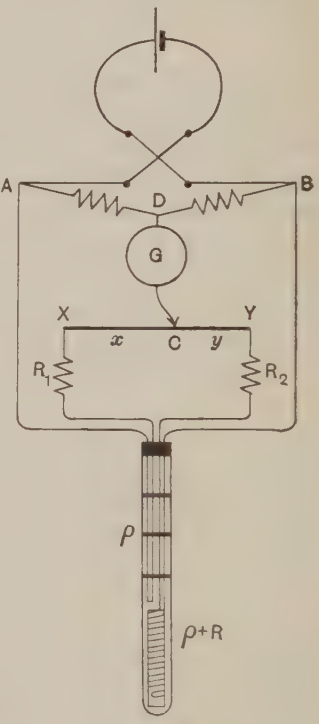


Fig. 96

Langley's *bolometer* is an instrument for the measurement of radiant heat depending on the variation of resistance of a blackened platinum strip.

72. Currents in solid conductors. We have now to inquire how currents are specified when they flow in solid conductors which cannot be even approximately regarded as wires. Consider a solid metallic cylinder in which the flow is uniform over the cross-section and parallel to the axis of the cylinder. Then the most important quantity with which we are concerned is the current per unit area of the cross-section; this is called the *current-density*. We may define current-density more generally as follows.

To fix ideas, suppose that only the negative electrons move, that each carries a charge of e electrostatic units, and that their average velocity near a point (x, y, z) of a solid conductor is $v\ (v_x, v_y, v_z)$. The stream of electrons is practically equivalent to a moving electric fluid of electric volume-density $\rho = ne$, where n is the number of electrons per c.c. near (x, y, z). Then it follows from Art. 5 that the current through any surface-element dS described in the conductor is $(l\rho v_x + m\rho v_y + n\rho v_z)\, dS$ electrostatic units per second, or $\left(l\frac{\rho v_x}{c} + m\frac{\rho v_y}{c} + n\frac{\rho v_z}{c}\right) dS$ electromagnetic units of current. The vector j with components

$$j_x = \frac{\rho v_x}{c}, \quad j_y = \frac{\rho v_y}{c}, \quad j_z = \frac{\rho v_z}{c} \ldots\ldots\ldots\ldots (18)$$

is called the current-density at the point (x, y, z), and now the current through the element dS is

$$j_n dS \ldots\ldots\ldots\ldots\ldots\ldots (19).$$

If positive charges are supposed to move as well as electrons, equations (18) become

$$j_x = \frac{1}{c}(\rho v_x + \rho' v_x'), \quad j_y = \frac{1}{c}(\rho v_y + \rho' v_y'), \quad j_z = \frac{1}{c}(\rho v_z + \rho' v_z'),$$

where ρ' and v' refer to the positive charges. The expression (19) remains unaltered.

There is an important relation connecting current-density with electric force in a solid conductor. Firstly, the direction of flow at every point of an isotropic conductor is clearly that of the impressed electric force. Imagine a small cylinder PQ of cross-section a and length l described with its axis parallel to the electric force E (Fig. 97). Then the cylinder may be regarded as a short piece of wire carrying a current ja, the difference of potential between

its ends being lE. The resistance is $l/\sigma a$ where σ is the conductivity, so that Ohm's law gives

$$lE = \frac{l}{\sigma a} ja, \quad \text{or} \quad j = \sigma E \ldots (20).$$

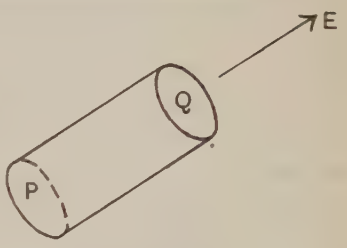


Fig. 97

This form of enunciation of Ohm's law corresponds more closely than the former one to Ohm's original ideas on conduction, which proceeded on an analogy between electric and thermal conduction. The electric force in this formula must be measured in electromagnetic units, true or derived.

The conditions that hold good while crossing the boundary separating two different conductors in contact are

(1) the tangential component of electric force is continuous,

(2) the normal component of current-density is continuous.

To prove (1), let AB represent the surface of separation, and draw a rectangle $PQRS$ whose sides PQ, RS parallel to the surface

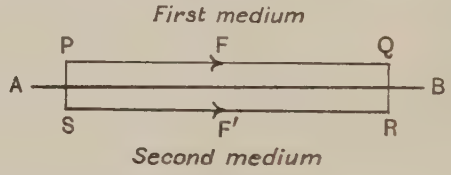


Fig. 98

are of the first order of small quantities, PS and RQ being of the second order. Let F, F' be the tangential components of electric force at P, S respectively. By the principle of energy, no work must be done by the forces of the field when a unit charge is taken round $PQRS$. Hence $F \,.\, PQ - F' \,.\, PQ = 0$, or $F = F'$. To prove (2), imagine a surface-element dS drawn on the interface, and let j_n, j_n' be the normal components of current-density in the first and second media respectively. The charge entering (or leaving) the first medium through dS is $j_n dS$, and that leaving (or entering) the second medium $j_n' dS$, per second. Hence $j_n = j_n'$, since there can be no accumulation of charge on the boundary.

The theory of conduction in solid conductors has important applications to the question of electric wiring. The current in a

wire must not be so great as to cause a dangerous rise of temperature in it. Now it is easy to prove that the rate of development of heat is $\frac{1}{2}\sigma E^2$ per c.c. per second, measured in ergs if σ and E are measured in true electromagnetic units. In derived units this becomes $\frac{1}{2}\sigma E^2$ watts, or $\frac{\sigma E^2}{8{\cdot}36}$ calories per c.c. per second, which may be written $j^2/8{\cdot}36\sigma$, where j is the current-density in amperes per sq. cm. Obviously j must not exceed a certain amount unless there is some special arrangement, such as a water bath, for removing the heat supplied and keeping down the temperature. A current-density of 200 amperes to the square centimetre is a safe limit with large copper cables, but larger current-densities can be used with small wires, in which the radiating surface is greater in proportion to the volume.

The size of wires in the United Kingdom is usually specified in terms of the standard wire gauge (S.W.G.) numbers, and cables are made up of separate wires by stranding. Thus a 7/22 cable means one composed of 7 strands of wire of No. 22 S.W.G. The following table gives the metric dimensions of the S.W.G. wires, and the resistances per metre in copper, manganin and German silver.

S.W.G.	Diameter in mm.	Copper		Manganin	German silver
		Safe current (amperes)	Ohms per metre	Ohms per metre	Ohms per metre
12	2·64	15·0	·0032	·077	·041
14	2·03	9·8	·0054	·131	·070
16	1·63	6·8	·0083	·204	·109
18	1·22	4·2	·0148	·361	·193
20	·914	2·6	·0260	·645	·345
22	·711	1·7	·0435	1·07	·57
24	·559	1·1	·070	1·73	·92
26	·457	·7	·105	2·58	1·38
28	·376	·5	·155	3·82	2·02
30	·315	·4	·222	5·45	2·90
32	·274	·3	·293	7·18	3·83
34	·234	·2	·404	9·90	5·27
36	·193	·15	·590	14·5	7·74
38	·152	·1	·950	23·2	12·4
40	·122	·06	1·48	36·3	19·4
42	·102	·05	2·10	53·4	27·8
44	·081	·03	3·30	81·7	43·5
46	·061	·02	5·90	145·5	77·4

CHAPTER V

MAGNETIC EFFECT OF CURRENTS

73. Rectilinear currents. In this chapter we shall study closely the laws of the action of currents on magnets and of magnets on currents, beginning with the simplest case of a thin straight wire carrying a current. Iron filings strewn on a flat surface encircling the wire tend to set along circles with the wire as centre. This suggests that the lines of magnetic force due to the current are circles round the wire, the direction of magnetic force being related to that of current by the right-handed screw rule.

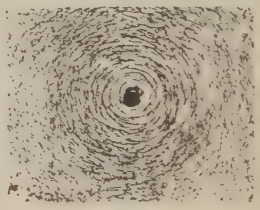


Fig. 99

Quantitative results are easily obtained by using a compass-box with a graduated circular scale, such as those used in magnetometers. If the wire is set in a vertical position magnetically east or west of the needle, either no deflexion is observed or else the needle turns through two right angles. Thus the magnetic force at any point due to the current is accurately at right angles to the radius vector from the wire to the point. A horizontal wire in the plane of the needle may tend to lift it somewhat at one end, but does not cause any scale deflexion in any position, so that there is no component of magnetic force parallel to the wire. Now suppose that the wire is placed in a vertical position either magnetically north or south of the needle. Just as in the case of the tangent galvanometer, a deflexion θ of the needle will result, which is connected with the magnetic force F due to the current by the relation $F = H \tan \theta$.

Using this method of determining F we find that F is inversely proportional at any point to the perpendicular distance r of the point from the straight wire. The following table gives the result of a series of observations:

r	θ	F in arbitrary units	rF
8·05	61° 30′	1·842	148
10·4	55° 20′	1·446	150
12·9	49° 5′	1·154	148
15·4	44°	·966	149
20·4	35° 45′	·720	147
22·9	32° 30′	·637	145

This experiment is however not very easy in practice, on account of the considerable length of wire required. A more convenient way of verifying the law, which is called the law of Biot and Savart, is the following. Let the wire be set vertically (Fig. 100), and a plate pierced with a hole arranged horizontally round it, being suspended by three equal strings attached to the wire. A magnet laid on the plate will take up a certain position of equilibrium under the action of the earth's field, but this position will be unaltered when a current is sent through the wire. For if we have an elementary pole of the magnet distant r from the wire, it is acted on by a force mF perpendicular to r, resulting in a couple mrF round the wire. Since rF is constant the total couple tending to turn the whole magnet round the wire is $rF\Sigma m$, and this vanishes because the total strength of all the poles of a magnet is zero.

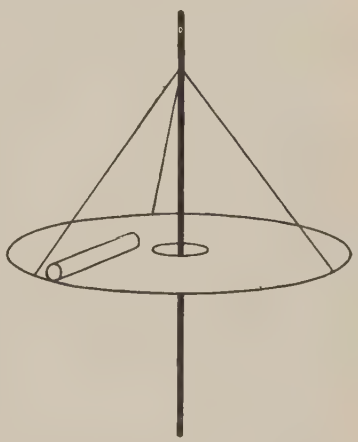


Fig. 100

Since F varies directly as the current and inversely as r, we may write

$$F = \frac{2\lambda i}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where λ is a constant, as yet undetermined, depending on the unit chosen for the measurement of current.

To find the magnetic potential of a straight current, take cylindrical co-ordinates (r, θ, z) in space, and let the infinitely long wire coincide with the axis of z, the positive direction of current being the positive direction of the axis. If P is the point (r, θ, z) and Q a near point $(r + dr, \theta + d\theta, z - dz)$, the work done by the magnetic force of the current when a unit pole is moved from P to Q is

$$0 \,.\, dr + \frac{2\lambda i}{r} \,.\, r d\theta + 0 \,.\, dz = 2\lambda i d\theta.$$

This is the perfect differential of $2\lambda i\theta$: hence there exists at P a magnetic potential

$$\Omega = \text{const.} - 2\lambda i\,\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

due to the current.

The magnetic potential differs in one essential respect from that due to a distribution of magnets—it is not single-valued. Quite apart from the arbitrary constant, Ω is indeterminate since θ might equally well be written $\theta \pm 2\pi$, $\theta \pm 4\pi$, etc. In fact, the magnetic potential decreases by $4\pi\lambda i$ whenever the point considered describes a curve like that shown in the figure, where the current i is supposed to flow vertically upwards from the paper. This process of going once round the current in a right-handed screw direction we shall describe as *interlacing the current positively*, so that the work done by the current on a unit magnetic pole in one positive interlace is $4\pi\lambda i$. When the pole describes a curve which does not interlace the current, θ resumes its original value without increase of 2π, and no work is done.

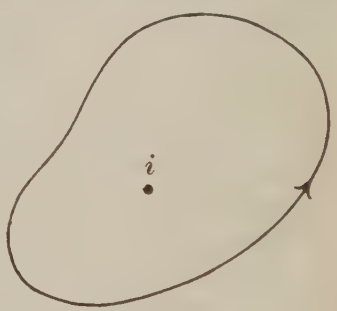


Fig. 101

We can obtain another representation of the magnetic potential

in terms of the area which the current appears to subtend when seen from P. Although we speak of an infinite straight current, the circuit must in reality be closed, and we can imagine the ends of the long straight wire to be joined to the battery by wires which are always at a great distance away. For instance, if the current flows along the axis of z in Fig. 102, the circuit may be

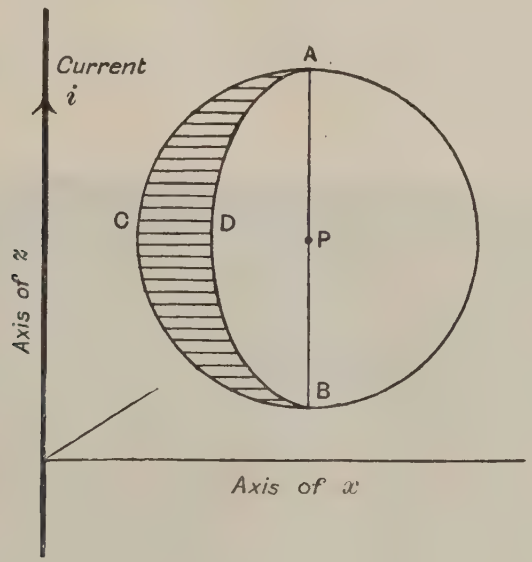


Fig. 102

supposed to be completed by an infinitely distant line described in the negative part of the xz plane. That is, the circuit may be regarded as a plane one enclosing the region given by $y = 0$, $x < 0$. The distant return wires will have no magnetic effect. Now describe a sphere of radius 1 cm. with the point P as centre, and let APB be a diameter parallel to the axis of z. Let ACB be a plane parallel to the plane $y = 0$, and ADB a plane through the axis of z. In this case an observer at P would regard the shaded area $ACBD$ as the projection of the circuit, supposed completed as above, on the unit sphere. Such an area of projection is called the *solid angle* subtended at the point P. If now P is the point (r, θ, z), the angle $CAD = \theta$, and therefore the

shaded area has the value $\phi = 2\theta$. Thus the magnetic potential at P is

$$\Omega = -\lambda i \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where ϕ is the solid angle subtended by the circuit at P.

74. Circuits of any form. The magnetic effects of currents in wires of general shape are very complicated from the quantitative point of view. The general effect can however be illustrated by means of iron filings, the figure showing the lines of magnetic force due to a circular current at points in a plane through its axis.

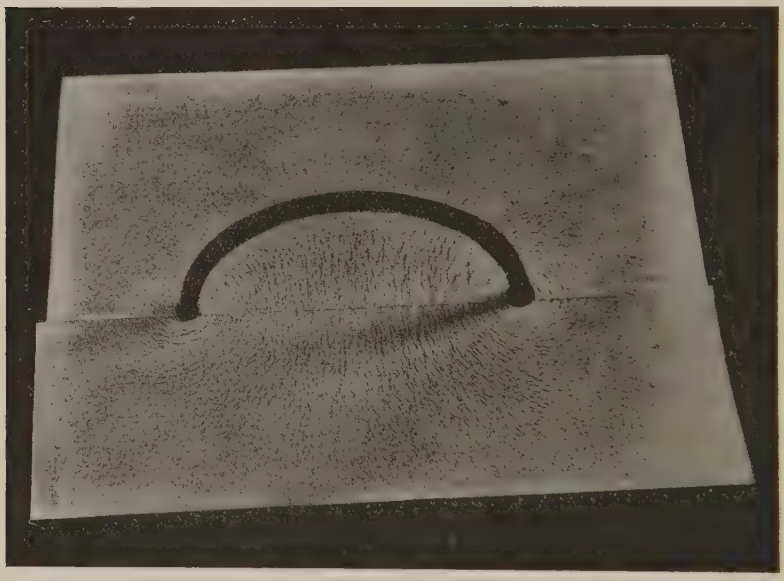


Fig. 103

The most remarkable phenomenon is the tendency of lines of force to interlace the circuit, which acts, so to speak, like a girdle binding the tubes of force together in a sheaf. This may readily be explained by the remark that although the current is no longer straight, yet the tendency to twist the lines of force round the wire persists in the bent state. It is easy to show, by means of a

small compass-needle, that the lines of force pass through the circuit in the direction shown in Fig. 104, that is, they interlace

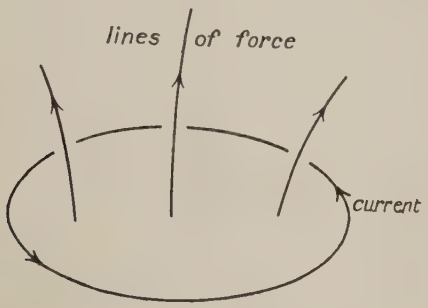


Fig. 104

the circuit positively in the sense in which the term is used in the last article.

The great difficulty in constructing a theory of electromagnetism from experimental data lies in the fact that we must deal with the whole circuit, as there is no way of examining the action of any part separately. We shall therefore proceed at once to a generalised hypothesis about the magnetic potential, which will of course have to be tested by experiment subsequently. The current of any shape is so far analogous to the rectilinear current in that lines of force interlace it, so that it is possible to have a closed line interlacing the current and such that at every point of it the magnetic face tends to urge a unit pole round the line. If the line does not interlace the current a unit pole taken round the line will be assisted in part of its path and resisted in the other. It is thus possible that the law already found for the straight current holds for any current, namely that work $4\pi\lambda i$ is done on a unit magnetic pole in one positive interlace of the current, while no work is done in a closed path not interlacing it. We shall go further and suppose that the magnetic potential at a point P due to *any* circuit is

$$\Omega = -\lambda i\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where ϕ is the solid angle subtended by the circuit at P. The

solid angle ϕ is shown in Fig. 105, which will be seen to correspond exactly with Fig. 102 drawn for the rectilinear current.

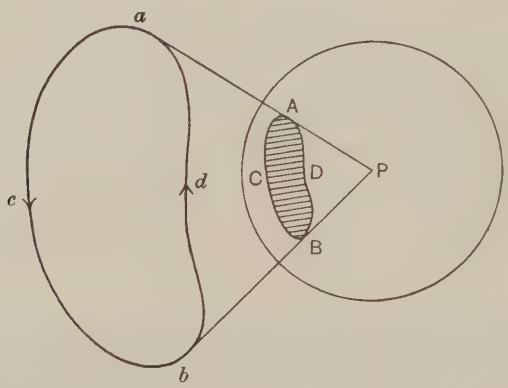


Fig. 105

The direction of the current with reference to P should be noticed, as it is regulated by the right-handed screw law. An observer at P reckons a solid angle as positive when the current circulates round in a right-handed screw direction with reference to the direction in which he looks. If the positive direction had been taken otherwise we should have had to write $+\lambda i\phi$ for the magnetic potential.

When the point P is at infinity, $\phi = 0$ since the circuit, if finite, subtends no angle at P. Thus no arbitrary constant is to be added to Ω, if we suppose one of its values to vanish at infinity. As P moves in towards the current from its position in Fig. 105, the solid angle increases. The reader can convince himself that when P describes a closed curve interlacing the current positively, the shaded area can increase continuously until it has covered the whole unit sphere and its original area as well. Thus the magnetic potential has decreased by $4\pi\lambda i$, as we should expect. If P describes a curve which does not interlace the current, the increase of the shaded area is followed by a decrease which leaves it at its original value.

If the current-carrying wire consists of n turns close together, the magnetic potential is $-n\lambda i\phi$. The theory also explains the absence of magnetic effect of a wire wound back on itself, as the

complete circuit subtends practically no solid angle at any point, so that $\Omega = 0$ everywhere.

75. Experimental verifications of the theory.

(1) *The magnetic force due to a plane circuit of any form at a point in its plane is at right angles to the plane.* This is easily verified by laying the circuit vertically with its plane magnetically east and west and observing a compass-needle whose centre is in the plane. If P is any point in the plane of the circuit and inside it, the solid angle at P has the constant value 2π. Hence no work is done on a unit pole in moving it about inside the circuit, i.e. there is no tangential component of magnetic force. Similarly for points in the plane and outside the circuit.

(2) *Magnetic force at points on the axis of a circular wire carrying a current.*

If O is the centre and P a point on the axis distant x from O, the solid angle subtended by the circular wire at P is $2\pi\ (1 - \cos\theta)$,

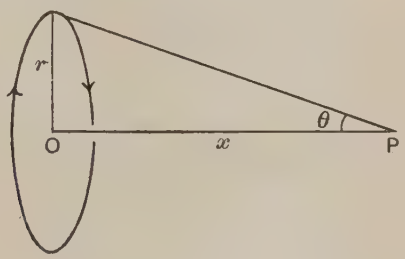


Fig. 106

where $\tan\theta = r/x$, r being the radius of the circle. Hence

$$\Omega = -2\pi\lambda i\,\{1 - x\,(r^2 + x^2)^{-\frac{1}{2}}\}.$$

The magnetic force is $-\partial\Omega/\partial x$, that is a force

$$F = 2\pi\lambda i\, r^2\,(r^2 + x^2)^{-\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

directed along PO.

To verify this set up a circular coil in the magnetic meridian and find the deflexions δ for various distances x of the centre of a compass-needle from the centre of the coil, with the compass-box magnetically east or west. The following numbers derived from an experiment show that F, or $\tan\delta$, varies as $(r^2 + x^2)^{-\frac{3}{2}}$.

$r = 12{\cdot}8$

x	δ	$(r^2 + x^2)^{\frac{3}{2}} \tan \delta$
0	53° 12′	2800
8·15	39° 24′	2870
11·5	29° 6′	2830
14·1	22° 15′	2820
17·1	16°	2800
20·1	11° 39′	2850
23·1	8° 48′	2850

Fig. 107 shows the lines of force at points in a plane through the axis. It may be compared with the filings diagram in Fig. 103.

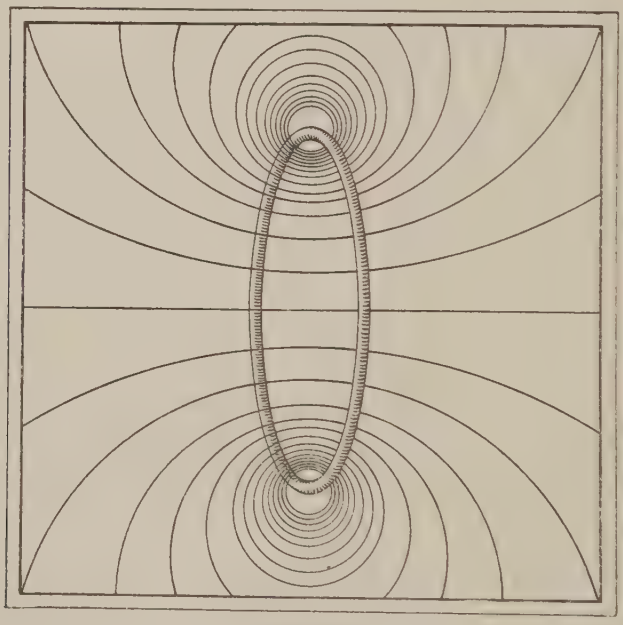


Fig. 107

Putting $x = 0$ in equation (5) we find that a circular coil of radius r produces at its centre a magnetic force $2\pi\lambda i/r$. Comparing this with the discussion of Art. 54, on which the definition of the electromagnetic unit of current was based, we see that

we have merely to put $\lambda = 1$ to make the formulae applicable to currents measured in true electromagnetic units. With this understanding we may recapitulate and say that the magnetic potential of a current at any point is

$$\Omega = -i\phi \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

while the work done by the field on a unit pole when the latter interlaces the circuit once positively is 4π times the current.

76. Magnetic force due to a current in a long cylindrical wire. Let O be the centre of a cross-section of the wire, $OA = a$ its radius, the current being supposed to flow vertically upwards in the figure. The magnetic force at a point P distant r from O will always be perpendicular to OP; for since the current is distributed symmetrically round O, similar filaments on opposite sides of OP will neutralise one another as regards the radial component of magnetic force. If H is the magnitude of the force at an *external* point P, $2\pi r H$ is the work done in taking a unit magnetic pole once round the current in a positive direction. This is equal to $4\pi i$ where i is the total current in the whole wire. Thus $H = 2i/r$, which is the same as if the current i was concentrated along the axis of the wire. Since Biot and Savart's law still holds for a finite circular wire, we need not trouble to choose thin wires in verifying the law experimentally.

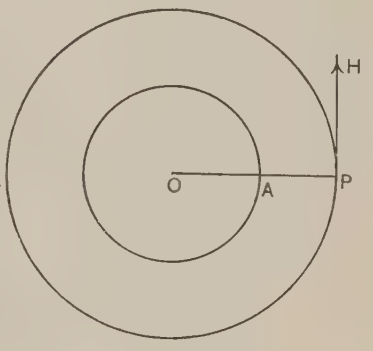


Fig. 108

The force at an *internal* point P can be found on the assumption that the current is distributed uniformly over the cross-section. In this case a circle of radius r interlaces a current of only $r^2 i/a^2$ instead of i: hence we have now $H = 2ri/a^2$, showing that the magnetic force is proportional to the distance of P from the central line of the wire. This theory shows that the difficulty as to the magnetic force due to a thin wire becoming apparently infinite as the wire is approached does not occur with the finite

wires used in practice. The magnetic force has its maximum value $2i/a$ at the surface of the wire, after which it decreases uniformly towards the centre.

77. Solenoids. The magnetic action of currents affords the most convenient means of producing a magnetic field of known strength. For example, let us have a ring of non-magnetic material, and wind insulated wire closely over it, covering the ring uniformly with n turns and leaving free ends by means of which a current i can be passed through it. Such an arrangement is called an endless solenoid: for theoretical purposes it may be replaced by n similar coils each carrying a current i and spaced uniformly round a ring-shaped region. Some of the coils are shown in Fig. 109.

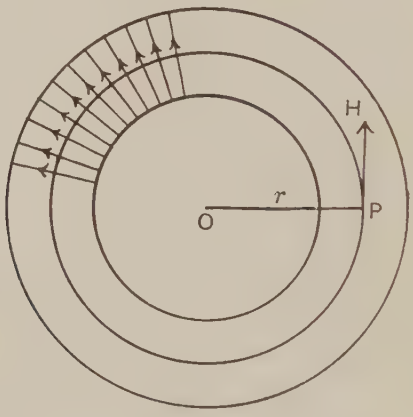


Fig. 109

To find the magnetic force at any point P, drop a perpendicular PO on the axis of symmetry of the ring. A little consideration will show that two coils symmetrically placed with respect to the plane through OP and the axis give a resultant magnetic force perpendicular to that plane: and since we may suppose all the coils arranged in pairs in this way, the magnetic force H at P is in this direction, as shown in the figure.

Imagine that a unit pole is carried round a circle, perpendicular to the axis, with O as centre and radius $OP = r$. If P is within the ring all the n turns are interlaced in one revolution, so that

the work done by the magnetic force is $4\pi ni$. Thus $2\pi r H = 4\pi ni$, that is

$$H = 2ni/r \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The lines of magnetic force inside the ring are circles round the axis, and the magnetic force varies inversely as the distance from the axis. If the cross-section of the ring is small compared with its radius, the magnetic force is therefore appreciably constant.

If P is in the space outside the ring no work is done in taking the unit pole round the circle, since no currents are interlaced. Hence the theoretical solenoid gives no magnetic force outside.

The magnetic force inside a long straight solenoid, which is practically realised by winding insulated wire closely round a long straight core, may be deduced from the preceding results. Consider a solenoid in the form of a ring of radius r, large compared with the cross-section. The number of turns of wire is n along the whole length $2\pi r$ of the circumference, i.e. the number of turns per unit length is given by $m = n/2\pi r$. If we make this substitution in the expression for H and then make r infinite, we find that the magnetic force inside an infinite straight solenoid is $4\pi mi$, where m is the number of turns per unit length. The force is constant and parallel to the solenoid, whatever the shape of the cross-section of the latter.

If we wish to measure current in amperes, the formula becomes

$$H = \frac{4\pi mi}{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Then H is in true magnetic units and m is the number of turns per centimetre.

The formulae will hold approximately for a finite solenoid provided that its length is great in comparison with the dimensions of the cross-section. We can moreover easily find an exact expression for the magnetic force at points on the axis of a finite solenoid of circular cross-section, which is of course the most usual type.

Let $2l$ be the length of the solenoid, a its radius and O its centre (Fig. 110). Taking the axis of the solenoid as axis of x, let us find the magnetic force at P, where $OP = x$. Consider the action of the group of windings between the abscissae ξ and $\xi + d\xi$.

Supposing the coils very close together, the number of turns of wire is $m d\xi$. Now from Art. 75 each turn produces at P a magnetic

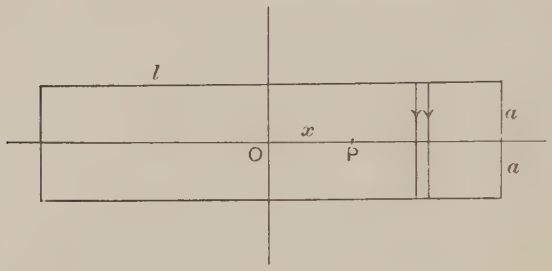
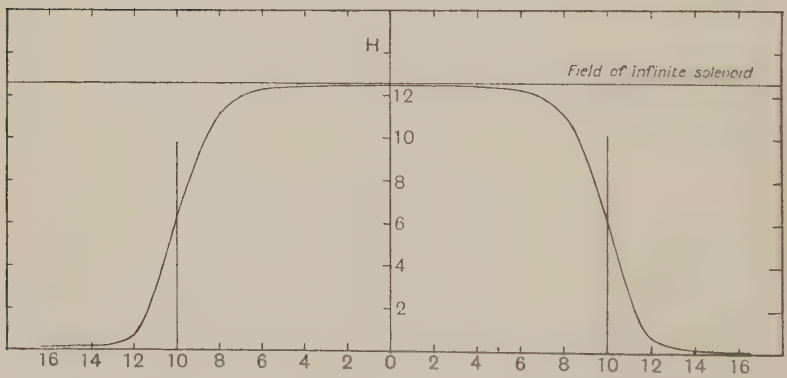


Fig. 110

force $2\pi i a^2 \{a^2 + (\xi - x)^2\}^{-\frac{3}{2}}$: hence the total magnetic force of the whole solenoid at P is

$$H = 2\pi m i a^2 \int_{-l}^{l} \{a^2 + (\xi - x)^2\}^{-\frac{3}{2}} d\xi$$

$$= 2\pi m i \left[\frac{l + x}{\{a^2 + (l + x)^2\}^{\frac{1}{2}}} + \frac{l - x}{\{a^2 + (l - x)^2\}^{\frac{1}{2}}} \right].$$



Length 20 cm., radius 1 cm., 10 turns per cm., current 1 ampere.

Fig. 111

Fig. 111 shows the variation of magnetic force along the axis of a solenoid of ordinary dimensions, calculated from the last formula. Thus the magnetic field inside a solenoid whose length is ten times its diameter remains constant to about 10 % over four-fifths of the length.

78. Action of magnets on currents. The mechanical force exerted by currents on magnets requires corresponding reactions of the magnets on the wires carrying the current. Some striking experiments can be arranged to show these effects.

Fig. 112 represents an arrangement for producing a continuous rotation of a wire round a magnet. A metal cylinder is partly filled with paraffin wax, by which a magnet is supported vertically inside it, and the cylinder is then filled with mercury until it nearly covers the magnet. A straight piece of copper wire is suspended by a short piece of thinner wire from a clamp, and dips in the mercury as shown in the figure. When a current passes through the apparatus in the direction shown, the straight wire moves continually round in the counter-clockwise direction, while on reversing the current it moves with equal speed in the opposite direction.

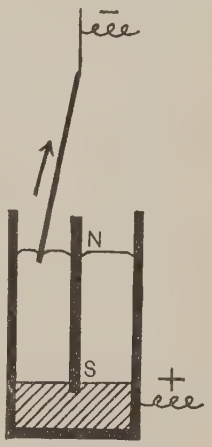


Fig. 112

A second experiment can be made with a light battery and a small coil of several turns. Connect the coil to the battery and fix the coil vertically, and then float both of them on a small board lying on water. Under the influence of the earth's magnetic field the plane of the coil will tend to set magnetically east and west. This arrangement is practically the same as that called *de la Rive's floating battery.*

A very interesting phenomenon is that known as the electromagnetic rotation of a conducting liquid. Take a shallow cylindrical metal vessel and pour a layer of mercury in it. Contact can be made by a central electrode and a wire attached to the vessel, or better a number of wires soldered to equidistant points on the circumference and joined together. By this means a radial current can be sent through the mercury layer. When a bar-magnet is held vertically over the central electrode and near the mercury, the latter will rotate quickly in the vicinity of the electrode, and more slowly in the outer parts. Its motion

can be easily followed by observing the movement of dust-particles on the surface. If the north pole of the magnet is nearer the liquid, and the current flows radially outwards, the rotation as seen from above is in the counter-clockwise direction.

Whatever the forces are which act on the parts of a wire carrying a current in a magnetic field, they can be shown to be at all points perpendicular to the wire. To prove this it is only necessary to make part of the circuit moveable in the direction of its length, for example a straight piece of copper rod sliding through two very slightly larger holes, or lying in two V-shaped grooves. Such a rod will not move in the direction of its length, no matter what magnetic field it is placed in.

79. Potential energy of a current and a system of magnetic poles. The potential energy of a unit pole at any point being $-i\phi$, that of a pole of strength m is $-mi\phi$, where ϕ is the solid angle under which the current is seen from the point. Now consider the *flux of magnetic force* through the circuit (Art. 5). The pole sends out lines of force equally in all directions, the total flux for a solid angle 4π being $4\pi m$ by Gauss' theorem. That is, if we write $N = m\phi$ for the flux of magnetic force through the current-carrying circuit due to the pole, the potential energy is $-iN$. In this expression the strength of the pole has disappeared, explicitly at any rate. For any number of poles the flux of force through the circuit is got by adding the fluxes due to the separate poles: hence we have the result that the potential energy of a system consisting of a current and any distribution of poles is $-iN$, where N is the flux of magnetic force through the current due to the poles, N being reckoned as positive when lines of force thread the current in the positive direction (Art. 74).

This theorem enables us to find the forces exerted by a magnetic field on a current. This action will of course be statically equivalent to the reversed resultant of the forces exerted by the current on the poles in the field. Further, without mentioning poles at all, we may speak of the potential energy of a current in a magnetic field, and find the forces experienced by such currents.

The motion of a rigid circuit, such as the floating battery, is covered by the rule that the current tries to embrace as many lines of force as possible. For since it moves so as to make its potential energy a minimum, its motion must be such as to make N a maximum.

As an example of the formula we may find the couple on a plane circuit carrying a current in a uniform magnetic field. Suppose that the circuit consists of a single turn of area A, and that it is moveable about a vertical axis and carries a current i. Let H be the uniform magnetic field, supposed in a horizontal

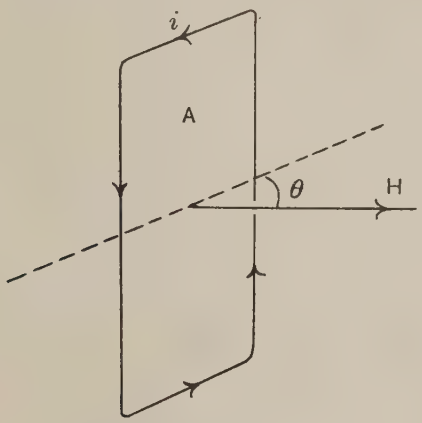


Fig. 113

direction, and let the direction of H make an angle θ with the plane of the circuit, as shown.

The flux of force through the circuit being $HA \sin\theta$, the potential energy of the current in the position θ is $W = -iHA \sin\theta$. Thus the couple tending to increase θ is

$$-\frac{dW}{d\theta} = iHA \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

If the current is left to itself it will take up the position $\theta = \frac{\pi}{2}$ normal to H. If, on the other hand, it only moves through a slight angle, so that θ is small, the couple is approximately iHA. The couple is n times as large when there are n windings close together, each of area A and carrying a current i.

The principle of the coil moving in a magnetic field is used in many electrical measuring instruments, as will be seen immediately.

80. The moving-coil galvanometer. Although the definition of current in electromagnetic units has been based on the tangent galvanometer, yet the disadvantages of that instrument are such as practically to prevent its regular use in laboratories. In order to get an absolute measurement of current it is necessary to measure the horizontal component of the earth's magnetic field in the place in which the galvanometer is to be used, as the presence of iron in modern buildings makes it quite impossible to take the value of H from the standard observation of a magnetic survey. Moreover the galvanometer constant is seriously affected by external magnetic fields of the order of ·01 absolute units, and such disturbances are unavoidable, now that strong electric currents are so constantly used. For example, a long vertical wire carrying a current of 100 amperes at a distance of 20 metres from a tangent galvanometer would produce a 5 per cent. error in its readings. For these reasons, and also to increase the sensitiveness of the instrument, galvanometers are now very largely constructed on the principle of the last article, namely by the movement of a coil carrying a current in a strong magnetic field. The so-called Deprez-d'Arsonval galvanometer (Fig. 114), which is really due to Lord Kelvin, is on this plan. A rectangular coil, of many turns of fine wire, is suspended from a fixed pillar by a fine metallic suspension, and hangs between the poles of a permanent steel magnet. There is very often a solid core of soft iron fixed inside the coil. The action of this will be better understood after we have considered the theory of induced magnetism (Art. 115), but for the present we may say that the core serves to

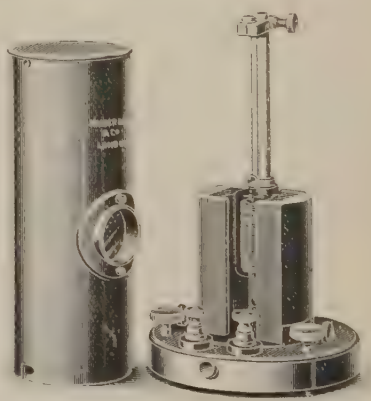


Fig. 114

concentrate the lines of force, so that they pass almost radially in the air-gap. Fig. 115 gives a vertical view of the instrument and shows roughly how the lines of magnetic force pass.

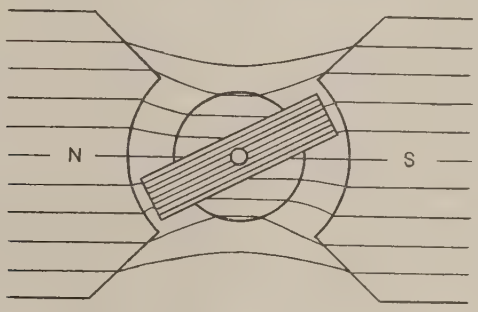


Fig. 115

The connexions to the coil are made through the suspension and through a fine wire leading to the lower end of the coil: the suspension is usually a fine strip of phosphor-bronze chosen so as to be as true as possible in its elastic properties. The considerable strength of field near the coil serves to increase the sensitiveness, and also renders the instrument practically independent of external magnetic fields. The coil is set so that its plane is approximately parallel to the magnetic field when no current is flowing. An ordinary galvanometer of this type will measure currents of the order of one-millionth of an ampere, while very sensitive instruments will give a deflexion for about 10^{-12} amperes.

It is hardly necessary to say that the couple on the coil of an actual galvanometer cannot, strictly speaking, be calculated by the formula of Art. 79, as the conditions do not coincide with the ideal state of affairs there contemplated. The effect of the radial field is to give a couple which is almost accurately proportional to i, say λi. (This would also hold for *small* deflexions in a uniform field, $\cos\theta$ being put equal to unity.) The restraining couple due to the twisting of the suspension is equal to $\mu\theta$, where μ is a constant, and the coil takes up a position for which these couples balance. Hence $\lambda i = \mu\theta$, or $i = k\theta$, where k is written for μ/λ. The quantity k is called the galvanometer constant, and it is usually understood to be expressed so that the formula gives

i in amperes. The determination of k when required is made by a separate experiment called *standardising*, explained in Art. 68.

The sensitiveness of the galvanometer can be increased by making the moving coil long and narrow, and by bringing the poles of the magnet close together. Galvanometers with a large number of turns are sensitive to current (and are therefore valuable for work on the discharge of small condensers), but on account of their greater resistance they are not necessarily most sensitive to potential (e.g. when used in conjunction with thermocouples).

For some purposes moving-coil galvanometers are made *dead-beat*, that is, so as to come to rest in their proper position as soon as possible. Galvanometers with the opposite property of swinging as long as possible without coming to rest are said to be *ballistic*.

In using the lamp and scale method with galvanometers we really measure the tangent of twice the angle turned through by the coil, provided the spot of light is originally in the centre of the scale. Now in ordinary use the scale is about 1 metre off the galvanometer, and the greatest deflexion is about 25 cm. from the centre of the scale. This gives $\tan 2\theta = \frac{1}{4}$, or $\theta =$ about 7°. The reduction to angle, that is from $\tan 2\theta$ to 2θ, amounts at most to about 2 per cent. It is not however desirable to make this reduction, as the uncertainty in the distribution of the lines of force near the moving coil produces in practice the same or even larger errors. Thus a moving-coil galvanometer when tested with a tangent galvanometer will sometimes show about 5 per cent. deviation from proportionality of scale-deflexion to current. It is therefore advisable, after having obtained a deflexion in any experiment, to standardise the galvanometer at the same deflexion.

81. Ammeters and voltmeters. The fundamental instrument required for measuring both current and potential in most cases is a *sensitive moving-coil galvanometer*, with a pointer moving over a fixed scale. This scale is graduated so as to read directly in convenient small units. A common type is the so-called millivoltmeter shown in the figure. The wire suspension is done away with, and the coil is mounted between agate bearings,

so that it can be carried about without injury. The torsion couple is produced by two light spiral springs above and below

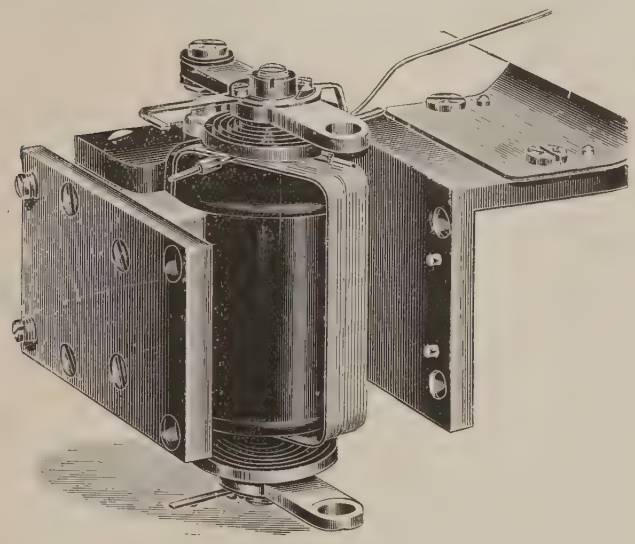


Fig. 116

the coil, which also serve as the wires leading the current in and out of the instrument. The millivoltmeter is so called because each division of the scale corresponds usually to a difference of potential of one-thousandth of a volt between the terminals.

If now such a sensitive galvanometer is shunted by a low resistance it becomes an ammeter, and if placed in series with a high resistance it becomes a voltmeter for measuring higher voltages. In the former case let R be the resistance of the shunt, G that of the galvanometer. Then an incoming current i divides in the ratio R to G, $Ri/(R+G)$ going through the galvanometer and the rest through the shunt. If R is small most of the current goes through the shunt, and the main current may be very high without there being much deflexion of the galvanometer. At the same time all the currents are cut down in the same ratio, and the same fixed-scale instrument may be used with various shunts in order to give different ranges of measurement.

If a galvanometer is placed in *series* with a resistance S,

only the fraction $G/(G+S)$ of the total drop of potential occurs between the terminals of the instrument. Thus a "millivoltmeter" of 1 ohm resistance and of 100 scale-divisions, reading up to $\frac{1}{10}$ volt, may be made to read up to 10 volts by placing a coil of 99 ohms in series with it.

Combined ammeters and voltmeters are on the market, and are very convenient. Fig. 117 shows the terminals and internal connexions of one of these. The reader will find it an interesting exercise to calculate the resistances R and T required, say, to make the instrument read as an ammeter up to 1 ampere (terminals $+$, $-A$) and as a voltmeter up to 10 volts (terminals $+$, $-V$), given that the full deflexion with the terminals $+$ and $-G$ corresponds to a current of 1/100 ampere and that both S and G are 1 ohm.

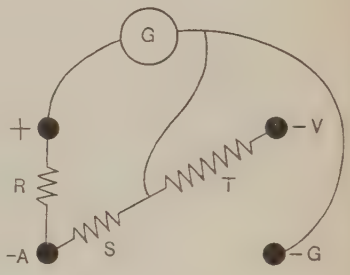


Fig. 117

The use and value of fixed-scale instruments dates from the discovery of methods of producing really permanent steel magnets, and they are now extensively used. There is practically no limit to the current that can be measured in this way. All that is necessary is to have a shunt of sufficiently low resistance that will carry the current without undue rise of temperature. Fig. 118 shows a simple form of shunt, consisting of a strip of manganin giving a large cooling surface.

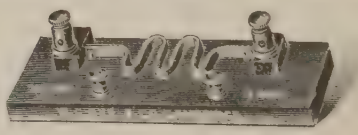


Fig. 118

82. The string galvanometer. For many purposes, for example for following the changes of current and voltage occurring with electrical machinery, we require a galvanometer which shall be at once very sensitive and very quick to take up its proper deflexion. This is attained in the *string galvanometer* invented by Einthoven. The principle of its action is shown in Fig. 119. CC is a fine quartz fibre, silvered to make it conduct, and carrying

the current to be measured. It lies between the poles *NS* of a strong electro-magnet, the poles being brought very near to the

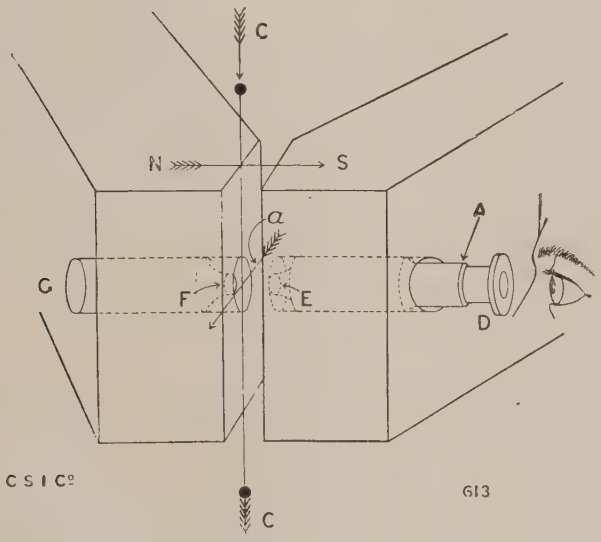


Fig. 119

fibre and perforated to allow the movement of the latter to be observed with a powerful microscope *ED*. When a current passes down the fibre from above to below a mechanical force is exerted on it in the direction of the arrow *a*, causing a deflexion of the fibre.

The instrument shares with the moving-coil galvanometer the advantage of being independent of external magnetic fields. The tension of the fibre makes the zero very steady, and also renders the readings practically independent of levelling. The quickness of the reading depends on the lightness of the fibre, which gives low inertia and also allows the air to damp it so much as to make it nearly dead-beat. As regards its sensitiveness, currents of 10^{-12} ampere can be detected with low tension on the fibre and high magnification (750). Increasing the tension of the fibre, of course, increases the quickness of response with a corresponding loss of sensitiveness.

83. The ballistic galvanometer. Whereas the dead-beat galvanometer is suitable for measuring constant currents, transient currents such as those produced by the discharge of condensers can be measured with the ballistic galvanometer. When a charge of e coulombs is sent suddenly through the galvanometer, the spot of light will begin to swing and move to and fro through a gradually decreasing arc with the original position as centre. The gradual decay of the swing, or *damping*, should be as small as possible. In the absence of damping the spot of light would swing through a distance τ on either side of the zero position : τ is called the *throw* of the galvanometer. The fundamental formula of the ballistic galvanometer is

$$e = \frac{T}{2\pi} k\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

where k is the galvanometer constant and T the time of swing of the coil.

To prove this, we have to complete the theory of Art. 80 by taking into account the angular inertia of the moving coil. Let I be its moment of inertia about a line coinciding with the suspending wire. Then the total couple tending to increase the deflexion θ is $\lambda i - \mu\theta$, and therefore the equation of motion of the coil in general is

$$I\frac{d^2\theta}{dt^2} = \lambda i - \mu\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

damping being neglected.

Equation (11) may be integrated with respect to t. Thus if the coil starts from rest in the position $\theta = 0$ we have

$$I\frac{d\theta}{dt} = \lambda \int_0^t i\,dt - \mu \int_0^t \theta\,dt.$$

The integral $\int_0^t i\,dt$ is the total charge e that has passed through the galvanometer from time $t = 0$. If this charge has passed through quickly the coil has had no time to move, so that $\int_0^t \theta\,dt$ is negligible. In this case the equation shows that a sudden angular velocity ω of the needle is produced, given by

$$I\omega = \lambda e \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

After the charge has once passed, the coil swings without any current flowing in it. Hence putting $i = 0$ in equation (11), the motion of the coil is regulated by the equation

$$I\frac{d^2\theta}{dt^2} + \mu\theta = 0,$$

with the condition that $\theta = 0$ and $d\theta/dt = \omega$ when $t = 0$. The equation of energy is

$$I\left(\frac{d\theta}{dt}\right)^2 + \mu\theta^2 = \text{constant}.$$

Applying this to the instants (1) when $t = 0$, (2) at the first stopping place, for which $\theta = \tau$ and $d\theta/dt = 0$, we have

$$I\omega^2 = \mu\tau^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

Further, the time of the simple harmonic oscillation of the coil is given by

$$T = 2\pi\,(I/\mu)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (14).$$

Combining the equations (12), (13) and (14) we have

$$e = \frac{I\tau}{\lambda}\left(\frac{\mu}{I}\right)^{\frac{1}{2}} = \frac{\mu\tau}{\lambda}\left(\frac{I}{\mu}\right)^{\frac{1}{2}} = \frac{T}{2\pi}\,k\tau \ldots\ldots\ldots (15),$$

since $k = \mu/\lambda$. This equation is very important as it gives e when the galvanometer constant and time of swing are known.

To allow for small damping assume a frictional couple proportional to the angular velocity. Then the equation of motion of the coil swinging freely is

$$\frac{d^2\theta}{dt^2} + 2\kappa\frac{d\theta}{dt} + n^2\theta = 0,$$

where κ is a small constant specifying the amount of damping. If we neglect κ^2 the solution is

$$\theta = e^{-\kappa t}\,(A\cos nt + B\sin nt).$$

In the case considered $\theta = 0$ when $t = 0$, so that $A = 0$. Since the damping in a few swings is small, the early stages of the motion are represented by $\theta = B\,(1 - \kappa t)\sin nt$; and if the damping is zero we should have a swing of amplitude B, so that B is the throw corrected for damping. In the actual case let θ_0 be the

first throw, θ_1 the next *on the same side*. These correspond to times $nt = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$ approximately, so that

$$\theta_0 = B\left(1 - \frac{\kappa\pi}{2n}\right)$$

and

$$\theta_1 = B\left(1 - \frac{3\kappa\pi}{2n}\right).$$

Thus

$$B - \theta_0 = \tfrac{1}{4}(\theta_0 - \theta_1),$$

i.e. the correction to true throw is got by adding one-quarter of the difference between the actual throws. Hence the true throw τ in equation (10) is given by

$$\tau = \theta_0 + \tfrac{1}{4}(\theta_0 - \theta_1) \quad \ldots\ldots\ldots\ldots\ldots (16).$$

Thus in a laboratory experiment in which the zero reading on the scale is (say) 24·2, if the galvanometer swings to the right and first comes to rest in the position 48·3, the next reading on the right being 47·4, we should have $\tau = 48{\cdot}5 - 24{\cdot}2 = 24{\cdot}3$. If $k = 10^{-6}$ and $T = 4$ seconds, the total charge passing through the galvanometer is $1{\cdot}55 \times 10^{-5}$ coulombs.

84. Measurement of a large capacity. The ballistic throw produced by the discharge is observable with a sensitive galvanometer if the condenser is of the order of a microfarad. We require a two-way key ABC such that B can be joined to A

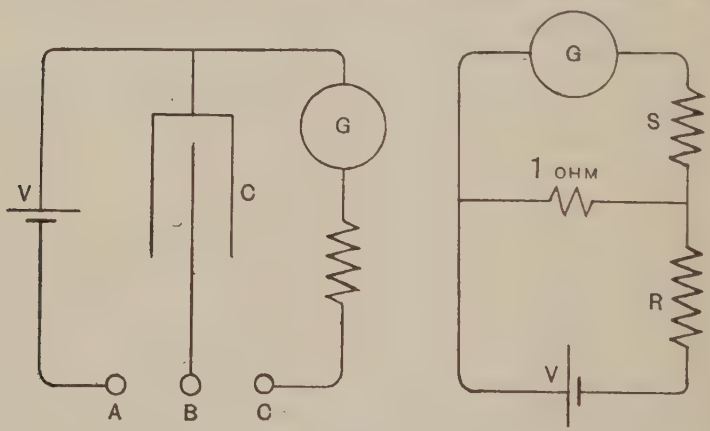


Fig. 120

or C at will. Start with AB joined, thus charging the condenser to potential V (Fig. 120). When the key is pressed down so as to

disconnect AB and join BC, a charge of CV coulombs is sent suddenly through the galvanometer, C being supposed expressed in farads and V in volts. Let τ be the ballistic throw, corrected for damping. Then

$$CV = \frac{T}{2\pi} k\tau.$$

The time of swing T can be observed with a stop-watch. It is safest to have a resistance of several thousand ohms in the galvanometer circuit: this does not affect the ballistic throw since the charge sent through is the same in all cases.

With the connexions as in the second part of the figure, shunt through a small resistance X (about 1 ohm), obtaining the same steady deflexion τ as the original throw, by adjusting the resistance R. If $S + G$ is considerably greater than unity (at least 100 ohms), we have very approximately

$$\frac{VX}{R} = (S + G)\, k\tau.$$

Dividing these equations, V and $k\tau$ go out, leaving

$$C = \frac{XT}{2\pi R\,(S + G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

A knowledge of the absolute magnitude of the potential V is unnecessary in this case.

85. Magnetic force of a plane current at large distances. Let θ be the angle between the normal to the plane and the radius vector OP drawn from a point O near the circuit to a distant point P. If A is the area of the circuit and r the distance P, the solid angle subtended at P is $\phi = -A \cos \theta / r^2$. Hence the magnetic potential at P is $\Omega = iA \cos \theta / r^2$, which is the same as that of a magnet of moment iA whose axis is perpendicular to the plane of the circuit. The relation

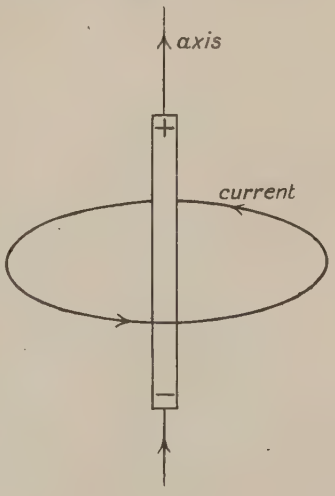


Fig. 121

between the two is easily seen to be that of the right-handed screw law, as shown in Fig. 121.

Consider now a solenoid of length l wound on a core of cross-section A, with n turns in all. The positive and negative poles of the magnets equivalent to the separate turns destroy each other in the interior just like the elementary magnets of a uniformly magnetised rod, leaving poles only at the ends. Let $\pm m$ be the pole-strengths when a current i is flowing. The total magnetic moment is equal to ml, where l is the length of the solenoid: it is also equal to the sum of the moments of the n separate turns, namely niA. Hence $m = niA/l$. Thus a solenoid acts on a distant point just like a magnet with poles at the ends of the solenoid, and this law only fails when the point comes within a distance of the solenoid comparable with the diameter of the core. This result is easily seen to be true for solenoids which are not straight. Further, two solenoids act on one another like bar-magnets.

Historically, the actions of currents on currents were the first to be considered in detail, the exact laws having been unravelled by Ampère very soon after Oersted's fundamental discovery. From the present point of view a current is supposed to produce a magnetic field in its vicinity, which acts on any second circuit which may be present. We proceed to obtain a general expression for the potential energy of the forces between two circuits carrying currents.

86. Mechanical forces between currents. Mutual inductance. Consider the case of two circuits of any form, carrying currents i, i' respectively, and let the flux of magnetic force through the second circuit due to unit current in the first be M. Then the flux through the second circuit in the actual case is $N = iM$. It follows from Art. 79 that the potential energy of the second circuit in the magnetic field of the first, that is the potential energy of the system, is $-ii'M$.

The symmetry of this result shows that M is also the flux of magnetic force through the first circuit due to a unit current in the second. M is called the *mutual inductance* of the two circuits: it evidently depends only on their form and relative position.

Let us calculate the force between a current i in a circular wire of centre O and radius a and a current i' in a parallel circular wire of small radius b whose centre P is on the axis of the first circle. Let $OP = x$, and let the currents circulate in the

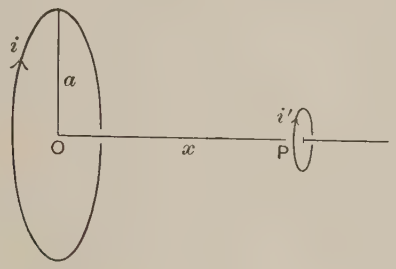


Fig. 122

directions shown. The magnetic force at P due to the large circuit has been found to be $2\pi i a^2 (a^2 + x^2)^{-\frac{3}{2}}$ directed along PO. The lines of force thread the second circuit positively, so that the potential energy is obtained by multiplying by $-i'.\pi b^2$. Thus the potential energy of the system is $W = -2\pi^2 i i' a^2 b^2 (a^2 + x^2)^{-\frac{3}{2}}$. The force tending to increase x is therefore

$$-\frac{dW}{dx} = -6\pi^2 i i' a^2 b^2 x (a^2 + x^2)^{-\frac{5}{2}}.$$

The circuits therefore attract one another when the currents are in the same direction, and repel one another when in opposite directions. It is easy to show that the force of attraction is greatest when $x = \frac{1}{2}a$.

The first really accurate observations on the mechanical forces between currents were those of W. Weber (1846). Two circular coils of many turns of wire were placed with their planes at right angles to one another, and one of the coils was supported by a bifilar suspension. By observing the small deflexions caused by a current passing through both coils in series, Weber was able to verify that the couple on the moveable coil was proportional to the square of the current. In a second experiment Weber placed the fixed coil in various positions with respect to the other, and showed that the observed couples agreed accurately with those to be expected on theoretical grounds.

87. Self-inductance of a circuit. By analogy with the mutual inductance of two circuits we may define the *self-inductance* L of a single wire C as the flux of magnetic force through it due to a unit current in C itself. Care is however necessary in defining and calculating self-inductances, because if C is taken as a geometrical line L is really infinite, in contrast with mutual inductances where the circuits can safely be taken as lines. Actual wires, however, are necessarily of finite cross-section, and L is defined as a mean magnetic flux as follows.

If dS is an element of cross-section of the wire, the electricity passing through dS traces out a tubular region of space which we may call a current-filament. Now when unit current is flowing in the whole wire, imagine the wire cut up into a large number n of filaments carrying equal currents. Then L is the mean of the magnetic fluxes through the filaments, i.e. if N is the flux through one of the filaments due to the current in the whole wire,

$$L = \frac{\Sigma N}{n},$$

where the summation extends over all the filaments. Writing $di = 1/n$ for the current in a filament, this becomes

$$L = \int N\, di \text{ where } \int di = 1.$$

It is not assumed in this definition that the current is uniformly distributed over the cross-section.

We can find an expression for the potential energy of the mutual electromagnetic actions between the various current-filaments of a wire in terms of L. Suppose in the first place that unit current is flowing in the wire. Then the potential energy of the filament di is $-N\,di$, and on integration the potential energy would appear to be $-L$. But in this way the force between each pair of filaments is counted in twice, so that the potential energy is only $-\frac{1}{2}L$. When a current i is flowing the force between each pair of filaments is i^2 times as great as before, and the potential energy in general is therefore $-\frac{1}{2}Li^2$. This quantity is essentially negative, as we should expect since the major part of the action between two filaments is the force of attraction between neighbouring portions.

As an example of self-inductance we may consider the case of a solenoid whose length l is great in comparison with its diameter. If A is the area of cross-section of the core and n the number of turns, the magnetic force inside the solenoid for unit current is $4\pi n/l$ except in the immediate neighbourhood of a winding. Here slight disturbances of the field occur, but the field always remains of the same order of magnitude. For if the wires are closely packed the radius of the cross-section of each is $l/2n$, and the magnetic force just outside one of the wires due to the unit current in the wire itself is $2 \div (l/2n) = 4n/l$. Now the disturbance of the field only occurs in a very narrow belt near the windings, so that N can safely be calculated on the assumption of an absolutely uniform interior field. Hence the flux through a single turn is $4\pi nA/l$, and the total flux $4\pi n^2 A/l$, since the effect of the ends is negligible. Thus

$$L = 4\pi n^2 A/l \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

absolute electromagnetic units. For the solenoid of Fig. 111 this formula gives $L = 79000$ E.M.U.

The solenoid is easier to treat than the simple circuit or loop of wire, because in the latter case the magnetic force near the wire is generally much greater than that elsewhere. We shall resume the calculation of self-inductances in Ch. VIII, when the reader will appreciate their importance better.

88. Electrodynamometers. Current balances. An electrodynamometer is a convenient instrument for measuring the forces between two circuits carrying currents. A simple form consists of a fixed coil placed vertically, and within it a moveable coil suspended like the coil of a moving-coil galvanometer from a torsion fibre. Initially the two coils are at right angles. If currents flow in the coils there is a couple on the moveable coil due to the magnetic field of the fixed coil, and the instrument shows a deflexion. By turning the torsion head round we can however bring the moveable coil back to its original position. The couple required to do this can be found, since it is proportional to the angle which the torsion head has turned through: it is also equal to the couple exerted by the fixed coil on the moveable coil in its original position, and this is proportional to the product

$i_1 i_2$ of the currents in the fixed and moveable coils. If the two coils are joined in series and the same current i passes through them, the couple is proportional to i^2. Electrodynamometers can therefore be used to measure a current irrespective of external magnetic fields, and are then known as current-balances.

Lord Kelvin designed an accurate form of current-balance known by his name, the principle being shown in Fig. 123. A, B, C, D represent four fixed horizontal coils wound so that the current circulates in them in the directions shown. The moveable coils X, Y are joined in series with the fixed coils, but

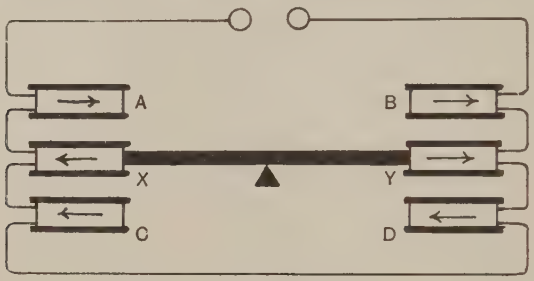


Fig. 123

arranged so that the current in Y is in the opposite direction to that in X. A horizontal beam connects X to Y and is supported somewhat after the manner of the beam of a balance. It has a graduated scale in front on which a rider of known weight can slide. When a current flows through the balance, X is repelled by A and attracted by C, while Y is attracted by B and repelled by D, all four forces combining to give a couple in the counter-clockwise direction in the figure, which can be balanced by known weights.

Fig. 124 shows an actual instrument, though the central coils are not visible. When no current is flowing a known weight is placed in the triangular trough on the right, and the rider, which is of the same weight, is placed at zero on the left of the scale. An auxiliary lever is then adjusted till the beam is in its correct position, as shown by the marks on the projections at its ends. On applying the current the rider is moved to the right till the beam is again in its normal position, and the current is then

read off on a scale opposite the pointer. Connexion with the moveable coils is made through the suspension of the beam, so that the latter may swing as freely as possible.

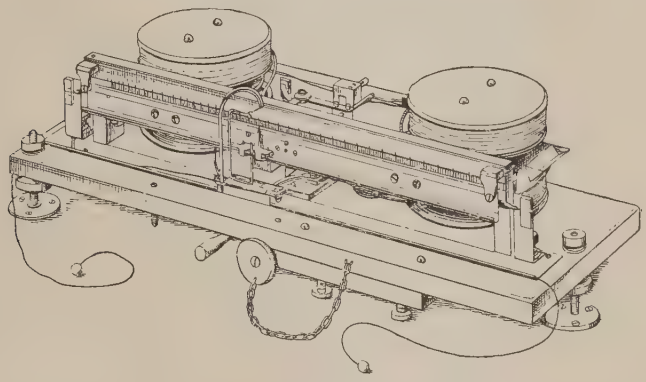


Fig. 124

The greatest sensitiveness and freedom from error is obtained by arranging the distances of the coils so that the couple exerted shall be a maximum. If the coils X, Y were small in comparison with the fixed coils, and if the latter consisted of a single turn each, we have seen in Art. 86 that the most favourable position is for X to be at a distance from A and C equal to half the radius of either coil. In the present case, however, the best distance is rather different.

If the dimensions of an electrodynamometer are accurately measured, the mutual forces can be calculated theoretically for given currents, so that the electrodynamometer will give a measurement of current in absolute electromagnetic units. One of the most perfect types of instrument for this purpose is that designed by W. E. Ayrton and J. V. Jones, the principle of which is illustrated by Fig. 125.

Two coils A, B, wound on insulating bobbins, are suspended from the beam of a sensitive balance. C, D represent hollow bobbins also wound with wire, which can be raised so as to enclose the coils A, B. The windings are arranged so that A repels C and B attracts D when a current passes in series through all the coils. The couple is then exactly compensated by adding

weights on a scale pan hanging from the left of the beam. This current-balance was used in finding the electro-chemical equivalent

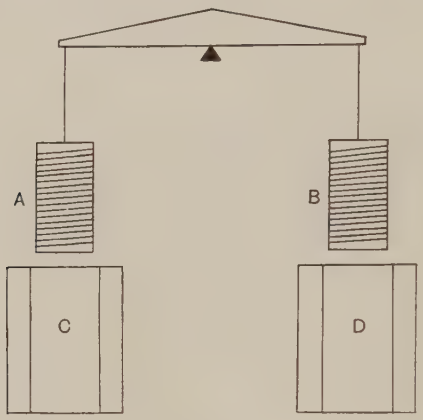


Fig. 125

of silver, and was considered to be accurate to 1 part in 50,000. In fact, the greatest source of error was thought to be the uncertainty in the value of g.

89. Action of a magnetic field on an element of current. We shall now show that the force exerted by a magnetic field on a closed circuit can be analysed into forces acting on its elements of length according to a definite law; and although we cannot really deal with the separate parts of a closed circuit, this representation of the forces will sometimes be found useful in calculations. The force that may be considered as acting on an element ds is a force of magnitude $ids H \sin\theta$ perpendicular to ds and H, where θ is the angle between ds and H: that is, the vector product of ids and H (Art. 2). In other words, we have to prove that a possible force on the element has components

$$i\,(H_z dy - H_y dz,\ H_x dz - H_z dx,\ H_y dx - H_x dy) \ldots (19).$$

If the circuit is given a small displacement (ξ, η, ζ) as a whole the above force would do work

$$i \begin{vmatrix} \xi, & \eta, & \zeta \\ dx, & dy, & dz \\ H_x, & H_y, & H_z \end{vmatrix}.$$

The work is therefore represented by i times the volume of the parallelepiped in the figure. Now the original element of arc

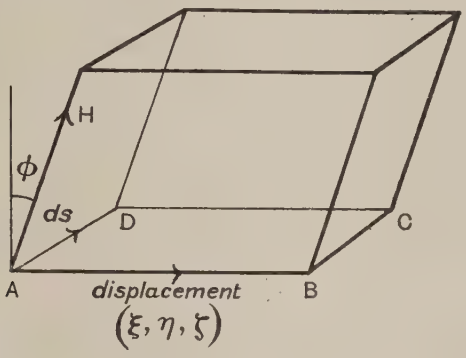


Fig. 126

AD is moved to BC by the displacement: hence the flux of magnetic force through the circuit has increased (as far as ds is concerned) by

$$\text{area } ABCD \times H \cos\phi,$$

the latter being the normal component of magnetic force for the new area added to the circuit. This latter expression is equal to the volume of the parallelepiped. Hence the decrease of the potential energy by the displacement (as far as ds is concerned) is i times the parallelepiped. The displacement (ξ, η, ζ) need not be the same for all the elements of current, but the preceding theory applies when the current is moved about in any way as a rigid body. In this case the mutual actions of the various parts of the current do no work on the whole, and the work done by the assumed system of elementary forces is equal to the decrease of potential energy. In other words, the assumed forces are equivalent over the whole circuit to the actual forces on the circuit, as regards the components of both force and couple.

This rule enables us to give an easy explanation of the actions of magnets on currents described in Art. 78. Thus in the case of electromagnetic rotation of liquids, a current radially outwards calls forth at every point a mechanical force tending to turn the carriers of electricity round the axis.

90. Equations of the electromagnetic field. If we have currents of a quite general kind in space, the law of work, namely that the work done in taking a unit magnetic pole round any circuit is 4π times the current enclosed, gives us an important set of equations connecting current-density with magnetic field. At the point (x, y, z) let H be the magnetic force and j the current-density. The work done in taking a unit pole round a circuit C is $\int_C H_x dx + H_y dy + H_z dz$. Also from Art. 72 the current flowing through the circuit is $\int_S (lj_x + mj_y + nj_z)\, dS$, integrated over any surface S with C as rim. Hence

$$4\pi \int_S (lj_x + mj_y + nj_z)\, dS = \int_C H_x dx + H_y dy + H_z dz$$
$$= \int_S \left\{ l\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + m\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + n\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \right\} dS$$

by Stokes' theorem. Since this equation holds for all possible surfaces S in the field, we must have

$$\left.\begin{aligned} 4\pi j_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ 4\pi j_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ 4\pi j_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (20).$$

It follows that $\partial j_x/\partial x + \partial j_y/\partial y + \partial j_z/\partial z = 0$, as we should expect, since the current-density in steady flow should be solenoidal.

Since work is now done in general whenever a pole is taken round a closed circuit, there will not be a magnetic potential. On the other hand the lines of magnetic force due to a current-filament are re-entrant curves, and it is natural to suppose that the magnetic force is solenoidal. Let us try to solve the last equations so as to give the magnetic force H in terms of j, with the condition

$$\partial H_x/\partial x + \partial H_y/\partial y + \partial H_z/\partial z = 0 \ldots\ldots\ldots (21).$$

Suppose that H is the curl of a vector A, that is,

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad H_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \ldots (22).$$

The equation (21) is then satisfied identically. Hence we have

$$4\pi\left(\frac{\partial j_z}{\partial y}-\frac{\partial j_y}{\partial z}\right)=\frac{\partial^2 H_y}{\partial x\partial y}-\frac{\partial^2 H_x}{\partial y^2}-\frac{\partial^2 H_x}{\partial z^2}+\frac{\partial^2 H_z}{\partial x\partial z}$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)-\frac{\partial^2 H_x}{\partial x^2}-\frac{\partial^2 H_x}{\partial y^2}-\frac{\partial^2 H_x}{\partial z^2},$$

or

$$\frac{\partial^2 H_x}{\partial x^2}+\frac{\partial^2 H_x}{\partial y^2}+\frac{\partial^2 H_x}{\partial z^2}=-4\pi\left(\frac{\partial j_z}{\partial y}-\frac{\partial j_y}{\partial z}\right).$$

Thus all the equations are satisfied if we put

$$\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}=-4\pi j_x,$$

and two similar equations. This equation for A_x is exactly of the same form as Poisson's equation for V in terms of ρ. Now in the latter case we know that one solution, at any rate, is given by taking for the potential at a point P the value $V=\int\frac{\rho d\tau}{r}$ integrated over the whole of space, where ρ is the density at a volume-element $d\tau$ distant r from P. Hence all the conditions are satisfied if we take for the components of A at the point $P(x, y, z)$ the quantities

$$A_x=\int\frac{j_x d\tau}{r},\quad A_y=\int\frac{j_y d\tau}{r},\quad A_z=\int\frac{j_z d\tau}{r}\quad\ldots\ldots(23),$$

where j is the current-density at the point $Q(\xi, \eta, \zeta)$ at which $d\tau$ is taken, and $r=PQ$. The vector A is called the *vector potential of magnetic force.*

Since the magnetic field is given by equation (22), it is thus completely determined. The solution thus obtained is not unique, since we can clearly superpose the magnetic field of any permanent magnetic distribution that may be present. The above solution, which gives $H=0$ when $j=0$ everywhere, may however fairly be described as giving the magnetic field *due to the currents*, and we shall neglect the subsidiary terms.

Thus the work law in its general form, with the aid of certain assumptions, determines the magnetic force of any distribution of currents in space. Applying the formulae to the linear circuit we shall obtain results equivalent to those already used.

91. Magnetic field of a linear current. Let $Q(\xi, \eta, \zeta)$ be a point on a closed circuit C, and let a thin wire, whose cross-section near Q is A, coincide with C and carry a current i. If ds is the element of arc of C, the distribution of current is equivalent to putting $j = 0$ except in the neighbourhood of C, where we write $j = \dfrac{i}{A}$ and $d\tau = Ads$, so that $ids = jd\tau$. Writing $id\xi$, $id\eta$, $id\zeta$ for $j_x d\tau$, $j_y d\tau$, $j_z d\tau$ respectively, the equations for the vector-potential at the point $P(x, y, z)$ reduce to

$$A_x = i\int \frac{d\xi}{r}, \quad A_y = i\int \frac{d\eta}{r}, \quad A_z = i\int \frac{d\zeta}{r},$$

the line integrals being taken round C.

Since $\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) = \dfrac{\xi - x}{r^3}$, etc., we have

$$\left.\begin{aligned} H_x &= i\int_C \frac{(\eta - y)\,d\zeta - (\zeta - z)\,d\eta}{r^3} \\ H_y &= i\int_C \frac{(\zeta - z)\,d\xi - (\xi - x)\,d\zeta}{r^3} \\ H_z &= i\int_C \frac{(\xi - x)\,d\eta - (\eta - y)\,d\xi}{r^3} \end{aligned}\right\} \quad \ldots\ldots\ldots(24).$$

The solid angle which the circuit subtends at P is given by

$$\phi = \int_S dS \cos\theta / r^2,$$

where S is any surface with C as its boundary and θ the angle between the radius vector from P and the normal to dS. Thus if (ξ, η, ζ) is any point on S,

$$\begin{aligned} \phi &= \int_S \left(l\frac{\xi - x}{r^3} + m\frac{\eta - y}{r^3} + n\frac{\zeta - z}{r^3}\right) dS \\ &= -\int_S \left\{l\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right) + m\frac{\partial}{\partial\eta}\left(\frac{1}{r}\right) + n\frac{\partial}{\partial\zeta}\left(\frac{1}{r}\right)\right\} dS. \end{aligned}$$

Since $\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) = -\dfrac{\partial}{\partial\xi}\left(\dfrac{1}{r}\right)$, we have

$$\frac{\partial\phi}{\partial x} = \int_S \left\{l\frac{\partial^2}{\partial\xi^2}\left(\frac{1}{r}\right) + m\frac{\partial^2}{\partial\xi\partial\eta}\left(\frac{1}{r}\right) + n\frac{\partial^2}{\partial\xi\partial\zeta}\left(\frac{1}{r}\right)\right\} dS.$$

Again writing H_x in the form

$$H_x = i\int_C \left\{0 \,.\, d\xi + \frac{\partial}{\partial\zeta}\left(\frac{1}{r}\right)d\eta - \frac{\partial}{\partial\eta}\left(\frac{1}{r}\right)d\zeta\right\}$$

and applying Stokes' theorem, we have

$$H_x = i\int_S \left\{-l\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)\frac{1}{r} + m\frac{\partial^2}{\partial\xi\partial\eta}\left(\frac{1}{r}\right) + n\frac{\partial^2}{\partial\xi\partial\zeta}\left(\frac{1}{r}\right)\right\} dS.$$

Hence $H_x = i\partial\phi/\partial x$, with similar expressions for H_y and H_z; that is, there is a magnetic potential $\Omega = -i\phi$, in agreement with the previous result.

The present investigation shows that the formula $\Omega = -i\phi$ is the only one consistent with the work law in its widest sense, and also shows the conditions under which a magnetic potential exists. In general, i.e. with currents distributed through space, the magnetic force is not derivable from a potential; but when the currents are located in limited tubular regions of space there exists a magnetic potential for points not in these regions. The peculiarity of this potential is that it is multiple-valued, permitting different work to be done on a unit pole when the latter is moved round a circuit which does or does not interlace the tubular regions.

92. Ampère's law of action of an element of current. We shall now show that the whole effect of a linear current is correctly accounted for by supposing that each element ds of the circuit produces a magnetic force at a point P of magnitude $ids\sin\theta/r^2$ perpendicular both to ds and to the radius vector r from P, θ being the angle between ds and r (Fig. 127). As regards the sign of the force, the element is supposed to tend to twist a positive pole round it in the right-handed screw manner. The rule stated above is equivalent to saying that the magnetic force at a point P due to the element ds is $\frac{i}{r^3}$ times the vector product of r and ds. The corresponding x-component of the elementary force is therefore

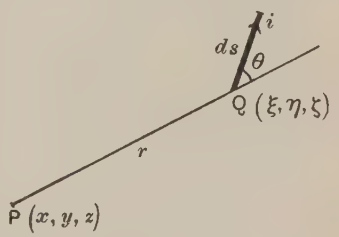


Fig. 127

$$\frac{i}{r^3}\{(\eta - y)\,d\zeta - (\zeta - z)\,d\eta\},$$

which is the element of the integral in equation (24) of the last article, so that the rule holds good.

Without attaching too much importance to the idea of element of current, we can use the law as a convenient one for certain calculations, particularly with straight or jointed circuits.

As an example let HK, KL be two long straight wires meeting at an angle $2A$, and let P be a point on the external bisector distant r from K. It is required to find the magnetic force at P, which is clearly perpendicular to the plane of the paper, when a current i passes along the wires. We have in the figure

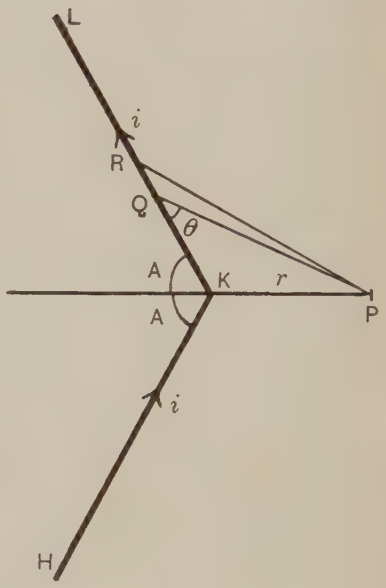


Fig. 128

$$\frac{KQ}{\sin(A-\theta)} = \frac{PQ}{\sin A} = \frac{r}{\sin\theta},$$

so that $PQ = \dfrac{r\sin A}{\sin\theta}$.

If QR is an element of the top half of the wire,

$$QR = d(KQ) = -\frac{r\sin A}{\sin^2\theta}\,d\theta.$$

Hence the magnetic force at P due to QR is

$$\frac{i\,.\,QR\sin\theta}{PQ^2} = -\frac{i\sin\theta\,d\theta}{r\sin A}.$$

Integrating from $\theta = A$ to $\theta = 0$ and doubling the result we have for the total magnetic force

$$\frac{2i(1-\cos A)}{r\sin A} = \frac{2i}{r}\tan\tfrac{1}{2}A.$$

Putting $A = \frac{1}{2}\pi$ gives the infinitely long straight circuit. Biot verified the formula in its more general form.

The accuracy of the electromagnetic laws, and of Ampère's elementary law, can be tested by setting up a circuit of known form and measuring the magnetic force at various points. We shall describe one final verification of this kind. It can be proved

that if we have a square circuit of side $2a$ and take a point distant x from the centre on a line through the centre parallel to one pair of sides, then the magnetic force is

$$\frac{2i}{a\,(a-x)}\{(a-x)^2+a^2\}^{\frac{1}{2}}+\frac{2i}{a\,(a+x)}\{(a+x)^2+a^2\}^{\frac{1}{2}}.$$

The following table gives the comparison between theory and experiment.

$a = 44{\cdot}6.$

x	Ratio of magnetic force to force at the centre	
	Obs.	Calc.
8·3	1·04	1·02
14·2	1·07	1·07
19·7	1·15	1·15
22·5	1·22	1·22
25·4	1·30	1·31
28·6	1·43	1·46
31·7	1·66	1·68

93. Neumann's formula for the mutual inductance of two circuits. When a unit current flows in a circuit C the magnetic force at the point P (x, y, z) is given by

$$H_x=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z},\quad H_y=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x},\quad H_z=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y},$$

where
$$A_x=\int_C\frac{d\xi}{r},\quad A_y=\int_C\frac{d\eta}{r},\quad A_z=\int_C\frac{d\zeta}{r}\quad\ldots\ldots(25).$$

The flux of magnetic force through a closed curve C' due to the unit current in C is $\int_{S'}(lH_x+mH_y+nH_z)\,dS$, where S' is any surface having C' as boundary. Hence if M is the mutual inductance of C and C',

$$M=\int_{S'}\left\{l\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+m\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+n\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\right\}dS$$

$$=\int_{C'}A_x dx+A_y dy+A_z dz \text{ by Stokes' theorem.}$$

Thus
$$M=\int_C\int_{C'}\frac{dx\,d\xi+dy\,d\eta+dz\,d\zeta}{r}\quad\ldots\ldots\ldots\ldots(26).$$

If ds is an element of C and ds' of C', the two making an angle ϵ with each other, this gives Neumann's formula, namely

$$M = \iint \frac{ds\, ds' \cos \epsilon}{r} \quad \ldots\ldots\ldots\ldots\ldots (27),$$

the integral extending round both circuits.

94. Mutual inductance of two parallel circles on the same axis. We shall consider the case in which the circles are in the same plane, and therefore concentric. Let a, b be the radii, and take an element ds at P on the circle of radius a and one of length $ds' = b\,d\psi$ on the circle of radius b. Then $\epsilon = \psi$ and $r^2 = a^2 + b^2 - 2ab\cos\psi$. The total contribution of ds to M is therefore

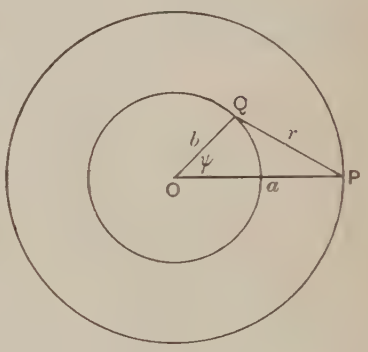


Fig. 129

$$ds \int_b \frac{\cos\epsilon\, ds'}{r} = ds \int_0^{2\pi} \frac{b\cos\psi\, d\psi}{(a^2 + b^2 - 2ab\cos\psi)^{\frac{1}{2}}}.$$

Evidently the mutual inductance is obtained by merely writing the whole circumference $2\pi a$ of the outer circle instead of ds, so that

$$M = 4\pi ab \int_0^{\pi} \frac{\cos\psi\, d\psi}{(a^2 + b^2 - 2ab\cos\psi)^{\frac{1}{2}}}.$$

Writing $\psi = \pi - 2\phi$ we have

$$M = -8\pi ab \int_0^{\frac{\pi}{2}} \frac{\cos 2\phi\, d\phi}{(a^2 + b^2 + 2ab\cos 2\phi)^{\frac{1}{2}}}.$$

This may be reduced to elliptic integrals by writing

$$k^2 = \frac{4ab}{(a+b)^2} \quad \ldots\ldots\ldots\ldots\ldots (28).$$

Thus we find after a little reduction that

$$M = \frac{8\pi ab}{a+b} \int_0^{\frac{\pi}{2}} \frac{(2\sin^2\phi - 1)\, d\phi}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}},$$

or

$$M = 4\pi\sqrt{ab}\left[\left(\frac{2}{k} - k\right)K - \frac{2}{k}E\right] \quad \ldots\ldots\ldots (29),$$

where K and E are complete elliptic integrals of the first and second kind to modulus k.

If the circles are no longer concentric but lie in parallel planes distant x apart, with the line joining their centres perpendicular to both planes, it is easy to show that equation (29) still holds, but (28) is replaced by

$$k^2 = \frac{4ab}{(a+b)^2 + x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The elliptic integrals have all been tabulated, and hence the quantity in square brackets is known in terms of k. Writing

$$k = \sin\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots (31),$$

$M/\sqrt{ab}$ can be tabulated for various values of γ, as was done by Maxwell. A table is given at the end of the chapter (p. 197).

In the case of concentric circles the calculation is simplified by observing that

$$\cos\gamma = \frac{a-b}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32),$$

which readily follows from (28) and (31). The following are two examples:

(1) $a = 16{\cdot}025$ cm., $b = 10{\cdot}550$ cm., $\gamma = 78^\circ\, 6'$, $M = 168{\cdot}61$ absolute E.M. units.

(2) $a = 16{\cdot}025$ cm., $b = 1{\cdot}750$ cm., $\gamma = 36^\circ\, 35'$, $M = 3{\cdot}79$ absolute E.M. units.

The exact calculation of a mutual inductance is of great importance in connexion with the determination of resistance in absolute measure (Art. 138). Similarly the calculation of the forces between two circuits is of importance for the absolute measurement of current. For the case of two circles we proceed as follows.

The potential energy of these forces is

$$W = 4\pi i i' \sqrt{ab}\left[\frac{2E}{k} - \left(\frac{2}{k} - k\right)K\right],$$

where

$$k^2 = \frac{4ab}{(a+b)^2 + x^2}.$$

Thus the attraction between the two circuits is

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial k}\frac{\partial k}{\partial x} = -\frac{k^3 x}{4ab}\frac{\partial W}{\partial k}.$$

The integrals $\partial E/\partial k$ and $\partial K/\partial k$ have to be calculated. We have

$$\frac{\partial E}{\partial k} = \int_0^{\frac{\pi}{2}} \frac{-k \sin^2\theta \, d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} = \frac{E-K}{k};$$

and also

$$\frac{\partial K}{\partial k} = k \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \, d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}.$$

Let $P \equiv \sin\theta\cos\theta\,(1-k^2\sin^2\theta)^{-\frac{1}{2}}$. Then

$$\frac{dP}{d\theta} = \frac{1-2\sin^2\theta+k^2\sin^4\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= -(1-k^2)\frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}} - \frac{1-k^2}{k^2}(1-k^2\sin^2\theta)^{-\frac{1}{2}} + \frac{1}{k^2}(1-k^2\sin^2\theta)^{\frac{1}{2}}.$$

On integration, since P vanishes both for $\theta = 0$ and for $\theta = \frac{1}{2}\pi$, we have

$$-\frac{1-k^2}{k}\frac{\partial K}{\partial k} + \frac{E-(1-k^2)\,K}{k^2} = 0, \quad \text{or} \quad \frac{\partial K}{\partial k} = \frac{E-(1-k^2)\,K}{k\,(1-k^2)}.$$

Making these substitutions we find for the pull between the circuits the expression

$$-\frac{\pi x k i i'}{(1-k^2)\sqrt{ab}}[(k^2-2)\,E + 2\,(1-k^2)\,K].$$

95. Action of a magnetic field on a moving charge. If we regard an electric current as a flow of electrons, and also regard Ampère's elementary law as actually true for the separate parts of a circuit, we have an extremely useful rule for the action of a magnetic field on a moving charge: namely that a charge e moving with velocity v in a magnetic field H experiences a force e/c times the vector product of v and H.

To prove this, let P, Q represent two sections of a wire distant ds from each other. Let n be the number of free electrons per cubic centimetre, A the cross-section of the wire and v the velocity of drift. The amount of electricity passing P per second is $neAv$ electrostatic units, so that the current i measured electromagnetically is equal to $neAv/c$. The total force

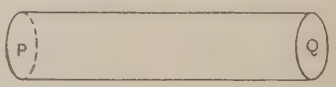


Fig. 130

exerted by the magnetic field H on the electrons between P and Q is $ids\,H \sin\theta = neAv\,ds\,H \sin\theta/c$, where θ is the angle between H and ds or v. The number of electrons concerned is $nA\,ds$. Hence the force on each is the vector product of ev/c and H; i.e. the components of mechanical force are

$$\frac{e}{c}(v_y H_z - v_z H_y,\quad v_z H_x - v_x H_z,\quad v_x H_y - v_y H_x),$$

or

$$\frac{e}{c}\begin{Vmatrix} v_x, & v_y, & v_z \\ H_x, & H_y, & H_z \end{Vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

Tables of Mutual Inductance (*see Art.* 94).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
5° 0′	·00166	8° 18′	·00753	11° 36′	·02068
6	·00175	24	·00780	42	·02124
12	·00185	30	·00809	48	·02179
18	·00196	36	·00839	54	·02234
24	·00207	42	·00871	12 0	·02291
30	·00219	48	·00900	6	·02350
36	·00231	54	·00929	12	·02409
42	·00244	9 0	·00961	18	·02469
48	·00256	6	·00994	24	·02531
54	·00270	12	·01028	30	·02594
6 0	·00284	18	·01063	36	·02657
6	00299	24	·01097	42	·02721
12	·00314	30	·01132	48	·02785
18	·00329	36	·01169	54	·02851
24	·00346	42	·01205	13 0	·02919
30	·00362	48	·01243	6	·02988
36	·00378	54	·01282	12	·03057
42	·00396	10 0	·01322	18	·03128
48	·00413	6	·01362	24	·03199
54	·00432	12	·01402	30	·03273
7 0	·00450	18	·01444	36	·03348
6	·00470	24	·01488	42	·03422
12	·00491	30	·01532	48	·03498
18	·00511	36	·01576	54	·03575
24	·00533	42	·01620	14 0	·03653
30	·00554	48	·01666	6	·03732
36	·00578	54	·01714	12	·03814
42	·00603	11 0	·01762	18	·03897
48	·00627	6	·01811	24	·03980
54	·00651	12	·01861	30	·04065
8 0	·00676	18	·01913	36	·04151
6	·00701	24	·01964	42	·04237
12	·00726	30	·02014	48	·04325

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
14° 54′	·04415	19° 54′	·106606	24° 54′	·212613
15 0	·04504	20 0	·108254	25 0	·215272
6	·04596	6	·109923	6	·217955
12	·04690	12	·111607	12	·220661
18	·04784	18	·113306	18	·223390
24	·04880	24	·115028	24	·226141
30	·04976	30	·116766	30	·228915
36	·05074	36	·118524	36	·231718
42	·05174	42	·120295	42	·234540
48	·05274	48	·122086	48	·237388
54	·05377	54	·123894	54	·240260
16 0	·05480	21 0	·125724	26 0	·243154
6	·05585	6	·127575	6	·246076
12	·05690	12	·129440	12	·249024
18	·05798	18	·131324	18	·251995
24	·05907	24	·133226	24	·254984
30	·06017	30	·135149	30	·258005
36	·06129	36	·137092	36	·261053
42	·06242	42	·139051	42	·264121
48	·06356	48	·141031	48	·267219
54	·06472	54	·143033	54	·270340
17 0	·06590	22 0	·145049	27 0	·273486
6	·06709	6	·147086	6	·276657
12	·06830	12	·149145	12	·279856
18	·06952	18	·151226	18	·283081
24	·07075	24	·153324	24	·286327
30	·07199	30	·155442	30	·289603
36	·07325	36	·157581	36	·292907
42	·07452	42	·159739	42	·296237
48	·07581	48	·161919	48	·299594
54	·07712	54	·164119	54	·302980
18 0	·07845	23 0	·166340	28 0	·306390
6	·07978	6	·168582	6	·309829
12	·08114	12	·170842	12	·313293
18	·08250	18	·173128	18	·316782
24	·08388	24	·175432	24	·320303
30	·08529	30	·177758	30	·323849
36	·08671	36	·180104	36	·327424
42	·08814	42	·182474	42	·331027
48	·08959	48	·184866	48	·334660
54	·09105	54	·187278	54	·338323
19 0	·09253	24 0	·189711	29 0	·342006
6	·09403	6	·192165	6	·345720
12	·09554	12	·194643	12	·349470
18	·09707	18	·197143	18	·353243
24	·09862	24	·199666	24	·357046
30	·100175	30	·202211	30	·360876
36	·101761	36	·204780	36	·364738
42	·103358	42	·207363	42	·368631
48	·104972	48	·209978	48	·372550

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
29° 54′	·376499	34° 54′	·615354	39° 54′	·950244
30 0	·380477	35 0	·621033	40 0	·958093
6	·384488	6	·626747	6	·965991
12	·388530	12	·632503	12	·973940
18	·392599	18	·638296	18	·981936
24	·396697	24	·644128	24	·989984
30	·400830	30	·650002	30	·998081
36	·404990	36	·655913	36	1·006230
42	·409184	42	·661863	42	1·014428
48	·413410	48	·667858	48	1·022678
54	·417663	54	·673886	54	1·030978
31 0	·421950	36 0	·679958	41 0	1·039329
6	·426268	6	·686070	6	1·047731
12	·430617	12	·692224	12	1·056187
18	·434999	18	·698418	18	1·064696
24	·439413	24	·704650	24	1·073255
30	·443861	30	·710924	30	1·081865
36	·448338	36	·717240	36	1·090529
42	·452849	42	·723597	42	1·099249
48	·457392	48	·729997	48	1·108021
54	·461968	54	·736438	54	1·116846
32 0	·466577	37 0	·742921	42 0	1·125726
6	·471220	6	·749446	6	1·134657
12	·475894	12	·756014	12	1·143646
18	·480604	18	·762626	18	1·152688
24	·485347	24	·769279	24	1·161785
30	·490123	30	·775979	30	1·170937
36	·494934	36	·782719	36	1·180150
42	·499780	42	·789501	42	1·189416
48	·504661	48	·796330	48	1·198735
54	·509574	54	·803204	54	1·208113
33 0	·514520	38 0	·810120	43 0	1·217547
6	·519504	6	·817080	6	1·227036
12	·524520	12	·824086	12	1·236585
18	·529576	18	·831137	18	1·246192
24	·534663	24	·838232	24	1·255856
30	·539786	30	·845373	30	1·265578
36	·544948	36	·852562	36	1·275360
42	·550143	42	·859794	42	1·285198
48	·555375	48	·867072	48	1·295098
54	·560642	54	·874397	54	1·305054
34 0	·565949	39 0	·881769	44 0	1·315071
6	·571290	6	·889186	6	1·325151
12	·576665	12	·896653	12	1·335288
18	·582079	18	·904165	18	1·345488
24	·587529	24	·911726	24	1·355747
30	·593020	30	·919329	30	1·366068
36	·598545	36	·926984	36	1·376450
42	·604110	42	·934688	42	1·386893
48	·609712	48	·942444	48	1·397398

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
44° 54′	1·407968	49° 54′	2·023803	54° 54′	2·846006
45 0	1·418598	50 0	2·038034	55 0	2·864981
6	1·429291	6	2·052351	6	2·884069
12	1·440051	12	2·066748	12	2·903266
18	1·450874	18	2·081230	18	2·922574
24	1·461762	24	2·095797	24	2·941994
30	1·472712	30	2·110445	30	2·961525
36	1·483725	36	2·125177	36	2·981170
42	1·494806	42	2·139995	42	3·000929
48	1·505954	48	2·154900	48	3·020802
54	1·517167	54	2·169889	54	3·040792
46 0	1·528443	51 0	2·184968	56 0	3·060898
6	1·539789	6	2·200130	6	3·081119
12	1·551202	12	2·215380	12	3·101456
18	1·562680	18	2·230719	18	3·121913
24	1·574225	24	2·246148	24	3·142488
30	1·585840	30	2·261665	30	3·163183
36	1·597524	36	2·277274	36	3·183998
42	1·609271	42	2·292970	42	3·204937
48	1·621077	48	2·308754	48	3·225992
54	1·632980	54	2·324631	54	3·247170
47 0	1·644941	52 0	2·340602	57 0	3·268474
6	1·656968	6	2·356664	6	3·289905
12	1·669069	12	2·372817	12	3·311460
18	1·681238	18	2·389063	18	3·333137
24	1·693479	24	2·405403	24	3·354941
30	1·705793	30	2·421839	30	3·376876
36	1·718177	36	2·438370	36	3·398936
42	1·730637	42	2·454995	42	3·421124
48	1·743166	48	2·471716	48	3·443444
54	1·755769	54	2·488532	54	3·465894
48 0	1·768440	53 0	2·505444	58 0	3·488473
6	1·781195	6	2·522459	6	3·511186
12	1·794021	12	2·539568	12	3·534033
18	1·806920	18	2·556777	18	3·557014
24	1·819895	24	2·574083	24	3·580129
30	1·832946	30	2·591490	30	3·603380
36	1·846071	36	2·608996	36	3·626768
42	1·859274	42	2·626608	42	3·650294
48	1·872554	48	2·644319	48	3·673957
54	1·885907	54	2·662131	54	3·697759
49 0	1·899338	54 0	2·680046	59 0	3·721703
6	1·912852	6	2·698064	6	3·745787
12	1·926444	12	2·716186	12	3·770017
18	1·940111	18	2·734412	18	3·794386
24	1·953859	24	2·752743	24	3·818899
30	1·967687	30	2·771182	30	3·843559
36	1·981595	36	2·789727	36	3·868363
42	1·995584	42	2·808378	42	3·893315
48	2·009653	48	2·827138	48	3·918417

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
59° 54′	3·943666	64° 54′	5·421532	69° 54′	7·451115
60 0	3·969066	65 0	5·455983	70 0	7·499113
6	3·994618	6	5·490649	6	7·547449
12	4·020317	12	5·525536	12	7·596130
18	4·046174	18	5·560646	18	7·645153
24	4·072183	24	5·595977	24	7·694527
30	4·098348	30	5·631535	30	7·744254
36	4·124669	36	5·667322	36	7·794338
42	4·151148	42	5·703333	42	7·844780
48	4·177783	48	5·739578	48	7·895588
54	4·204583	54	5·776057	54	7·946760
61 0	4·231539	66 0	5·812771	71 0	7·998308
6	4·258658	6	5·849716	6	8·050230
12	4·285939	12	5·886906	12	8·102533
18	4·313386	18	5·924333	18	8·155211
24	4·340998	24	5·962006	24	8·208281
30	4·368777	30	5·999921	30	8·261743
36	4·396721	36	6·038082	36	8·315598
42	4·424836	42	6·076494	42	8·369854
48	4·453121	48	6·115157	48	8·424513
54	4·481575	54	6·154071	54	8·479581
62 0	4·510204	67 0	6·193241	72 0	8·535060
6	4·539005	6	6·232671	6	8·590958
12	4·567985	12	6·272356	12	8·647276
18	4·597138	18	6·312306	18	8·704018
24	4·626471	24	6·352521	24	8·761194
30	4·655979	30	6·393001	30	8·818806
36	4·685669	36	6·433752	36	8·876855
42	4·715544	42	6·474772	42	8·935351
48	4·745597	48	6·516067	48	8·994303
54	4·775835	54	6·557639	54	9·053704
63 0	4·806260	68 0	6·599488	73 0	9·113567
6	4·836872	6	6·641619	6	9·173900
12	4·867673	12	6·684032	12	9·234702
18	4·898661	18	6·726735	18	9·295978
24	4·929841	24	6·769723	24	9·357741
30	4·961217	30	6·813005	30	9·419994
36	4·992783	36	6·856578	36	9·482734
42	5·024548	42	6·900450	42	9·545978
48	5·056507	48	6·944623	48	9·609731
54	5·088669	54	6·989098	54	9·673991
64 0	5·121026	69 0	7·033873	74 0	9·738770
6	5·153588	6	7·078964	6	9·804079
12	5·186352	12	7·124362	12	9·869919
18	5·219322	18	7·170072	18	9·936286
24	5·252501	24	7·216104	24	10·00320
30	5·285883	30	7·262452	30	10·07068
36	5·319477	36	7·309127	36	10·13871
42	5·353281	42	7·356128	42	10·20731
48	5·387300	48	7·403454	48	10·27647

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
74° 54′	10·34622	80° 0′	14·90704	85° 0′	23·15395
75 0	10·41655	6	15·02247	6	23·40052
6	10·48748	12	15·13925	12	23·65241
12	10·55902	18	15·25738	18	23·90984
18	10·63115	24	15·37691	24	24·17308
24	10·70392	30	15·49786	30	24·44234
30	10·77731	36	15·62026	36	24·71791
36	10·85134	42	15·74414	42	25·00006
42	10·92601	48	15·86953	48	25·28912
48	11·00133	54	15·99647	54	25·58540
54	11·07731	81 0	16·12498	86 0	25·88926
76 0	11·15397	6	16·25511	6	26·20108
6	11·23131	12	16·38688	12	26·52127
12	11·30933	18	16·52033	18	26·85026
18	11·38807	24	16·65552	24	27·18855
24	11·46750	30	16·79246	30	27·53665
30	11·54766	36	16·93120	36	27·89511
36	11·62855	42	17·07179	42	28·26458
42	11·71018	48	17·21427	48	28·64571
48	11·79256	54	17·35868	54	29·03922
54	11·87571	82 0	17·50507	87 0	29·44593
77 0	11·95963	6	17·65349	6	29·86672
6	12·04434	12	17·80399	12	30·30260
12	12·12985	18	17·95663	18	30·75463
18	12·21616	24	18·11146	24	31·22405
24	12·30330	30	18·26853	30	31·71218
30	12·39128	36	18·42790	36	32·22057
36	12·48011	42	18·58963	42	32·75093
42	12·56979	48	18·75380	48	33·30520
48	12·66035	54	18·92046	54	33·88560
54	12·75181	83 0	19·08970	88 0	34·49465
78 0	12·84417	6	19·26157	6	35·13533
6	12·93744	12	19·43614	12	35·81099
12	13·03165	18	19·61351	18	36·52565
18	13·12681	24	19·79376	24	37·28401
24	13·22294	30	19·97697	30	38·09171
30	13·32005	36	20·16323	36	38·95555
36	13·41816	42	20·35263	42	39·88383
42	13·51728	48	20·54529	48	40·88686
48	13·61744	54	20·74129	54	41·97762
54	13·71865	84 0	20·94075	89 0	43·17285
79 0	13·82093	6	21·14378	6	44·49455
6	13·92429	12	21·35051	12	45·97255
12	14·02877	18	21·56105	18	47·64863
18	14·13438	24	21·77555	24	49·58404
24	14·24113	30	21·99414	30	51·87366
30	14 34906	36	22·21696	36	54·67647
36	14·45818	42	22·44419	42	58·29054
42	14·56851	48	22·67598	48	63·38499
48	14·68008	54	22·91251	54	72·09483
54	14·79293				

Tables of Mutual Inductance (continued).

γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$	γ	$\frac{M}{\sqrt{ab}}$
89° 0′	43·1728	89° 19′	47·9513	89° 37′	55·2141
1	43·3840	20	48·2614	38	55·7738
2	43·5983	21	48·5792	39	56·3586
3	43·8167	22	48·9058	40	56·9718
4	44·0393	23	49·2402	41	57·6143
5	44·2649	24	49·5840	42	58·2905
6	44·4945	25	49·9379	43	59·0106
7	44·7291	26	50·3025	44	59·7778
8	44·9676	27	50·6769	45	60·5927
9	45·2127	28	51·0634	46	61·4582
10	45·4601	29	51·4624	47	62·3821
11	45·7141	30	51·8737	48	63·3850
12	45·9726	31	52·2998	49	64·4926
13	46·2370	32	52·7412	50	65·7189
14	46·5072	33	53·1986	51	67·0718
15	46·7832	34	53·6723	52	68·5662
16	47·0652	35	54·1639	53	70·2280
17	47·3533	36	54·6765	54	72·0948
18	47·6486				

CHAPTER VI

THERMOELECTRICITY

96. The Seebeck effect. It is a matter of everyday observation in a laboratory that no current flows in a circuit that does not contain an electric battery, provided that the temperature is the same throughout. That this is not true when the temperature is different in different parts was first shown by Seebeck in 1821. He found that if a circuit consisted of two metals, and one of the junctions was hotter than the other, then a current flowed round the circuit. The direction of the current depends on the pair of metals as well as on the temperatures of the junctions: in a copper-iron circuit in which one junction is at the freezing-point and the other at the boiling-point of water, the current flows from copper to iron across the hot junction and from iron to copper across the cold junction.

The electromotive force in the circuit, which is usually of the order of $\frac{1}{1000}$ volt, is most conveniently measured by a potentiometer method, as shown in the figure. The hot junction is placed in a hole bored in a massive iron cylinder C, and insulated from it by wrapping the wires round with asbestos. The cylinder is heated in a sand-bath over a Bunsen burner. Its temperature rises gradually, and is measured with a thermometer inserted in a second hole similar to the first. In this way observations can be easily extended to a temperature of 200° of the hot junction. The cold junction is immersed in a beaker of water, protected from the radiation of the cylinder by a sheet of asbestos.

Let v be the thermal E.M.F. and V the E.M.F. of the auxiliary battery. Then no current will pass through the galvanometer when both keys are down, provided that $v = XV/(X + R)$. Since X is small in comparison with R (X being most conveniently

$\frac{1}{10}$th ohm and R of the order of 100 ohms) we have approximately $v = XV/R$. It is, however, very difficult to maintain the balance

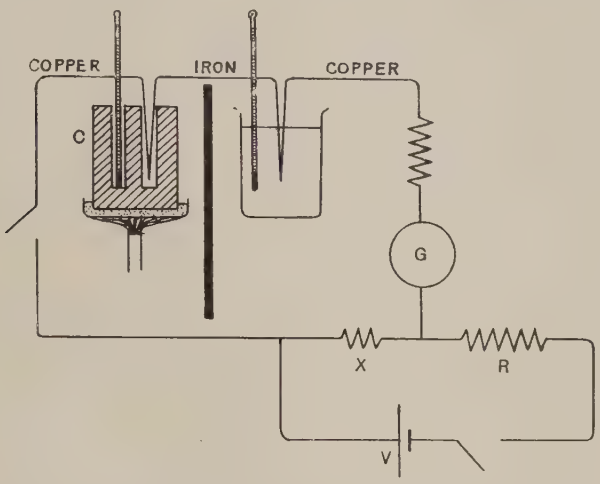


Fig. 131

in this way, as the temperature seldom remains constant over the time required for an observation. It is most convenient, after obtaining a rough balance, to diminish R slightly, thus throwing the galvanometer spot off the zero. The spot creeps back to zero as the cylinder heats up, and the temperature may be read at the time when the deflexion is zero. The process is then repeated with a smaller value of R, and so on.

The presence of two metals is in general necessary for the development of thermoelectric currents. For example, if the terminals of a sensitive galvanometer are joined by a long bare copper wire, no deflexion occurs when a Bunsen burner plays on the wire at any point. Similarly the total E.M.F. in a thermoelectric circuit consisting of two given metals depends only on the temperatures of the junctions, and not on the temperatures of all the parts of the circuit. This principle is of great use in thermoelectric measurements: for example in Fig. 131 we can keep the left-hand beaker at 100° C. and the right-hand beaker at 0° C. and thereby measure the E.M.F. of the circuit for these temperatures of the junctions, without keeping the galvanometer and the resistance X also at 0°.

97. Laws of the thermoelectric circuit. Let A, B be two metals, the temperatures of the hot and cold junctions being θ_1 and θ_2 respectively. The thermal voltage V will be reckoned as positive when the current flows from B to A across the hot junction and from A to B across the cold junction. We shall use the symbol (A, B) for V when it is desired to specify the metals used, and (θ_1, θ_2) when we consider more particularly the temperatures of the junctions. The laws of the thermoelectric circuit are:

(1) For a given pair of metals, the E.M.F. for temperatures θ_1 and θ_3 is the sum of the E.M.F.'s for θ_1 and θ_2 and for θ_2 and θ_3, or symbolically

$$(\theta_1, \theta_3) = (\theta_1, \theta_2) + (\theta_2, \theta_3).$$

(2) For given temperatures of the junctions, the E.M.F. of two metals A, C is the sum of the E.M.F.'s of A, B and of B, C, B being any third metal. Symbolically,

$$(A, C) = (A, B) + (B, C).$$

The first law can be verified by direct measurement, for example with a single copper-iron circuit and three beakers maintained at 0°, 50°, and 100° C. Its utility is obvious: instead of finding the E.M.F. of a given pair of metals for all temperatures of both junctions, we have only to make observations for various temperatures of the hot junction, keeping the cold junction at any convenient temperature such as 0° C., and then the thermal voltages in other cases come out as differences. The second law can also be verified directly, or more neatly by a null method, as shown in Fig. 132.

It is found that if the metals are joined in the order $ABCABC$, and the junctions made alternately hot and cold junctions, no current flows. But the circuit is clearly equivalent to three thermoelectric circuits with metals B, C; C, A; A, B connected in series, so that we have

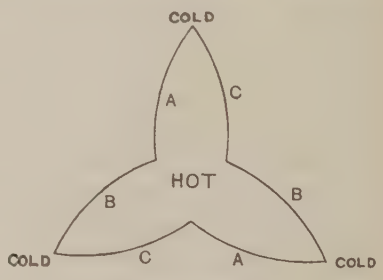


Fig. 132

$$(B, C) + (C, A) + (A, B) = 0,$$

which is the same as the previous enunciation, since (C, A) is clearly equal to $-(A, C)$.

Combining the two laws we have a simple rule for finding the thermal E.M.F. of any circuit for any temperatures of the junctions. Select a convenient temperature (e.g. 0° C.) for the cold junction and a standard metal (e.g. lead), and measure the thermal E.M.F. for various metals at different temperatures of the hot junction. If A_θ is this E.M.F. for a metal A against lead, with temperature θ of the hot junction, the E.M.F. of a thermoelectric circuit of metals A, B and temperatures θ_1, θ_2 is given by

$$V = A_{\theta_1} - A_{\theta_2} - (B_{\theta_1} - B_{\theta_2}) \ldots\ldots\ldots\ldots (1).$$

98. Dependence of thermal E.M.F. on temperature. Fig. 133 shows the thermal voltages of various metals against lead, one junction being maintained at 0° and the other at various temperatures. The figure is taken from the results of good

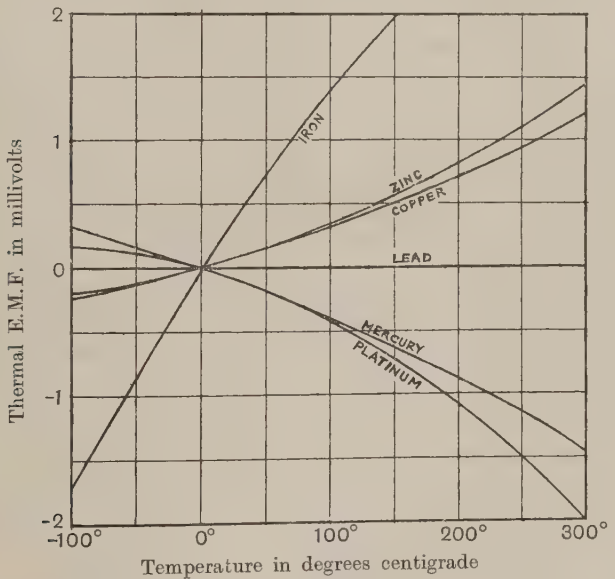


Fig. 133

modern experiments (Noll, Dewar and Fleming), in which great care was taken to obtain pure materials: but considerable discrepancies sometimes arise which are partly accounted for by

differences in the treatment to which the metals have been subjected. The range of accurate observation is not sufficient to decide the form of the curves in the figure: they may however be represented approximately by parabolas whose axes are vertical. This is equivalent to expressing the thermal E.M.F. V for a temperature of $t°$ centigrade in the form $V = at + bt^2$: but it should be remembered that this formula is not rigidly true and sometimes differs from the results of experiment by amounts exceeding the errors of observation.

The laws of the thermoelectric circuit given in the last article show that if the temperatures $(0°, t°)$ of the junctions are prescribed we can arrange the metals in a *thermoelectric series* such that each has a positive E.M.F. when used with the next one in the series. For small values of t the series is as follows:

Antimony, iron, zinc, copper, silver, lead, mercury, platinum-rhodium, platinum, nickel, bismuth.

It was found by Cumming in 1823 that the order of the metals is not invariable, but can be changed by the use of higher temperatures for the hot junction. This phenomenon is known as *thermoelectric inversion*. For example, iron is positive with respect to copper for temperatures less than 550° C., but negative above this temperature.

The choice of the pair of metals in a thermal junction naturally depends on the uses to which the junction is to be put. Thus an antimony-bismuth couple is used when great sensitiveness is required: for temperatures 0° and 100° this gives a thermal voltage of about $1{\cdot}2 \times 10^{-2}$, which is about twelve times that of an iron-copper circuit at the same temperatures. At high temperatures, on the other hand, sufficient sensitiveness can easily be obtained, and the chief desideratum is high melting-point of the metals. For example, a circuit of platinum and 10 % platinum-rhodium alloy can be used up to 1500°, where it gives a potential of about 16 millivolts.

The thermoelectric behaviour of iron is very anomalous. Thus for example Belloc found that $dV/d\theta$ for an iron-platinum circuit fell until the temperature rose to 380° C., then rose to a maximum value for a temperature of 830°, and fell to a second minimum value at 950°, after which it rose again. The three bends in

the curve were very sharp, the two highest being apparently connected with the transition temperatures of iron.

99. The Peltier effect. The question as to the source of the energy of thermoelectric currents was partly settled by the discovery of Peltier (1834) that heat is in general absorbed or liberated when a current crosses the junction of two metals. This is easily demonstrated by the apparatus shown in Fig. 134. The two junctions A, B of a composite circuit are enclosed in two glass bulbs joined by a tube containing a thread of mercury, the whole forming a differential air-thermometer. The thread of mercury is seen to move on turning on the current, and to move in the other direction when the current is reversed. The heating effect of the current in the wires is the same in both bulbs and has no effect on the mercury thread.

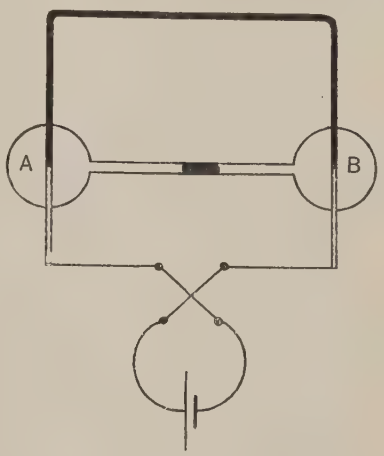


Fig. 134

In a copper-iron circuit, heat is developed when the current flows from iron to copper and absorbed when the current flows from copper to iron. Thus in a thermoelectric circuit, in which the current flows from copper to iron across the hot junction, heat is abstracted from the hot junction by the passage of the current and given up to the cold junction, so that the thermoelectric circuit develops electrical power at the expense of the heat supplied, and acts like a heat engine. This point of view will be elaborated later.

If heat is absorbed by a current flowing across a junction, an equal amount of heat is liberated in the same time by an equal current in the opposite direction. More generally, the heat absorbed per second by a current i flowing from a metal B to a metal A is found to be proportional to the current i, and equal to Πi, where Π is called the Peltier coefficient of A with

respect to B. In this formula the case of liberation of heat is included if we agree that negative absorption means liberation: the convention as to sign is important and will be adhered to throughout.

The proportionality of heating effect and current can be shown roughly by means of the air-thermometer described, using only small movements of the thread of mercury. Careful work is necessary if an accurate measurement is to be made, because in addition to the Peltier heat there is the usual heat developed by the mere passage of the current through the wires. The amounts of heat to be measured are also very small: for example, that developed by 1 ampere in a copper-iron junction would, if utilised to boil an ordinary-sized kettle, require about 18 months.

Very careful experiments were made by Jahn in 1888, by means of which the Peltier heats of various metals with copper were determined in absolute measure. His plan was to enclose the junction in a Bunsen ice calorimeter. The heating effect due to resistance was considerably greater than the Peltier heats, and had to be estimated and allowed for by reversing the current. The proportionality of Peltier effect and current was accurately verified, since the same Peltier coefficient was obtained with various currents.

100. The Kelvin effect. In 1851 Lord Kelvin showed that the Peltier absorption of heat was not the only effect of its kind, but that there must be thermal effects depending on the passage of a current along a single unequally heated conductor. He found that heat was liberated when an electric current passed from hot to cold in copper, and absorbed when it passed from cold to hot. The effects in iron were found to be exactly opposite.

The Kelvin effect in copper can be shown by the following method due to Callendar, depending on the variation of resistance of the metal with temperature. ACB is a very thin wire fastened into thicker wire at the end, the middle point being soldered to another wire attached to a sensitive low-resistance galvanometer. Copper wires AD, BD complete a low-resistance Wheatstone's bridge made entirely of copper, and A, B are connected to a battery through a resistance and a reversing key. Let a current be sent

through the bridge so as to pass from A to B in both branches, and suppose that the bridge is accurately balanced by means of

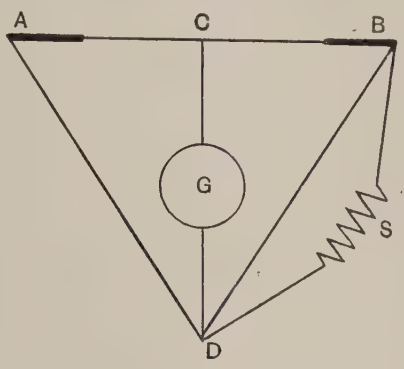


Fig. 135

a low-resistance adjustable shunt S across BD or AD. On reversing the current the balance will be found to be upset, and it is easy to verify that the effect is to increase the relative resistance of AC with respect to CB.

The wires AD, BD are not appreciably heated by the current, but the thin wire ACB does become hot. Thus there is an upward temperature-gradient from A to C and a downward gradient from C to B along the branch. Assuming for the present that heat is liberated in passing from hot to cold in copper, there is originally absorption of heat in AC and liberation in CB, in addition to the heat caused by resistance. On reversal of the current heat begins to be liberated in AC and absorbed in CB, so that the temperature of AC on the whole begins to rise and that of CB begins to fall. This causes an increase in the resistance of AC and a decrease in that of CB, which destroys the balance of the bridge in the sense actually observed. Thus the assumption as to the sign of the effect in copper is verified. In performing the experiment the bridge should be lagged with cotton-wool or other non-conducting material in order to prevent irregular deflexions of the galvanometer caused by local changes of temperature.

The laws of the Kelvin effect may be expressed in terms of a coefficient σ, sometimes called the specific heat of electricity, but

which we shall call the *coefficient of the Kelvin effect*. It is defined as the heat absorbed per second when a current of 1 ampere flows in the given metal from one point of the metal to another whose temperature is 1° higher, over and above the heat developed according to Joule's law. The Kelvin effect has been found to be proportional to the current: thus when a current of i amperes flows for time t from one point to another differing from it by the small temperature $d\theta$ the heat absorbed is $\sigma i t d\theta$.

The determination of σ in absolute measure is very difficult on account of the smallness of the heat to be measured, and the disturbing effects of Joule heat, thermal conductivity and emissivity in the metal. King has used the bridge method already described to measure σ for copper, and more recently Berg has made absolute measurements by compensating the heat of the Kelvin effect by the Joule heat due to the resistance of the metal.

101. Theory of the thermoelectric circuit. We shall now develop the theory of the thermoelectric circuit considered as an engine for converting heat into electrical work, and as a preliminary Fig. 136 shows the positive direction of measurement

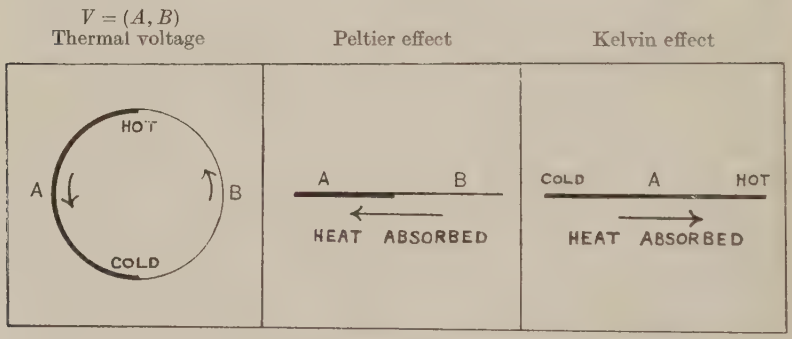


Fig. 136. Positive directions of thermoelectric effects

of the various thermoelectric quantities already mentioned, the direction of the current being shown by the arrows in each case.

The work done in carrying the electric charges round the thermoelectric circuit is accounted for by the Peltier absorption and liberation of heat at the hot and cold junctions and the

Kelvin effect in the two metals. Both these thermal effects are *reversible**, that is, they are changed in sign and unaltered in magnitude when the current flows in the opposite direction. The irreversible heat developed on account of resistance depends on the square of the current, but if we suppose the current to be small enough it is negligible in comparison with the Peltier and Kelvin effects, both of which are proportional to the first power of the current. We may now apply the laws of thermodynamics to the thermoelectric circuit considered as a reversible heat engine, as was first done by Lord Kelvin.

Let a small current of i amperes pass round the circuit for a time t, the hot junction being at absolute temperature θ and the cold junction at temperature θ_0. The Kelvin coefficients for metals A and B are taken as σ and σ' respectively at temperature θ. In time t the work done in conveying charge round the circuit is Vit joules, or $10^7\,Vit$ ergs. If J is the mechanical equivalent of heat this requires that $\dfrac{10^7\,Vit}{J}$ calories should be abstracted in the form of Peltier and Kelvin heat. Heat of amount Πit calories is absorbed at the hot junction and $\Pi_0 it$ rejected to the vessel containing the cold junction: further, during the passage of the current from hot to cold in metal A the heat absorbed is $-it\int_{\theta_0}^{\theta}\sigma d\theta$, while heat of amount $it\int_{\theta_0}^{\theta}\sigma' d\theta$ is absorbed when the current flows from cold to hot in metal B. Hence applying the first law of thermodynamics and dividing out by it we have

$$\frac{10^7\,V}{J} = \Pi - \Pi_0 - \int_{\theta_0}^{\theta}(\sigma - \sigma')\,d\theta.$$

The second law of thermodynamics states that if dQ is a small element of heat taken up at any part of the cycle at temperature θ, then $\int dQ/\theta = 0$, where the integral means summation over all the elements of heat taken up from outside in the whole operation, heat liberated being of course regarded as negative. This gives similarly

$$\frac{\Pi}{\theta} - \frac{\Pi_0}{\theta_0} = \int_{\theta_0}^{\theta}\frac{\sigma - \sigma'}{\theta}\,d\theta.$$

* For the conditions of reversibility of a heat engine, see Buckingham's *Thermodynamics*, pp. 94–5, 113–4.

Differentiating the last two equations with respect to θ, we have

$$\frac{10^7}{J}\frac{dV}{d\theta} = \frac{d\Pi}{d\theta} - (\sigma - \sigma') \quad \ldots\ldots\ldots\ldots\ldots (2),$$

and

$$\frac{d}{d\theta}\left(\frac{\Pi}{\theta}\right) = \frac{\sigma - \sigma'}{\theta} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Substituting for $\sigma - \sigma'$ in (2) we have

$$\frac{10^7}{J}\frac{dV}{d\theta} = \frac{d\Pi}{d\theta} - \theta\frac{d}{d\theta}\left(\frac{\Pi}{\theta}\right),$$

or

$$\Pi = \frac{10^7}{J}\theta\frac{dV}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

This value when substituted in (3) gives further

$$\sigma - \sigma' = \frac{10^7}{J}\theta\frac{d^2V}{d\theta^2} \quad \ldots\ldots\ldots\ldots\ldots (5).$$

V is supposed to be measured in volts, and the factor $10^7/J$ has the value $1/4{\cdot}18 = 0{\cdot}239$.

102. Experimental verifications of the theory. The equations (4) and (5) give a theoretical means of calculating the Peltier and Kelvin coefficients from observations on thermal voltage at various temperatures. It is however very desirable that the various quantities should be measured for the same specimens of the metals used, as slight impurities or differences in physical state have a great effect on thermoelectric properties. In Jahn's experiments, already mentioned, the values of $dV/d\theta$ and Π were measured for the same specimens, and the results are given in the following table:

Temperature 0° C. = 273 abs.

Metals	$dV/d\theta$ in volts per degree	$\frac{10^7}{J}\theta\frac{dV}{d\theta}$	Peltier coefficient calories per second per ampere
Silver-copper	$+ 2{\cdot}12 \times 10^{-6}$	$+ 1{\cdot}38 \times 10^{-4}$	$+ 1{\cdot}15 \times 10^{-4}$
Iron-copper	$+ 11{\cdot}28 \times 10^{-6}$	$+ 7{\cdot}35 \times 10^{-4}$	$+ 8{\cdot}79 \times 10^{-4}$
Platinum-copper	$- 1{\cdot}40 \times 10^{-6}$	$- 0{\cdot}912 \times 10^{-4}$	$- 0{\cdot}889 \times 10^{-4}$
Zinc-copper	$+ 1{\cdot}51 \times 10^{-6}$	$+ 0{\cdot}997 \times 10^{-4}$	$+ 1{\cdot}62 \times 10^{-4}$
Cadmium-copper	$+ 2{\cdot}64 \times 10^{-6}$	$+ 1{\cdot}72 \times 10^{-4}$	$+ 1{\cdot}71 \times 10^{-4}$
Nickel-copper	$- 20{\cdot}03 \times 10^{-6}$	$- 13{\cdot}1 \times 10^{-4}$	$- 12{\cdot}1 \times 10^{-4}$

The agreement of the numbers in the last column gives a satisfactory verification of equation (4) except in the case of

copper and zinc. The necessity for working with the same specimens, when special precautions are not taken to secure purity, is seen from the fact that Jahn's values for $dV/d\theta$ differ from those of more recent experiments by much more than the differences of the last two columns in the above table.

The equation (5) has not been verified so accurately, for several reasons. In the first place, the Kelvin effect is exceedingly difficult to measure, and determinations of it and the thermal voltage have not been made on the same specimens. Secondly, the values of $d^2V/d\theta^2$ given by the best modern experiments show considerable differences. In the last column of the following table values of $d^2V/d\theta^2$ at 100° C. have been used which are the mean of those of several observers, but the second differential of V cannot very well be determined accurately from experiments over a limited range of temperature. The values of σ are derived chiefly from the recent experiments of Cermak and Berg.

Kelvin coefficients at 100° C. = 373 abs.

	σ in calories per coulomb per degree		$\sigma - \sigma'$ observed	$\sigma - \sigma'$ calculated from (5)
Copper	$+ 4{\cdot}7 \times 10^{-7}$	Copper-lead	$+ 3{\cdot}7 \times 10^{-7}$	$+ 7{\cdot}1 \times 10^{-7}$
Iron	$- 29 \times 10^{-7}$	Iron-lead	$- 30 \times 10^{-7}$	$- 27 \times 10^{-7}$
Mercury	$- 5{\cdot}5 \times 10^{-7}$	Mercury-lead	$- 6{\cdot}5 \times 10^{-7}$	$- 7{\cdot}7 \times 10^{-7}$
Platinum	$- 22 \times 10^{-7}$	Platinum-lead	$- 23 \times 10^{-7}$	$- 22 \times 10^{-7}$
Zinc	$+ 8 \times 10^{-7}$	Zinc-lead	$+ 7 \times 10^{-7}$	$+ 13 \times 10^{-7}$
Cadmium	$+ 24 \times 10^{-7}$	Cadmium-lead	$+ 23 \times 10^{-7}$	$+ 28 \times 10^{-7}$
Lead	$+ 1 \times 10^{-7}$			

Considering these difficulties we can only regard the agreement of the numbers in the last column as highly satisfactory.

103. Thermopiles and their efficiency. We can use the thermoelectric circuit as a source of potential, arranging thermocouples in series as shown in Fig. 137. Such an arrangement is called a thermopile, and considerable voltages may be attained if enough couples are used:

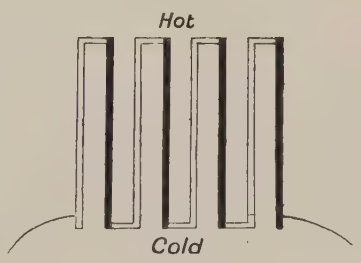


Fig. 137

for example if the hot junctions of a constantan-manganin pile are maintained at 550° C. and the cold junctions at 0° the voltage of each pair is about 1/20, so that with 100 couples a 5-volt battery is obtained.

The thermopile considered as a source of power has however a low efficiency; that is, the ratio of the electrical work obtainable from it is but a small fraction of the mechanical equivalent of the heat supplied. To illustrate this fact consider the case of the constantan-manganin pile just mentioned, supposing that the wire used throughout is of 2 mm. diameter, the length of each segment of wire being 20 cm. We shall neglect the radiation of heat by the hot metals to begin with, and compare the electrical work that can be extracted from the pile with the heat required to maintain the temperatures constant against conduction from the hot to the cold junctions.

The temperature gradient $-d\theta/dx$ in each wire is of the order 30 degrees per cm., and the cross-section of each wire is $\pi/100$ sq. cm. $= 1/30$ (say). Hence the flow of heat down each wire is approximately K calories per second, where K is the thermal conductivity of the metal. The value of K for both constantan and manganin being about 1/20, the total flow of heat from the hot to the cold junction by way of the 200 connecting strips of wire is about 10 calories per second. Now since the specific resistance of either metal is about $4{\cdot}5 \times 10^{-5}$ the total internal resistance of the battery is $4000 \times 4{\cdot}5 \times 10^{-5} \times (100/\pi)$ ohms, i.e. about 5 ohms.

If the resistance of the external circuit is R ohms the current is $i = 5/(R + 5)$ amperes, and the energy utilised outside the battery per second is Ri^2 joules $= \dfrac{Ri^2}{4{\cdot}18}$ calories $= \dfrac{6R}{(R+5)^2}$ calories, approximately. This has the maximum value 3/10 when $R = 5$. Hence the efficiency of this thermopile, considered as a heat engine, can in no case exceed 3 per cent. In this calculation we have neglected not only the heat radiated from the wires (which may amount to several hundred calories per second), but also the Peltier and Kelvin effects in the metals. It follows that the thermopile must always be a very inefficient source of electrical power.

The thermopile is, however, of great use for measuring either high temperatures or small differences of temperature. In the former case it has obvious advantages over the thermometer: it can be made extremely durable, and the metals can be chosen to have a very high melting-point. Temperatures up to 1500° C. may be measured with a thermocouple of platinum and platinum-rhodium; and with tungsten and molybdenum it would be possible to go much higher. By using a very sensitive galvanometer, and a pair of metals far apart in the thermoelectric scale, great sensitiveness can be obtained for small differences of temperature between the junctions. Instruments of this kind, called *radiometers*, are valuable for detecting and measuring thermal radiation.

If the detecting instrument is a galvanometer of comparatively high resistance there is every advantage in multiplying the number of junctions up to a certain point. It does not however follow that the greatest sensitiveness is obtained with a large number of couples, because after a certain point the increase of voltage is neutralised by the ever-increasing resistance of the circuit. The greatest sensitiveness, in fact, is obtained by the proper use of a *single* junction, as in Boys' *radio-micrometer*, the principle of which is shown in the figure. The antimony-bismuth thermocouple is connected to a small loop of wire lying between the poles of a magnet after the manner of the coil of a sensitive moving-coil galvanometer. The loop is attached to a fine glass rod *G* carrying a mirror *M*, and the whole is suspended from a fine quartz fibre. Radiation falling on the couple, or on a blackened copper disc attached to it, sends a current round the loop, which is deflected like the moving coil of a galvanometer. The extremely low resistance

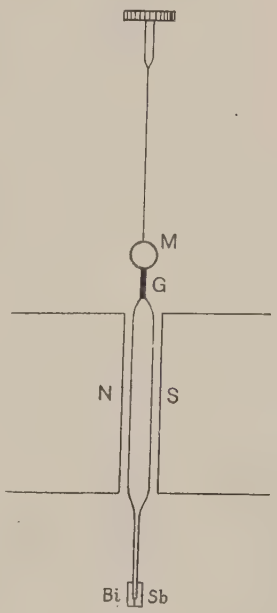


Fig. 138

of the whole loop more than compensates the disadvantage of having only one junction.

104. Thermal galvanometers. Thermal galvanometers in general depend on the heating effect of the current to be measured. We shall see later that there is a class of fluctuating currents (Arts. 129, 131) which would give no reading on an ordinary moving-coil galvanometer. The heat produced may however be used to actuate thermoelectric couples, the resulting current being then measured in the usual way.

A very simple and effective arrangement is one devised by Klemenčić, consisting of two fine wires crossing one another as in Fig. 139. When a current is passed between the terminals *AA* the wires are heated, particularly at the point where they cross. This point therefore forms the hot junction of a thermopile; and if a galvanometer is joined to the terminals *BB* a current is sent round it which can be measured in the usual way. The strength of the current depends on the temperature to which the junction will rise: hence the sensitiveness may be considerably increased by placing the whole instrument in a vacuum (as was first done by Lebedew), since the radiation of heat from the junction is thereby diminished.

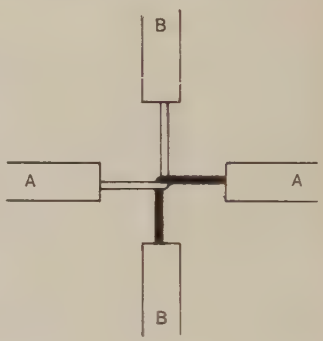


Fig. 139

In Duddell's *thermo-galvanometer* (Fig. 140) the current passes through a fine filament placed directly under the junction of a radio-micrometer. The filament, or "heater," is usually a platinised quartz fibre of from 4 to 100 ohms resistance: its distance from the junction can be adjusted by a screw shown at *F* in the figure. The steadiness of the zero depends on maintaining reasonably constant temperature near the thermal junction, and this is secured by closing the chamber containing the heater and couple with a heavy block *E* of copper, the whole instrument being contained in a second copper case provided with a glass

window. Readings are made by the lamp and scale method, the small square mirror being shown at H, while the instrument is made portable by means of the clamp B which grips the moveable loop.

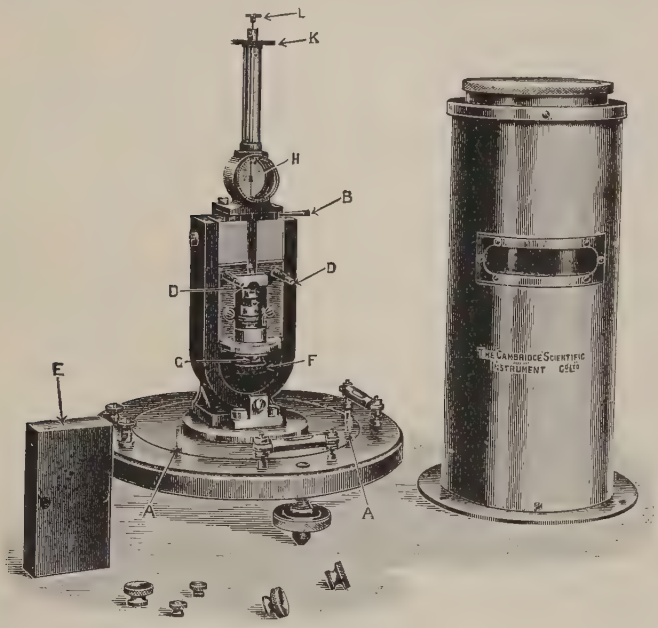


Fig. 140

A less sensitive but very convenient instrument on the same principle is Duddell's thermo-ammeter, in which the thermo-couple is attached to a moving coil mounted between bearings after the manner of a millivoltmeter. The instrument is graduated so as to read directly in milliamperes.

An obvious difficulty in the way of attaining great sensitiveness with thermal instruments is that their deflexion is proportional to the square of the current, so that if the current is halved the deflexion is reduced to one-quarter. Duddell's thermo-galvanometer will detect currents of 10^{-6} amperes with a resistance of the order of 100 ohms, but in general thermal galvanometers are considered sensitive if a current of 1 milliampere yields an appreciable deflexion. It is difficult to give exact data since the sensitiveness may depend very much on the direct-current galvanometer used in conjunction with it.

CHAPTER VII

INDUCED MAGNETISM

105. Variation of magnetisation with impressed magnetic force. (H, I) curve. We have hitherto regarded all magnets as practically permanent, neglecting the change in their magnetism which actually occurs whenever they are placed in magnetic fields. We have now to investigate the magnetic properties of iron and other substances more generally, and find out what circumstances determine the strength of magnets. The first step is to find out how the intensity of magnetisation I of a body varies when the field H in which it lies is changed. In examining these changes the magnetic action of currents is of the greatest assistance; for we have here a means of producing magnetic fields of considerable strength, which moreover can be calculated accurately. The following is a simple experimental arrangement for testing the magnetic properties of thin wires composed of magnetisable material (Fig. 141).

A and B are two similar vertical solenoids placed magnetically east and west of a compass-box, A being fixed and B moveable in a groove pointing towards the compass-needle. The solenoid A is the magnetising solenoid, while B serves to compensate the magnetic effect of the solenoid itself, leaving the magnetised wire only to affect the compass-needle. To effect compensation a strong current is first of all passed through both solenoids in the same direction, and B moved in its groove until there is no deflexion of the needle.

The wire to be tested is now passed down the centre of the fixed solenoid A, so that its lower end is on a level with the needle. The rheostat in circuit should have considerable resistance in

addition to large carrying-capacity (say 200 ohms, capable of carrying 2 amperes continuously), so that we can apply currents

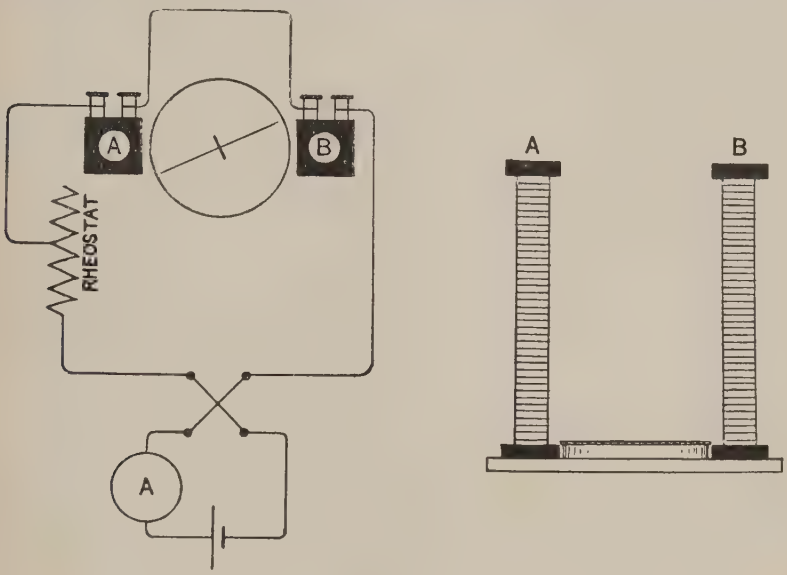


Fig. 141

differing considerably in magnitude. The direction of the reversing key determines the *sign* of the current i in any case; the positive direction of current will be taken to be that which produces a magnetising force vertically downwards. If H is the magnitude of this force, n the number of turns of the solenoid and l its length, we have

$$H = \frac{4\pi ni}{10l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

i being measured in amperes.

Suppose that A is to the west of the compass-needle, and that the deflexion of the needle is θ, measured from north towards east. If I is the corresponding intensity of magnetisation and a the radius of the wire, this deflexion is due to the poles $\pm \pi a^2 I$ at the ends of the wire. Thus if r is the distance of the lower pole from the needle and h the earth's horizontal component, we have

$$\pi a^2 I \left\{ \frac{1}{r^2} - \frac{r}{(r^2 + l^2)^{\frac{3}{2}}} \right\} = h \tan \theta.$$

Hence I is known if h is known. Instead of making a separate determination of h at the place of observation, it is enough to make a single subsidiary observation. When the experiment is finished take away the wire and the moveable solenoid, and observe the deflexion α caused by a known current i_0 in A alone. Let b be the mean radius of the cross-section of the *solenoid*; then it follows from Art. 85 that the magnetic effect of A in this case is that of a bar-magnet whose poles are $\pm ni_0\pi b^2/10l$, provided that r is large compared with b. Since this condition is easily attained in practice, we have

$$\frac{ni_0\pi b^2}{10l}\left\{\frac{1}{r^2}-\frac{r}{(r^2+l^2)^{\frac{3}{2}}}\right\}=h\tan\alpha.$$

Dividing the last two equations we obtain

$$I=\frac{nb^2i_0}{10la^2\tan\alpha}\tan\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Equations (1) and (2) thus give H and I in any given case by observations of i and θ. The method is not excessively accurate on account of the finite length of the experimental solenoids, but is sufficiently so for most purposes.

In making the observations we first cut off the current and pass the wire, previously demagnetised, down the centre of A. A weak positive current i is then passed through the solenoids. As the resistance is slowly cut out the intensity of magnetisation

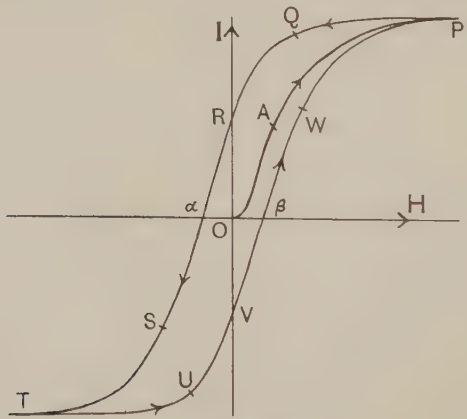
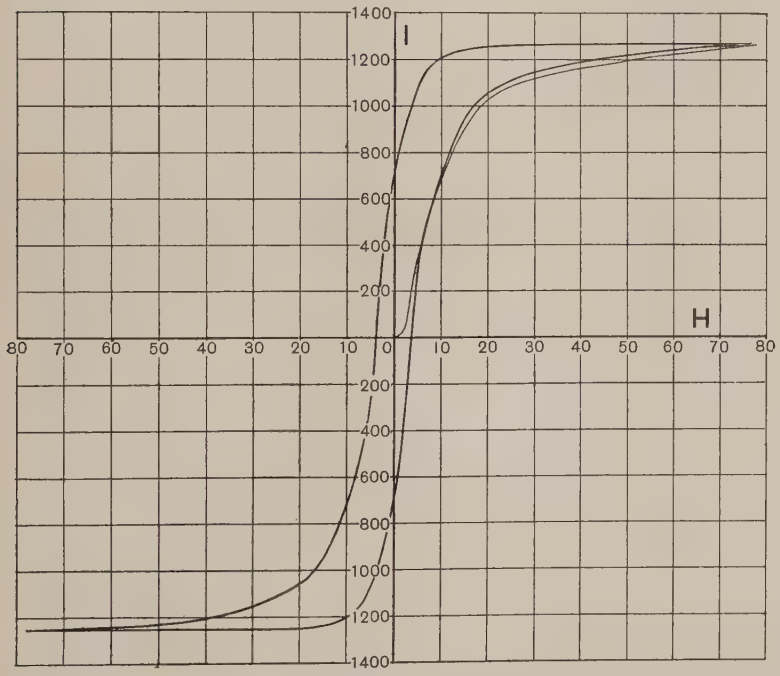


Fig. 142

will rise as shown by the line OAP in Fig. 142. By the time the point P is reached I has become nearly constant. The iron is then said to be *saturated*, and P may be called a saturation point.

It is convenient to arrange matters so that the point P is reached when the rheostat is short-circuited. If we begin to put resistance in the circuit again by means of the rheostat, the magnetisation does not resume its original value for any given current, but is consistently greater. At the point R the current is zero and may next be reversed. If, after reversal, resistance is cut out of the circuit continuously, the magnetisation falls constantly as shown in the line $RaST$, changing sign at a and approaching negative saturation at T. This latter point is the exact counterpart of P, and the remaining part $TUV\beta WP$ of the curve is delineated in exactly the same way as the first half.

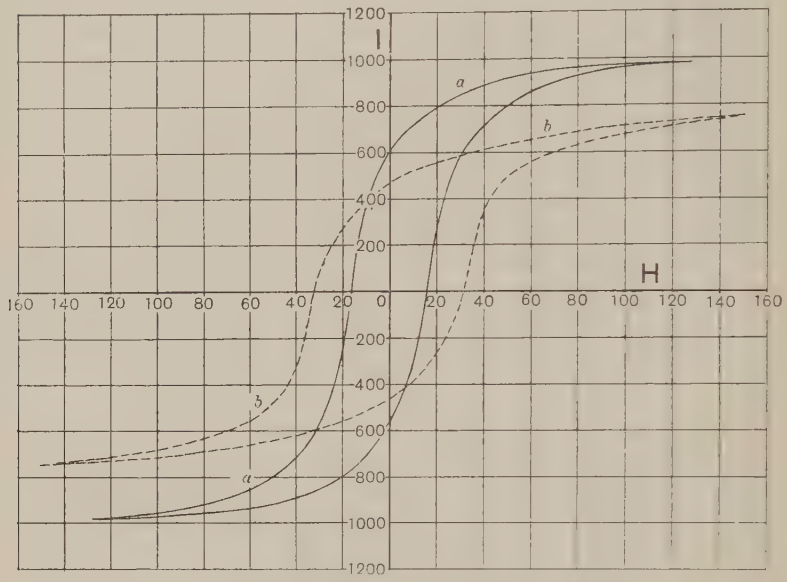
Fig. 143 shows an actual curve obtained in this way with a



Soft iron wire

Fig. 143

soft iron wire. In Fig. 144 the continuous curve a is drawn for a steel rod (bicycle spoke), the curve b showing the considerable



a, drawn steel wire; *b*, the same glass-hardened

Fig. 144

effect caused when the spoke is glass-hardened by plunging it suddenly into cold water after heating it to a bright red heat.

106. Hysteresis. Residual magnetism. The preceding experiments show that the magnetisation of iron is not in general proportional to the magnetising force. It is, in fact, not even possible to assign *a priori* a value of I when H is given, as the behaviour of every specimen depends on the previous magnetic forces to which it has been subjected as well as on that actually present. If the curves of the last article are examined they will exhibit the characteristic tendency of the magnetisation to lag behind the impressed magnetic force: thus along PQR in Fig. 142 the magnetisation tends to retain the large amount it has acquired at P, while the lowness of the ascending branch βWP shows that the iron has some difficulty in ridding itself of the effects of its previous negative saturation. The phenomenon, first described

by Warburg in 1881, has been named *hysteresis* by Ewing, who has examined it in great detail.

A great variety of hysteresis curves can be drawn by varying the manner in which the magnetic force is applied, some of these being shown roughly in the figure. The first diagram shows the

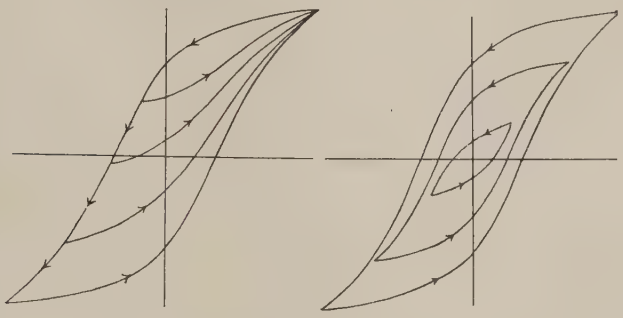


Fig. 145

effect of not proceeding to negative saturation before beginning to increase the magnetising current again, the second some cyclic curves obtained by varying the magnetic force continuously between fixed limits, the limits being too small to give the normal hysteresis loop. In general, the abnormal curves lie within the original saturation loop, though certain curves may lie outside it to a slight extent, as for example the "initial" curve in Fig. 143.

The hysteresis loop enables us to give an exact definition of certain terms to which no precise meaning has previously been attached. In Fig. 142 the length OR is taken as the measure of the *residual magnetism* of the rod, representing the magnetisation when the magnetic force has been removed after saturation. Similarly $O\alpha$, the magnetic force in the opposite direction necessary to remove the magnetism altogether from the rod, is called the *coercive force*. It is a measure of the power of the specimen to resist the abstraction of its magnetism when placed in a magnetic field.

We thus see that a study of the hysteresis curve enables us to judge of the permanence of ordinary magnets. Since the magnetic forces to which magnets are normally subjected do not exceed one unit, no serious loss of magnetisation is to be expected

in hard steel magnets. In time, however, the magnetism will gradually weaken on account of accidental shocks.

The hysteresis curve for soft iron (Fig. 143) has been traced from an experiment in which care was taken not to shake the specimen. Soft iron is very sensitive to shock, particularly in the early stages of magnetisation: in fact the small residual power of soft iron in practice is largely due to this cause.

Ewing introduced the convenient method, now generally used, of demagnetising iron by subjecting it to an alternating magnetic field whose amplitude diminishes steadily to zero. A rod can be demagnetised in the solenoid used to determine its (H, I) curve, with the aid of a small alternating-current generator and a rheostat of considerable range. Initially an alternating current is sent through the solenoid whose maximum value is sufficient for saturation: the current is then gradually reduced as low as possible by means of the rheostat, and if the operation is carefully performed the rod will be quite free from magnetism. Fig. 146 illustrates the changes undergone by the iron in this process.

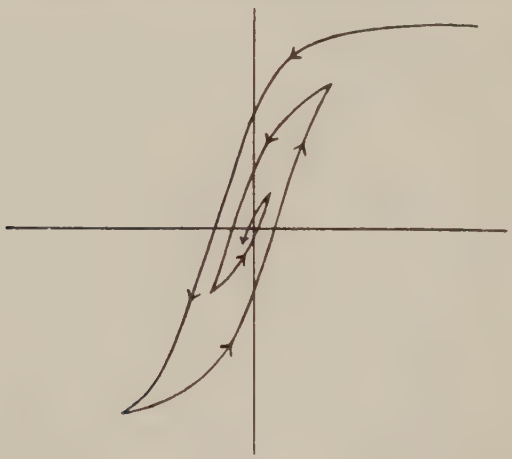


Fig. 146

As the extreme magnetising force at each reversal is gradually reduced the hysteresis loops gradually diminish and finally vanish into the origin, leaving the iron in a state in which I and H are both zero. It might perhaps be thought that the same effect

could be produced in a single cycle, e.g. by choosing the point of reversal in the left-hand diagram of Fig. 145 so as to make the return loop pass through the origin. In this case, however, it is found that the iron still shows the effect of its previous magnetic history, in that it responds differently to positive and negative magnetising forces.

In addition to the ordinary hysteresis effects iron takes some little time to attain its magnetisation in any given case. This effect has been named *Nachwirkung* (magnetic after-effect): but it is not very observable when the variations of magnetisation are slow, as in the method of Art. 105.

107. Work of magnetisation. Hysteresis loss. The lagging of magnetisation behind the impressed force when we go round a cycle can be accounted for by supposing that some force inside the molecule prevents the molecular magnets from responding completely to a change of magnetic force. Thus when a piece of iron is taken round a hysteresis cycle the external field on the whole does work on the molecular forces in the body.

Let us have a rod of length l and cross-section A uniformly magnetised by a field H to intensity I. The equivalent surface-distributions are $\pm AI$ as shown. When the magnetisation I is changed to $I + dI$, H remaining unaltered, work is done equal to that done in transferring pole-strength AdI from one end of the rod to the other, that is $lAHdI$. The effect of increasing H to $H + dH$ without altering I would be to do no work on the molecular forces of the body, since the position of the molecular magnets is unchanged. Work would be done merely on the current or external magnet producing the field. Hence when the force and magnetisation are changed from H and I to $H + dH$ and $I + dI$, work $lAHdI$ is done on the magnetised rod. In taking the rod round a complete hysteresis cycle the work done is $lA \int HdI$. Since lA is the volume of the rod, the work per unit volume is equal to $\int HdI$, the area of the hysteresis loop.

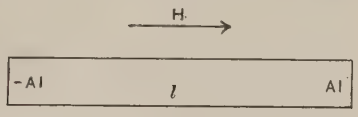


Fig. 147

If the rod is taken repeatedly round the cycle, for example

by means of an impressed alternating current, the supply of work results in a continual rise of temperature of the rod, until checked by radiation. Thus when magnetic substances are taken round hysteresis cycles by external power, part of this is lost in heating the substances. Such losses of available energy are called *hysteresis losses*.

Hysteresis losses can be found beforehand by measuring the (H, I) loop with a planimeter, a process which gives accurate values corresponding to any assigned maximum value of H. In Ewing's *hysteresis tester* the loss is given by a single observation, which practically consists in measuring the mean couple required to rotate an iron rod between the poles of a magnet. If C is this couple and v the volume of the rod, $2\pi C$ is, in the absence of friction, the work required to take the magnetism of the rod round its cycle, that is $v \int H dI$. The hysteresis loss is therefore $2\pi C/v$.

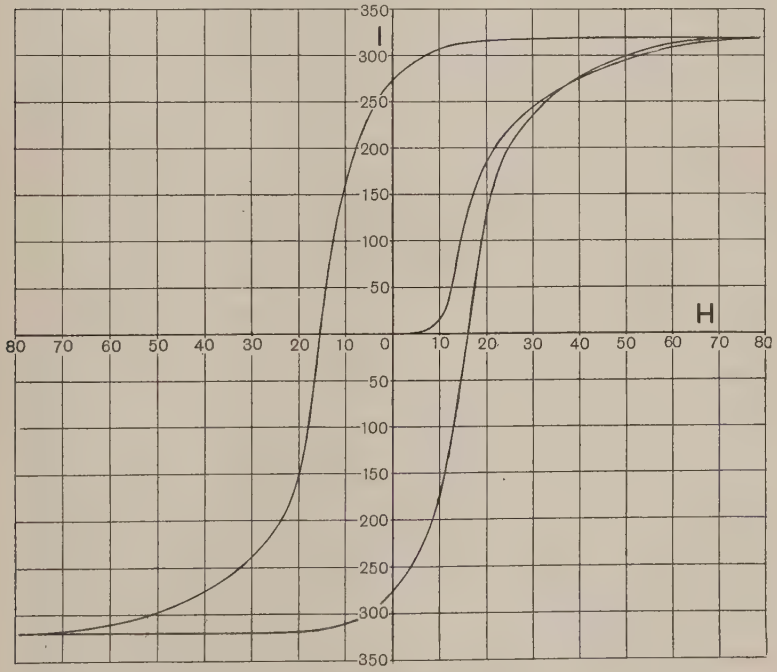
The energy dissipated in a single cycle in soft iron is of the order of 10,000 ergs per cubic centimetre, that in steel being about ten times as large.

108. Magnetic properties of iron, nickel and cobalt. Heusler alloys. The fact that specimens of iron and steel prepared in different ways have vastly different magnetic qualities is now very well known and utilised in various ways. In cases where iron is taken round its hysteresis cycle many times a second, as for example in the alternate current transformer (see Ch. IX), the chief desideratum is low hysteresis loss, combined of course with great magnetisability. This condition is fulfilled by pure well-annealed iron, the comparatively small area of whose hysteresis curve is seen from Fig. 143.

Of late years the magnetic properties of iron alloys have been carefully examined with a view to the production of steels of commercial value. The presence of small quantities of other elements has very great effect on the hysteresis loss and coercive force, while frequently leaving the maximum magnetisation practically unchanged. For example, an iron alloy containing silicon, patented by Sir R. Hadfield under the name of Stalloy, has a hysteresis loss per c.c. of only 3000 ergs per cycle, together with great magnetisability.

A hysteresis curve of small area means either low coercive force or low residual magnetism, or both, so that materials suitable for the cores of transformers are entirely unsuited for making permanent magnets. Hard tungsten-steel, or iron alloyed with from 3 to 8 per cent. of tungsten, has a very large coercive force (about 60 units), and is extensively used for magnets where great permanence is required. In fact, the modern ammeters and voltmeters of the moving coil type, with graduated scale, have only become possible with the perfection of the manufacture of magnets which resist shock and demagnetisation by external agency. The hysteresis loss per cycle in hard tungsten-steel has the high value of 260,000.

The precise effect of the substance with which iron is alloyed is imperfectly understood, and many surprising peculiarities have been brought to light, of which we may mention two. An alloy containing 12 % of manganese has its magnetisability



Nickel wire

Fig. 148

almost entirely removed, the intensity of magnetisation being proportional to the magnetising force, and given approximately by the equation $I = \frac{1}{30}H$. There is no residual magnetism, even after exposure to the strongest fields. Even more remarkable is the very similar behaviour of the iron-nickel alloy with 68 % of iron and 32 % of nickel, since nickel as well as iron is a strongly magnetic substance.

The (H, I) curve for a soft nickel wire is shown in Fig. 148, the curve being very similar to that of steel. The maximum magnetisation that nickel can acquire varies from 300 to 400; that is, about one-third that of iron. The coercive force is comparatively high, ranging from 10 to 20.

Cobalt is intermediate in magnetic properties between iron and nickel, and can acquire a magnetisation of about 700. The coercive force is much the same as for nickel.

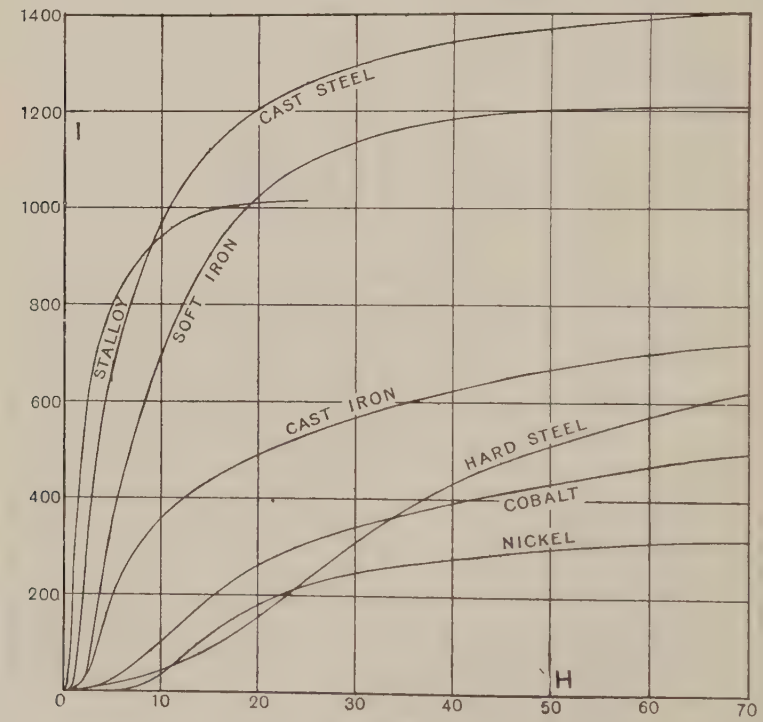


Fig. 149

The remarkable discovery was made by Heusler in 1903 that certain alloys of copper, manganese and aluminium, all of which metals are practically non-magnetisable by themselves, had strong magnetic qualities. For example, an alloy containing 61·5 % of copper, 23·5 % of manganese, and 15 % of aluminium, has magnetic properties closely resembling those of nickel. The hysteresis curves of all strongly magnetic substances have much the same shape, and exhibit the phenomena of saturation and remanence, the residual magnetisation varying from one-half to two-thirds the maximum.

The comparative behaviour of iron, nickel and cobalt is shown in Fig. 149, which shows the magnetisation curves obtained on the first application of the magnetic force. The range of H is not great enough to show the whole of the curve for hard steel, the magnetisation of which rises under high force to almost the amount acquired by soft iron. The accompanying table gives some approximate numerical data concerning the various substances hitherto mentioned. The column headed B (magnetic induction) is here given for reference: it will be better understood later.

Magnetisation of strongly magnetic substances.
(Approximate numbers only, temperature normal.)

	Intensity of magnetisation I		Magnetic induction B		Coercive force	$\int I\,dH$, energy loss per cycle, ergs per c.c.
	max.	residual	max.	residual		
Cast steel ...	1400	700	18000	9000	2	20000
Tungsten steel, hardened	1300	800	16000	10000	60	260000
Soft iron ...	1200	900	15000	11000	2	10000
Hard steel ...	1000	700	12000	9000	40	120000
Transformer iron	1000	600	12000	8000	1·5	8000
Stalloy (silicon steel)	1000	500	12000	6000	0·7	3000
Cobalt	700	250	9000	3000	10	20000
Nickel	400	200	5000	2500	10	12000
Heusler alloy: Cu 61·5 %, Mn 23·5 %, Al 15 %	300	200	4000	2500	8	8000

109. Electro-magnets. The simplest form of electro-magnet consists of an iron or steel rod uniformly wound with wire. Such a rod acts like a permanent magnet, only to a considerably greater degree, as the saturation intensity of magnetisation can be maintained instead of the residual intensity.

Electro-magnets are chiefly used in laboratories to produce strong and tolerably uniform magnetic fields. A common type

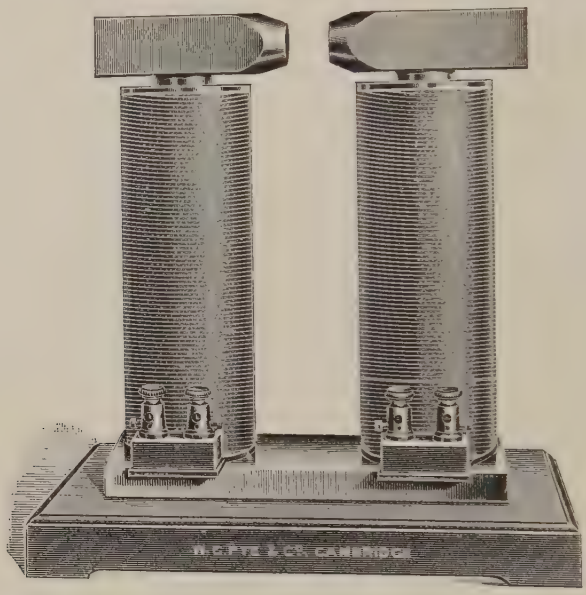


Fig. 150

is shown in Fig. 150. Two soft iron pillars are firmly fixed in a strong iron base, a solenoid of many turns of thick wire being slipped over each. In this form the electro-magnet is essentially a horse-shoe magnet kept excited by external current. Strong magnetic fields are formed between the poles if the latter are brought close together, and this is effected by means of two masses of soft iron, called pole-pieces, which lie on the fixed pillars. The pole-pieces are moveable and afford a means of varying the strength of the field, an additional control being given by the use of various magnetising currents. If very strong fields are required the pole-pieces must be brought near together, and are preferably of conical form. Fig. 151 shows

an electro-magnet of du Bois' type, capable of giving fields of 40,000 to 50,000 C.G.S. units.

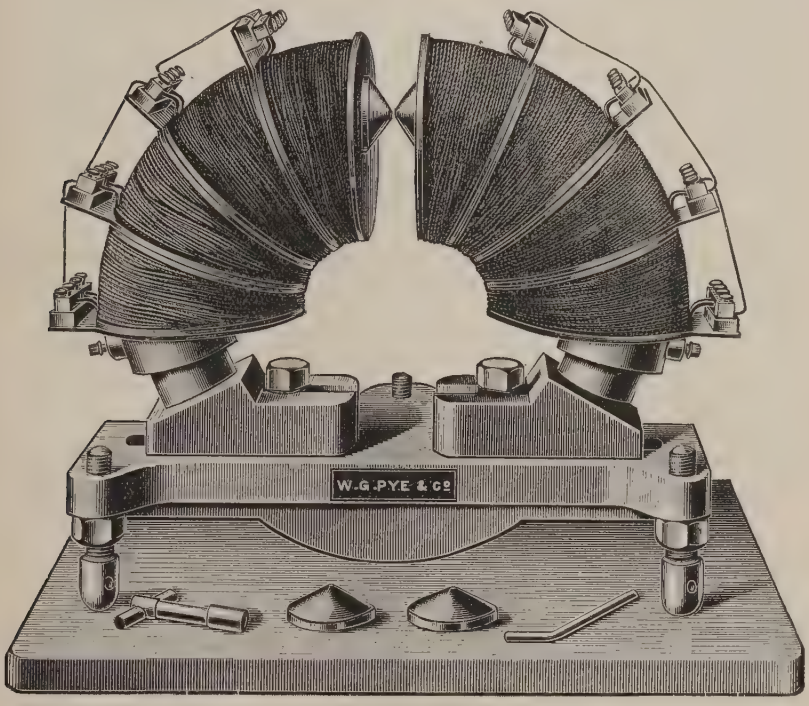


Fig. 151

The production of a uniform magnetic field of any considerable strength is a difficult matter. In order to see what fields can be obtained without the use of iron, consider the case of a solenoid of n turns per centimetre, each coil carrying a current of i amperes. The field inside is $4\pi ni/10$ C.G.S. units, so that to obtain a field-strength of 1000 with 10 amperes the number of turns per centimetre would have to be about 80. Such a coil would have considerable size. The ordinary electro-magnet gives a field of from 2000 to 10,000 units. The field naturally varies with the current used to excite the magnet, the variation often being so great that it is difficult to reproduce accurately a field of given strength.

The order of magnitude of the force with which the poles of

an electro-magnet pull each other together can be estimated by considering the case of a very long iron bar, permanently magnetised to intensity I, and then cut in two and the parts slightly separated from one another. There is a surface-layer of magnetism of surface-density $\pm I$ on the opposing faces, the whole arrangement being the magnetic analogue of a charged parallel-plate condenser. Applying the theory of Art. 34, we see that the mechanical force per sq. cm. of either face is $2\pi I^2$ dynes. If $I = 1200$ this is about 10 kilogrammes' weight per sq. cm.

110. Mechanical force on a magnetised body in a non-homogeneous magnetic field. A magnet placed in a uniform magnetic field is acted on in general by a couple, but no resultant force. When the field is not uniform there is however a force tending to move the magnet as a whole in a definite direction. It is instructive to calculate the force exerted by a given non-homogeneous field on a small magnetised body placed at any point, when the magnetic force due to the body itself is neglected.

Let (x, y, z) be the position of the negative pole of one of the elementary magnets of the body, $(x + lr,\ y + mr,\ z + nr)$ the position of the positive pole. If μ is the strength of the positive pole, the components of moment are

$$M_x = \mu lr, \quad M_y = \mu mr, \quad M_z = \mu nr.$$

If (H_x, H_y, H_z) is the magnetic force due to external causes at the point (x, y, z), the x-component of magnetic force at $(x + lr,\ y + mr,\ z + nr)$ is approximately

$$H_x + lr\frac{\partial H_x}{\partial x} + mr\frac{\partial H_x}{\partial y} + nr\frac{\partial H_x}{\partial z}.$$

Multiplying the components by $-\mu$ and μ and adding, the resultant force on the elementary magnet is found to be F, where

$$F_x = \mu lr\frac{\partial H_x}{\partial x} + \mu mr\frac{\partial H_x}{\partial y} + \mu nr\frac{\partial H_x}{\partial z}.$$

The coefficients μlr, μmr, μnr are the components of the magnetic moment M, and since the magnetic field is derivable from a potential, the above expression can be written

$$F_x = M_x\frac{\partial H_x}{\partial x} + M_y\frac{\partial H_y}{\partial x} + M_z\frac{\partial H_z}{\partial x}.$$

The total force on a small body is obtained by addition of the forces for its component magnets. Hence if v is the volume and I the intensity of magnetisation the force is given by

$$F_x = v\left(I_x \frac{\partial H_x}{\partial x} + I_y \frac{\partial H_y}{\partial x} + I_z \frac{\partial H_z}{\partial x}\right),$$

and two similar equations.

An important case is that in which I is parallel to the resultant magnetic field H, though not necessarily proportional to it. In this case we have

$$\frac{I_x}{I} = \frac{H_x}{H},$$

so that $$F_x = \frac{vI}{H}\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_y}{\partial x} + H_z \frac{\partial H_z}{\partial x}\right),$$

and since $H^2 = H_x^2 + H_y^2 + H_z^2$ this gives

$$\left.\begin{aligned} F_x &= vI\frac{\partial H}{\partial x}, \\ \text{Similarly} \quad F_y &= vI\frac{\partial H}{\partial y}, \\ \text{and} \quad F_z &= vI\frac{\partial H}{\partial z} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The resultant force is in the direction of greatest increase of H, which however does not usually coincide with the direction of H itself. From this result we draw the general conclusion that magnetic bodies* try to move as far as possible into the strongest parts of the field, an explanation which accounts for the attractions exerted by magnets on soft iron and other bodies in their neighbourhood.

111. Magnetic substances in general. Paramagnetism and diamagnetism. Although the substances already mentioned are the only ones known with strong magnetic properties, all bodies are magnetic to some degree. In view of this fact, it is as well to define the unit magnetic pole at the outset as that which repels a similar pole distant 1 cm. away *in vacuo* with a force of 1 dyne. The magnetisability of air, though small, is finite and measurable, and has to be taken into account in determinations of the magnetic properties of other substances.

* I.e. if paramagnetic: diamagnetic bodies tend to move to the weakest parts of the field.

Most weakly magnetic substances behave in a very simple manner in a magnetic field: the intensity of magnetisation I is proportional to H, and there is no hysteresis.

For isotropic bodies the directions coincide, and we can write

$$I = \kappa H \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where κ is a constant characteristic of the body, though depending of course on its physical state and temperature. κ is called the *susceptibility*. If it is positive the body is said to be *paramagnetic*: but there are also substances, called diamagnetic substances, which are magnetised by induction in a direction diametrically opposite to that of the impressed field.

Curie's method of determining κ consists in measuring the forces acting on small pieces of the substance when placed in a non-homogeneous magnetic field. The principles involved are illustrated by the following example. Suppose that we have a small hollow vessel capable of containing the solid, liquid or gas to be examined, the outside volume being v_1 and the inside volume v_2. Let κ_1 be the susceptibility of the material of the vessel, κ_2 of the substance under examination, and κ_3 of air under normal conditions. Suppose that the mechanical forces on the vessel are observed when it is (1) exhausted, (2) filled with air at normal temperature and pressure, and (3) filled with the substance, and let these be denoted by F_1, F_2, and F_3 respectively. In case (1) there is a force on the vessel arising from a volume $v_1 - v_2$ of a substance of susceptibility κ_1; but since a volume v_1 of air is displaced its effect has to be subtracted. Hence equation (3) of the last article gives

$$F_1 = \{(v_1 - v_2)\,\kappa_1 H - v_1 \kappa_2 H\} \left\{\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2\right\}^{\frac{1}{2}},$$

or, writing $1/\lambda$ for $H\{(\partial H/\partial x)^2 + (\partial H/\partial y)^2 + (\partial H/\partial z)^2\}^{\frac{1}{2}}$,

$$(v_1 - v_2)\,\kappa_1 - v_1 \kappa_2 = \lambda F_1 \ldots\ldots\ldots\ldots (5).$$

The results of the second and third experiments are similarly expressed in the equations

$$(v_1 - v_2)\,\kappa_1 - v_1 \kappa_2 + v_2 \kappa_2 = \lambda F_2 \ldots\ldots\ldots (6),$$

and

$$(v_1 - v_2)\,\kappa_1 - v_1 \kappa_2 + v_2 \kappa_3 = \lambda F_3 \ldots\ldots\ldots (7).$$

The equations (5), (6) and (7) give κ_1, κ_2 and κ_3 in terms of

measurable quantities. In a single experiment with a small solid body it is important to remember that what is really measured is the difference of the susceptibilities of the body and air, so that some such process as the above is necessary in order to obtain absolute values.

In Curie's experiments the substance to be examined was contained in a bulb C attached to a delicate torsion balance and lying between the poles of an electro-magnet. The magnetising coils are turned through a small angle, and C lies on the line of symmetry a little away from the strongest part of the field. The magnetic force at points on the line of symmetry is parallel to the axis of y but variable in the direction of the axis of x. Curie chose the point at which HdH/dx had its maximum value, in order that the force on the body might be only slightly altered by a small change in its position. The bulb C was moreover surrounded by an electric furnace so that observations could be taken at various temperatures. The following table gives the approximate value of κ for certain substances (at ordinary temperatures).

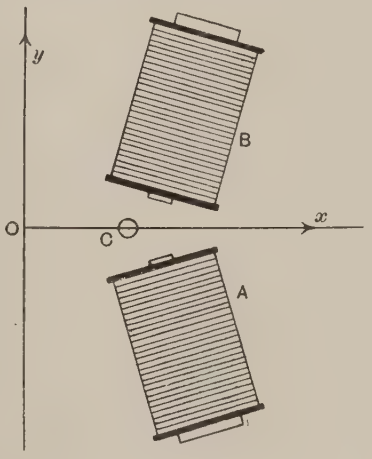


Fig. 152

Values of $10^7 \kappa$			
Paramagnetic substances		Diamagnetic substances	
Platinum ...	200	Bismuth	−130
Liquid oxygen ...	30	Mercury	− 26
Aluminium ...	18	Silver	− 15
Oxygen	1·6	Lead	− 11
Air	·32	Water	− 7·5
Carbon dioxide ...	·17	Copper	− 7
Hydrogen ...	·08		

Certain experiments seem to indicate a slight falling off of κ with increasing force for some paramagnetic substances, showing that the proportionality of I to H is not absolutely strict. This has not been observed with diamagnetic bodies. Townsend showed that solutions of iron salts are truly paramagnetic, and that, when allowance is made for the diamagnetism of the water of solution, the susceptibility is proportional to the mass of iron in solution.

Iron itself acts like a true paramagnetic body if demagnetised and only subjected to very small magnetic forces, the susceptibility being about 7.

We must pass over the interesting theories that have been proposed to account for the phenomena of induced magnetism, and make only a few general remarks. It is generally agreed that diamagnetism is a fundamental property of all matter, depending on a direct effect of the magnetic field on the atom. In certain substances the neighbouring molecules of a magnetised body influence one another in such a way as to mask the general diamagnetism and convert it into paramagnetism. The conditions under which this takes place are not perfectly understood: but they are evidently somewhat precarious and depend on some peculiarity in molecular structure which appears periodically in elements arranged in the order of their atomic weight. Apart from other difficulties, a satisfactory theory of induced magnetism would have to show why the possession of strong magnetic quality is so rare in the series of elements.

112. Effect of temperature on magnetisation. Stresses in magnetised bodies. The maximum intensity of magnetisation of iron is almost independent of the temperature, provided the latter does not exceed 600° C. Above this point it decreases rapidly, until it practically vanishes at about 780°, which is called the *critical temperature*. At this temperature, which coincides with the temperature of recalescence, iron seems to undergo considerable change in all its physical properties.

The effect of temperature on magnetism was carefully studied by Curie with a view to discovering the conditions under which a body was strongly magnetic, paramagnetic or diamagnetic,

and the steps in the change from one state to another. Above the critical temperature iron seems to be truly paramagnetic, the susceptibility however decreasing with rise of temperature.

The most important discovery made by Curie was the difference in the behaviour of paramagnetic and diamagnetic bodies. The susceptibility of many diamagnetic bodies is independent of the temperature, while paramagnetic bodies in general show a decrease at high temperatures. No general law holds for all paramagnetic substances, but an approximate law is that if ρ is the density at absolute temperature θ, then κ is proportional to ρ/θ. This law holds accurately for oxygen, and for nickel at high temperatures.

Considerable changes take place in the magnetisation of iron and other metals in given fields when subjected to compressive or tensile stress, the phenomena being in part very complicated. Similarly the mere fact of magnetising an iron rod produces strains in the rod, as was first observed by Joule in 1847. Initially the rod becomes longer, the increase of length assuming its greatest value of about $2\frac{1}{2}$ parts in a million for a field $H = 80$. On further increasing the force the elongation decreases, and finally changes sign, the final effect for very strong fields being a contraction of about one part in 200,000. The effect in cobalt is precisely opposite, namely contraction in weak fields and extension in strong fields, while nickel contracts constantly during the whole process.

Quincke found that a paramagnetic liquid, contained in a capillary tube, rose in the tube when a strong magnetic field was applied near the meniscus. This action is readily explained by the theory given in Art. 110 to account for the movement of magnetic bodies in non-homogeneous magnetic fields. Let A be the cross-section of the tube, and consider the forces on the slice of liquid between heights z and $z + dz$ above the normal surface. If ρ is the density and p the pressure at height z, the downward force due to gravity is $g\rho A\,dz$ and the thrust due to the difference of pressure on the faces is $A\,dp$. Again by equations (3) of Art. 110 the upward force due to the magnetic field is

$$vI\,\frac{dH}{dz} = A\,dz\,.\,\kappa H\,.\,\frac{dH}{dz}\,.$$

Hence for equilibrium

$$\kappa A H\, dH = g\rho A\, dz + A\, dp,$$

which on integration gives

$$\tfrac{1}{2}\kappa H^2 = g\rho z + p + \text{const.}$$

Let T be the surface-tension and a the radius of the meniscus. Then if h_1 is the height of the meniscus, and the normal surface of the liquid is in a region devoid of magnetic force,

$$\tfrac{1}{2}\kappa H^2 = g\rho h_1 - \frac{2T}{a},$$

where H stands for the magnetic force opposite the meniscus. If h_2 is the height without any magnetic field,

$$0 = g\rho h_2 - \frac{2T}{a}.$$

Subtracting and writing $h = h_1 - h_2$ for the rise of level caused by the field, we have

$$\kappa H^2 = 2g\rho h.$$

Thus κ can be found by measurements of h in known fields, and this was done by Quincke for numerous liquids.

113. Magnetic force and magnetic induction. Poisson's analysis of magnetism (Art. 20) shows that the magnetic effect of a body at an external point P is the same as that of certain volume- and surface-distributions. When the point P is inside the body there is some doubt as to how to define the magnetic force. We shall take it as that due to the surface-distribution σ over the external boundary of the magnetised body and the volume-distribution ρ, which may of course extend right up to P without giving rise to difficulty. With this understanding there exists a magnetic potential Ω, and the components of magnetic force are given by

$$H_x = -\partial\Omega/\partial x, \quad H_y = -\partial\Omega/\partial y, \quad H_z = -\partial\Omega/\partial z \ldots (8).$$

Further, we may apply Poisson's equation (Art. 32) and obtain the result

$$\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2} = -4\pi\rho = 4\pi\left(\frac{\partial I_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z}\right) \ldots (9).$$

We shall now define the important vector B having the components

$$B_x = H_x + 4\pi I_x, \quad B_y = H_y + 4\pi I_y, \quad B_z = H_z + 4\pi I_z \ldots (10).$$

The vector B is called the *magnetic induction*, and the last equation may be written $B = H + 4\pi I$, where addition signifies addition of rectangular components. Further, it follows easily from (8) and (9) that

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \ldots\ldots\ldots\ldots (11).$$

In vacuo, where $I = 0$, B is of course identical with H; and this is practically true for air, for which I in the strongest fields ($H = 40{,}000$) is only about ·0013. The vectors H and B enjoy characteristic properties, which are very important in the theory. Thus from (11) it follows that B is always solenoidal, but in general is not derivable from a potential. On the contrary, H is derivable from a potential, but in general is not solenoidal.

The vectors B and H are exactly analogous to the vectors D and E occurring in the theory of dielectrics (Art. 50), and much of the above theory agrees word for word with the corresponding theory in electrostatics.

The reader will appreciate the full importance of the vector B after reading the next chapter: but in order to justify its introduction we may say that it is the determining quantity in a large class of important phenomena on which most of the modern science of electrical engineering is based. Further, methods can be devised, based on the theory of the next chapter, for measuring B directly without previous knowledge of I. In the case of iron, where I is large in comparison with H, B is nearly equal to $4\pi I$, so that if I is given we obtain B without serious error by multiplying by 12·6.

114. Tubes of force and tubes of induction. A line of induction is defined as a line giving at every point the direction of B at that point, just as in the analogous case of a line of force. Similarly we can imagine tubes of induction drawn filling the magnetic field (cf. Art. 18). The property of lines of magnetic force there proved, namely that the product of magnetic force and cross-section of a small tube is constant, fails when we pass

into regions containing poles. Lines of magnetic force, in fact, begin or end on Poisson's equivalent magnetic distributions. On the other hand, if a small tube of induction is drawn the product of B and the cross-section of the tube is constant, since this fact depends simply on the solenoidal property enjoyed by B even inside magnetised bodies. Thus lines of induction never begin or end, but circulate in closed curves.

The figure shows diagrammatically the lines of force and lines

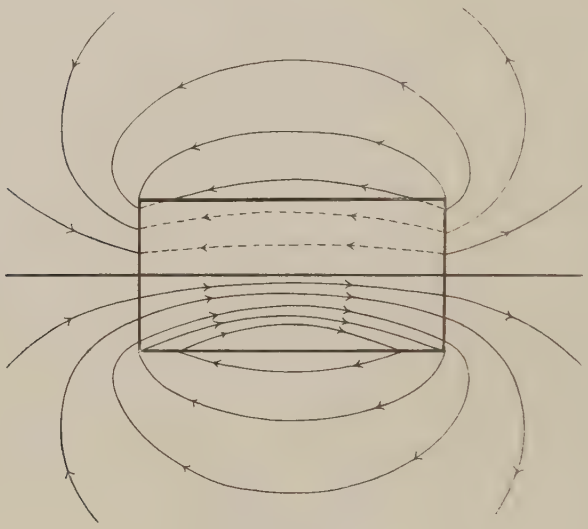


Fig. 153

of induction for a short bar-magnet magnetised from left to right. Only one half of each diagram is shown, the top giving the lines of force and the bottom the lines of induction. In the space outside the magnet the lines of force and induction coincide. Inside, we must remember that H is the force due to the impressed field (if any) and Poisson's magnetic distributions, so that in this case the lines of force and induction run for the most part in opposite directions. This is of course entirely due to the property of remanence: if B was simply proportional to H and hysteresis was absent the lines of force and induction would coincide.

When the material is in the form of a long rod the lines of

magnetic force due to the ends of the rod hardly pass down the rod at all, so that the impressed field is the actual field. Thus with long rods we get the full magnetic force magnetising the rod, while in short rods the opposition of the magnetic force due to the ends has a demagnetising effect.

115. Conditions at the boundary of two media. Magnetic screening. Bearing in mind the analogy of electric and magnetic force and induction, the reader will have no difficulty in proving the following boundary conditions at the surface of separation of two different magnetic media (cf. Arts. 49, 50):

(1) The tangential component of magnetic force is continuous;

(2) The normal component of magnetic induction is continuous, on crossing the boundary.

These conditions necessitate refraction of both lines of force and lines of induction in crossing the boundary. They also give definite boundary conditions which correspond to the condition of constant potential all over a conductor in electrostatics, thereby enabling us to find the magnetic force theoretically in certain cases.

We shall illustrate this by considering the case of a spherical shell of paramagnetic material lying in a uniform magnetic field, which more or less illustrates the behaviour of a soft iron shell under the same conditions. Take the centre of the shell as origin, and the direction of the field as axis of a system of polar co-ordinates in space. The inner radius of the shell is taken as b, the outer radius as a, and the susceptibility of the material of the shell κ. The magnetic potential will be represented by different analytical expressions in the three regions: let it be Ω_1 inside the shell, Ω_2 in the material of the shell, and Ω_3 outside. Evidently Ω_1 and Ω_3 satisfy Laplace's equation, i.e.

$$\frac{\partial^2\Omega_1}{\partial x^2}+\frac{\partial^2\Omega_1}{\partial y^2}+\frac{\partial^2\Omega_1}{\partial z^2}=0 \quad \ldots\ldots\ldots\ldots(12)$$

and

$$\frac{\partial^2\Omega_3}{\partial x^2}+\frac{\partial^2\Omega_3}{\partial y^2}+\frac{\partial^2\Omega_3}{\partial z^2}=0 \quad \ldots\ldots\ldots\ldots(13).$$

We can show that Ω_2 also satisfies Laplace's equation: for since

$B_x = H_x + 4\pi I_x = (1 + 4\pi\kappa) H_x = -(1 + 4\pi\kappa) \dfrac{\partial \Omega_2}{\partial x}$, and κ is the susceptibility, the solenoidal condition

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

gives

$$\frac{\partial^2 \Omega_2}{\partial x^2} + \frac{\partial^2 \Omega_2}{\partial y^2} + \frac{\partial^2 \Omega_2}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (14).$$

The condition of continuity of the tangential component of magnetic force at the inner shell gives

$$\left(\frac{\partial \Omega_1}{\partial \theta}\right)_{r=b} = \left(\frac{\partial \Omega_2}{\partial \theta}\right)_{r=b} \quad \ldots\ldots\ldots\ldots\ldots (15).$$

Similarly the continuity of the normal component gives

$$\left(\frac{\partial \Omega_1}{\partial r}\right)_{r=b} = (1 + 4\pi\kappa) \left(\frac{\partial \Omega_2}{\partial r}\right)_{r=b} \quad \ldots\ldots\ldots\ldots (16).$$

The conditions holding at $r = a$ are found in just the same way to be

$$\left(\frac{\partial \Omega_2}{\partial \theta}\right)_{r=a} = \left(\frac{\partial \Omega_3}{\partial \theta}\right)_{r=a} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

and

$$(1 + 4\pi\kappa) \left(\frac{\partial \Omega_2}{\partial r}\right)_{r=a} = \left(\frac{\partial \Omega_3}{\partial r}\right)_{r=a} \quad \ldots\ldots\ldots (18).$$

Finally for large distances from the origin the field has to tend to uniformity, i.e.

$$\lim_{r=\infty} \Omega_3 = -Hz = -Hr \cos\theta \quad \ldots\ldots\ldots (19),$$

where an arbitrary constant can be put on the right if desired. The mathematical problem is to find expressions for Ω_1, Ω_2, Ω_3 which satisfy all these conditions. Try a solution of the form

$$\left.\begin{aligned} \Omega_1 &= Ar \cos\theta \\ \Omega_2 &= Cr \cos\theta + \frac{D \cos\theta}{r^2} \\ \Omega_3 &= -Hr \cos\theta + \frac{E \cos\theta}{r^2} \end{aligned}\right\}.$$

Then all the conditions are easily seen to be fulfilled if the constants A, C, D, E satisfy the equations

$$\left.\begin{aligned} A &= C + D/b^3 \\ A &= (1 + 4\pi\kappa)\,(C - 2D/b^3) \\ C + D/a^3 &= -H + E/a^3 \\ (1 + 4\pi\kappa)\,(C - 2D/a^3) &= -H - 2E/a^3 \end{aligned}\right\},$$

which are sufficient to determine the constants.

The figure shows the lines of induction* in this case for the

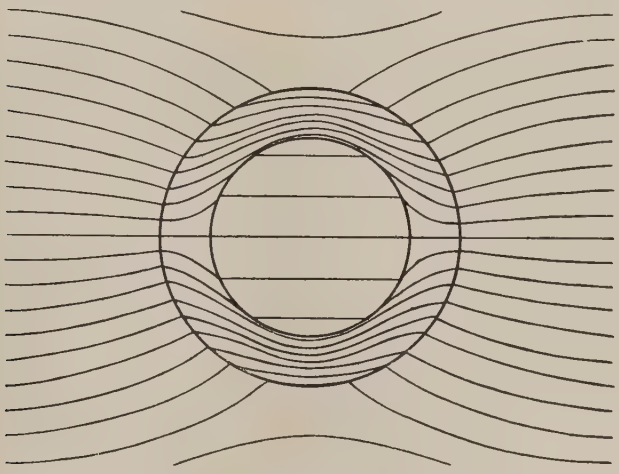


Fig. 154

value $\kappa = 1$, b/a being taken as 2/3. The theory is of course not really applicable in such cases, because such high magnetisability is only found in *ferromagnetic* substances, i.e. substances for which I is not proportional to H. The results are however qualitatively true for soft iron under moderate magnetising forces, and give us some idea of the trend of the lines of induction *inside* the metal, which are not directly observable. The point to be especially noticed is the tendency of the lines of induction to crowd in the material of the shell, a fact which results in scarcity of lines of induction inside the shell. Remembering that the density of

* The equation of the lines of force (or induction) corresponding to the potential $\Omega = Ar\cos\theta + B\cos\theta/r^2$ is $Ar^2\sin^2\theta - 2B\sin^2\theta/r = \text{const.}$

lines of induction (in air) is a measure of the strength of the field, we see that the magnetic force inside the shell is small compared with the impressed field. This is expressed by the statement that soft iron acts as a screen against external magnetic action, but the screening power is not so perfect as that possessed by an earthed conductor in electrostatics. In any case an iron sphere for use as a magnetic screen must be well annealed, carefully demagnetised before use, and not exposed to very great magnetic forces. Galvanometers have been enclosed in soft iron spheres to protect them from disturbance.

116. Work done in taking a unit magnetic pole round a current in a magnetic medium. The general work theorem requires no modification when the current is in a region containing magnetisable bodies. In all cases the work done by the magnetic force when a unit pole is taken round a closed circuit is $4\pi i$ if the circuit interlaces the current positively, $-4\pi i$ if negatively, and 0 if not at all. The total magnetic force H consists of two parts, H_0 due to the current and H_1 due to Poisson's equivalent magnetic distributions. The latter is derived from a single-valued potential, so that no work is done by it in any closed circuit. Hence the total work of H is equal to that of H_0, which proves the result. Expressed analytically, the integral

$$\int H_x dx + H_y dy + H_z dz$$

taken round a closed contour C is 4π times the current flowing through C in a positive direction with reference to the direction of integration. It is important to notice that the theorem does not hold in general when B is substituted for H.

117. Potential energy of a current in a magnetic medium. It is B, on the other hand, and not H, that determines the mechanical forces on the current. We can prove that the potential energy of a circuit carrying a current i is $-iN$, where N is the flux of *induction* through the circuit in the positive sense. By this we mean that the work done by the magnetic system on the current in any displacement of the latter is i times the increase of N, the current and the magnetised bodies being understood to be kept unchanged during the process.

Fig. 155 shows a section of the field, the current being supposed to come up through the paper at C_1 and pass down through it at C_2. S is a surface with the circuit as rim, and an air-space is

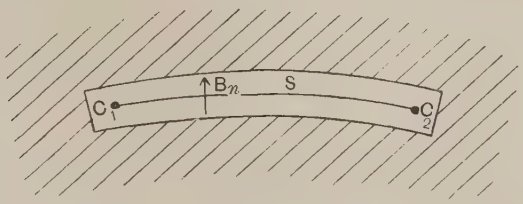


Fig. 155

cleared round S by surfaces parallel and near to it. The removal of the indefinitely small amount of magnetic matter involves negligible work, and we can suppose that the magnetic state of the rest of the body is unaffected by this removal. The circuit and the cap S are now in a region free from magnetic matter, so that the potential energy is $-iN$, where N is the flux of magnetic force through the circuit. But it follows from Art. 115 that the normal component of magnetic force in the gap is equal to B_n the normal component of magnetic induction in the neighbouring parts of the magnetic matter: hence the potential energy is $-i\int B_n dS$, which was to be proved.

The concentration of lines of magnetic induction in iron cores may be utilised to increase the mechanical forces acting on a circuit slipped over these cores, as in the case of the moving-coil galvanometer.

CHAPTER VIII

INDUCTION OF CURRENTS

118. Faraday's experiments. In 1831 Faraday showed that currents could be produced in closed circuits not containing any battery by making or breaking other circuits in their vicinity, and by the motion of circuits and magnets relative to one another. The qualitative laws of the phenomenon were completely unravelled by Faraday in two memoirs*, which well repay careful study on account of the exhaustive nature of the experiments and the admirable clearness with which they are described. Faraday first prepared two solenoids of many turns of wire on the same core, insulated from one another, one being connected to a battery and the other to a sensitive galvanometer. There was not the slightest deflexion of the galvanometer while a steady current flowed through the battery circuit; but on first closing the circuit the galvanometer showed a momentary deflexion, and again in the opposite direction when the current was cut off. These temporary induced currents were found to be capable of magnetising steel needles. As regards the direction of the induced currents, imagine the solenoids straightened out so as to form parallel wires A, B (Fig. 156). Then on closing the battery circuit (or *primary*) the current in the galvanometer circuit (or *secondary*) was in the

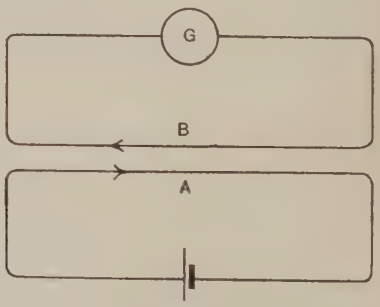


Fig. 156

* Faraday, "Experimental Researches in Electricity," Ser. 1, *Phil. Trans.*, vol. 122, p. 125 (Nov. 24th, 1831), Ser. 2, *Phil. Trans.*, vol. 122, p. 163 (Jan. 12th, 1832).

opposite direction as shown in the figure: on cutting off the current in the primary that induced in the secondary was in the direction of the original primary current. Precisely similar effects were obtained on moving the circuits relatively to each other. Thus if the wire A, carrying a constant current, approached the wire B it induced a current in the opposite direction as long as the motion continued, the current being reversed when the wire receded.

Much greater effects were obtained with smaller currents by winding the primary and secondary helices over the same closed iron ring (Fig. 157). In order to elucidate the effect of the iron ring

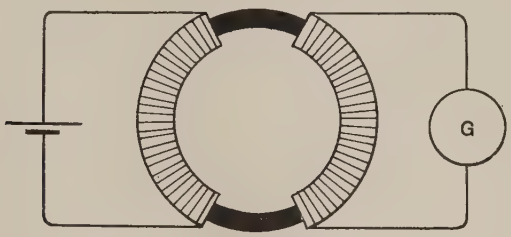


Fig. 157

Faraday wound two solenoids on a hollow pasteboard cylinder, in which metallic cores could be inserted. A copper core had no effect on the magnitude of the induced current, while an iron core produced a great increase, but relatively not so much as a complete ring. Further, currents were induced by suddenly inserting a bar-magnet into the core, the direction of the momentary current in the solenoid being related to that of the axis of the magnet by the *left-handed* screw law. An equal and opposite deflexion occurred on suddenly removing the magnet.

It appears from the preceding that the effects obtained from currents and magnets are similar in kind, the only requisite being the presence of a magnetic field. It occurred to Faraday that the earth's magnetic field might be utilised, and he succeeded in fact in obtaining currents by turning a coil of wire suddenly through two right angles in the earth's field.

Faraday was further able to show the induction of currents in solid masses of metal moving in a magnetic field and thus to

construct the first machine for the mechanical generation of electricity. In one experiment a metal disc D rotated in the direction shown in a magnetic field, and conducting brushes were pressed against the axle and circumference. On joining these brushes by a wire a steady current was found to flow in the wire from the circumference inwards, so that the current in the disc itself was radially outwards. The effect was greatest when the resultant field of the earth was perpendicular to the disc, and vanished when the disc was in the magnetic meridian, suggesting that the current is proportional to the component H of magnetic force perpendicular to the plane of the disc. Faraday also obtained currents from the rotation of a bar-magnet in a similar manner, by making connexion with the side and end of the magnet respectively. Here the magnet acts at once as the producer of the field and as the moving conductor.

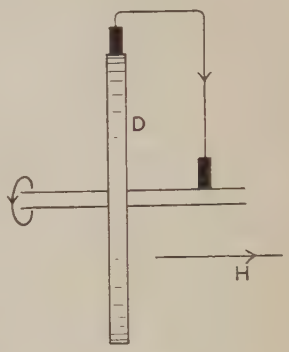


Fig. 158

By using wires of different metal in the ordinary induction experiments Faraday came to the conclusion that the currents were in the ratio of the conducting powers of the metals, and that the "force" (electromotive force of induction) was, *caeteris paribus*, independent of the nature of the wire. The condition necessary for the induction of currents was clearly enunciated by Faraday, namely that there should be a cutting of "magnetic curves" by the circuit; or in modern phraseology that there should be a change in the flux of magnetic induction through the circuit.

119. The law of electromagnetic induction. The exact laws of electromagnetic induction were first enunciated by F. E. Neumann in 1845. We shall consider the case of a single circuit, for which the law is given by considerations of energy. Let us have a circuit C of self-inductance L (Art. 87) carrying a current i at time t, in the presence of a system of magnetic poles. For the sake of generality suppose that it contains a cell of E.M.F. V as well as being affected by the motion of the magnets. From

what we know of the mechanism of the voltaic cell it appears likely that the energy changes in the cell during the passage of a given charge round the circuit are the same under all circumstances, so that in time dt work $Vidt$ is done on the current by the cell. This work is expended in three ways:

(1) In heating the circuit. It appears likely from the electronic theory of conduction that this amounts to Ri^2 in work-units per second in all cases, so that work Ri^2dt is used up in this way.

(2) When the relative position of the circuit and the magnetic system changes, the forces between the circuit and the system do work idN, where N is the flux of magnetic induction through the circuit due to the magnetic system (Art. 79).

(3) When the current is changing the work done by the forces between the elementary filaments of the current is $d(\frac{1}{2}Li^2) = Lidi$, at any rate for a rigid circuit. Hence we have

$$Vidt = Ri^2dt + idN + Lidi,$$

or

$$Ri = V - \frac{dN}{dt} - L\frac{di}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now Li is in a certain sense the flux of magnetic force (or induction) through the circuit due to the current itself, being the mean flux through elementary filaments carrying equal currents. Hence the total flux *due to all causes, including the current itself*, is $N + Li$, and writing this briefly N, we have the following enunciation of the law of electromagnetic induction:

In any circuit electromagnetic induction gives rise to an electromotive force equal to the rate of decrease of the flux of magnetic induction through the circuit.

In this enunciation the flux of induction is supposed to have sign as well as magnitude (cf. Art. 79), and electromotive force is reckoned as positive when it tends to send a current round the circuit in the standard direction. It will be seen from the last article that the directions of the currents in Faraday's experiments are all correctly accounted for by this theory. The case of the rotating disc (Fig. 158) is not quite so clear: but the question will be considered later on the lines of the theory of electrons (Art. 137), which throws additional light on the question of induction in moving conductors.

When there is more than one circuit considerations of energy do not suffice to determine the laws, as only one equation is obtained in this way. We shall however assume that the law of electromagnetic induction here enunciated holds for each circuit separately; that is, the induced E.M.F. is $-dN/dt$, where N is the flux of magnetic induction due to all causes, inclusive of the circuits present in the field.

In the case of two circuits let L_1, L_2 be the self-inductances, M the mutual inductance, R_1, R_2 the resistances and i_1, i_2 the currents. The flux of magnetic induction through the first circuit is $L_1 i_1$ due to itself and $M i_2$ due to the second circuit, so that the E.M.F. of electromagnetic induction is

$$-L_1 di_1/dt - M di_2/dt.$$

It follows that if in addition electromotive forces V_1, V_2 are acting in the two circuits, the equations for the currents are

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + R_1 i_1 &= V_1 \\ M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} + R_2 i_2 &= V_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2).$$

The importance of self and mutual inductance will now be understood, as their presence modifies the effects with changing currents very greatly, and for very rapid changes the phenomena are generally determined much more by inductance than by resistance.

120. Experimental verifications of the theory. If a sensitive ballistic galvanometer is available, some of the laws of electromagnetic induction may be easily verified in the laboratory by a compensation method due to Felici. A (Fig. 159) is a coil of about 20 turns of thick wire mounted on a board, the dimensions being about 50×40 cm. The coil X consists of a single turn of wire stretched round four small brass screws at the corners, so that it can be replaced by another wire of different material or diameter, but of the same configuration. B represents a small solenoid of thick wire in three layers, each layer having about 10 turns to the centimetre; and Y is a small secondary coil of about 20 turns sliding inside B, and wound on a wooden core of

about 2 sq. cm. sectional area. The solenoid C and core Z, to which we shall refer later, are similar to B and Y.

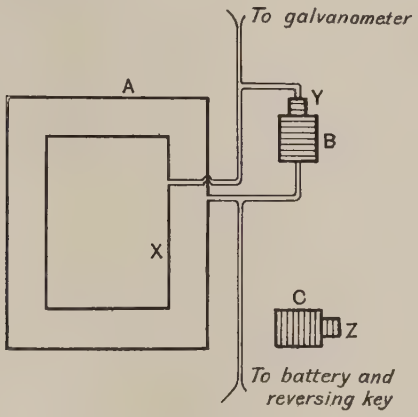


Fig. 159

In order to obtain a fair test of the theory, it is necessary to ensure that there shall be no flux of induction through the leads, and also that the three pairs of coils do not act inductively on one another. The latter condition is practically secured by the arrangement of the coils in the figure, and the former by making all the connexions of flexible wire whose two strands are twisted round one another, shown in the figure by the parallel lines near together.

The core Y is first removed from B and a current of about 10 amperes passed through A and B in series. On reversing there will be a throw of the galvanometer due to the inductive effect of A on X. If Y is now inserted the effect of B on it is added, and the throw is greater or less according as the two electromotive forces assist or counteract each other. We can arrange that the latter shall be the case and also adjust the position of Y inside B so that exact compensation occurs and the galvanometer is not affected by any changes in the battery circuit. The pair of coils C, Z is then adjusted in the same way (the connexions not being shown), and we are now ready to make the following experiments.

(1) With the connexions of the figure, replace the wire X by one of the same configuration but different material, or different cross-section. The galvanometer still shows no deflexion. Hence the E.M.F. of induction is independent of the nature and cross-section of the wire forming the circuit X, as we should expect from the theory.

(2) Interchange the connexions of A and X, so that X is in the battery circuit and A in the galvanometer circuit. No throw occurs on making or reversing the current, showing that the inductive effect of A on X for a particular current is equal to that of X on A.

(3) Connect the primary coils B, C in parallel so that the current through A divides between B and C, and arrange the three secondary coils X, Y, Z in series. Again there is no throw of the galvanometer under any circumstances. It follows that the inductive effects of B on Y and of C on Z, which are equal for the same currents, are the same when the current is divided between them. Hence the inductive effect is proportional to the current.

121. Rule for calculating self-inductances. A convenient rule can be given for the self-inductance of a wire of circular cross-section bent into the form of a curve whose radius of curvature is large compared with the radius of the cross-section.

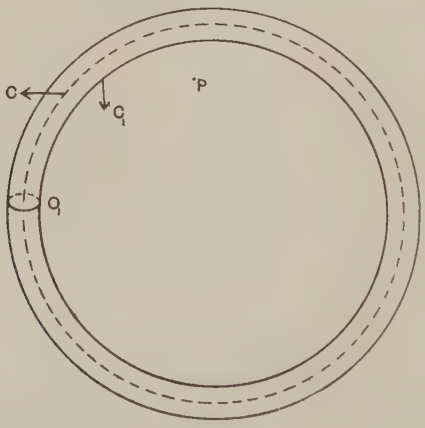


Fig. 160

a few isolated points or corners excepted. Consider in the first place a circular circuit as shown, C being the line of centres of the cross-sections and C_1 the inner rim of the wire. If P is any point inside the circle C_1 it is easy to show that the magnetic force at P can be calculated without serious error by concentrating the current in the wire along its axis C. This is evidently true if P is not near the wire: and if P is close to the wire the greater part of the magnetic force arises from the neighbouring part of the circuit, which acts on it practically like a long straight wire carrying a current. If P is at a short distance r from the line of centres the magnetic force is therefore nearly equal to $2i/r$, which is again the same as if the current i was concentrated along the line of centres.

Now take a particular cross-section of the wire as shown in Fig. 161, and let dS be an element of area distant r from the

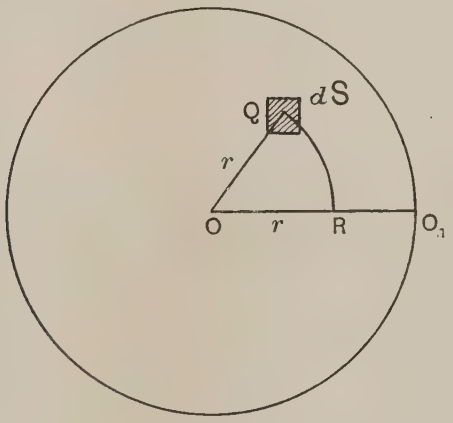


Fig. 161

centre. Let unit current flow in the wire, and let M be the mutual inductances of the circuits C and C_1. The flux of magnetic force N through the filament on dS is equal to M together with the flux through the band-like space between the filament and C_1. As before, the magnetic force inside is practically all due to the neighbouring part of the current, being therefore $2\rho/a^2$ at distance ρ from O, where a is the radius of the cross-section.

Let l be the total length of the wire. Then since the lines of magnetic force at points in the cross-section are circles with O as centre, all lines of force passing between dS and O_1 pass between R and O_1, and the flux of force between the filament and C_1 is equal to

$$l\int_r^a \frac{2\rho}{a^2}\,d\rho = \frac{l\,(a^2-r^2)}{a^2}.$$

Hence
$$N = M + \frac{l\,(a^2-r^2)}{a^2}.$$

Referring to the definition of the self-inductance given in Art. 87 and remembering that since the unit current is spread uniformly over the cross-section of the wire $di = dS/\pi a^2$, we have

$$L = \frac{1}{\pi a^2}\int N dS.$$

Hence
$$L = \frac{M+l}{\pi a^2}\int dS - \frac{l}{\pi a^4}\int r^2 dS$$

$$= M + l - \frac{l}{\pi a^4}\int_0^a 2\pi r^3 dr$$

$$= M + \tfrac{1}{2}l.$$

The method of reasoning is evidently general and applies to circuits of any shape. Hence we have the following rule of calculation:

Draw the line of centres of the cross-sections of the wire and also draw a line parallel to it in the surface of the wire. Calculate the mutual inductance of these two circuits and add to it half the total length of the wire. The result is the self-inductance of the circuit.

The preceding calculations give the inductance in absolute electromagnetic units. In the derived system the unit of inductance is the *henry*, or 10^9 true electromagnetic units. A circuit of inductance one henry is such that an E.M.F. of one volt is caused when the current is changing at the rate of one ampere per second: for in this case $di/dt = 1/10$ absolute E.M.U., and if $L = 10^9$ the E.M.F. is 10^8 E.M.U. $= 1$ volt. The henry is a rather large unit, and small inductances are often expressed in true electromagnetic units.

122. Calculation of self-inductance in particular cases.

(1) *Rectangular circuit.* Using the result of the last article, M is found by Neumann's formula (Art. 93). We begin by finding the value of $\iint ds\,ds'/r$ for two equal and parallel lines of length l distant h apart. Writing J for this double integral, we have

$$J = \int_0^l \int_0^l \frac{dx\,dy}{\{(x-y)^2 + h^2\}^{\frac{1}{2}}} = \int_0^l dx \left[\sinh^{-1}\frac{y-x}{h}\right]_{y=0}^{y=l}$$

$$= \int_0^l \left[\sinh^{-1}\frac{l-x}{h} + \sinh^{-1}\frac{x}{h}\right] dx = 2\int_0^l \sinh^{-1}\frac{x}{h}\,dx.$$

The integral is evaluated by the substitution $x = h\sinh\theta$, giving

$$J = 2\left[-(l^2+h^2)^{\frac{1}{2}} + l\log\frac{l+(l^2+h^2)^{\frac{1}{2}}}{h} + h\right].$$

Considering the two rectangles in Fig. 162, which corresponds to Fig. 160, we see that M receives no contribution from mutually

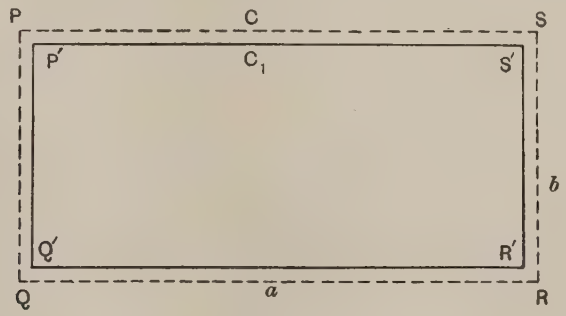


Fig. 162

perpendicular sides (for which $\epsilon = \frac{1}{2}\pi$), but that

$$M = \iint \pm\, ds\,ds'/r$$

for pairs of parallel sides only. Let the central line C be a rectangle (a, b) and let r be the (small) radius of the cross-section of the wire used. Then the contribution of PQ and $R'S'$ to M, which is negative, is

$$2\left[(a^2+b^2)^{\frac{1}{2}} - a - b\log\frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}\right].$$

The contribution of PQ and $P'Q'$ is

$$2\left[r - (b^2+r^2)^{\frac{1}{2}} + b\log\frac{b+(b^2+r^2)^{\frac{1}{2}}}{r}\right],$$

which is approximately $2\left\{-b+b\log\frac{2b}{r}\right\}$, since r is small. The sum of these terms gives the part of M arising from the side PQ, and dealing similarly with the other sides we find

$$M=4\left\{-2a-2b+2(a^2+b^2)^{\frac{1}{2}}+a\log(2a/r)+b\log(2b/r)\right.$$
$$\left.-a\log\frac{a+(a^2+b^2)^{\frac{1}{2}}}{b}-b\log\frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}\right\}.$$

The self-inductance is obtained by adding $a+b$. If we write d for the diagonal of the rectangle, $c=a+b$ for the semi-perimeter, we find a convenient formula due to Mascart, namely

$$L=9{\cdot}210\left\{c\log_{10}\frac{2ab}{r}-a\log_{10}(a+d)-b\log_{10}(b+d)\right\}-7c+8d\ldots(3)$$

absolute electromagnetic units.

Thus the self-inductance of a square circuit of side one metre, made of round-sectioned wire 1 mm. in diameter, is 5660 electromagnetic units.

(2) *Circular coil of wire of one turn.* Let a be the radius of the circle into which the wire is bent, r that of the cross-section of the wire itself. We have first to find an approximate expression for the mutual inductance of two concentric circles of radii a and $a-r$, when r is small. The formula for any radii is given in Art. 94. Here $k^2=4ab/(a+b)^2$ tends to unity; and if we write k' for $(1-k^2)^{\frac{1}{2}}$ we have $k'=(a-b)/(a+b)=r/2a$ nearly. The elliptic integral E tends to unity, while K tends to infinity. We have

$$K=\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{-\frac{1}{2}}d\phi$$
$$=\int_0^{\frac{\pi}{2}}(k'^2+k^2\cos^2\phi)^{-\frac{1}{2}}d\phi=\int_0^{\frac{\pi}{2}}(k'^2+k^2\sin^2\phi)^{-\frac{1}{2}}d\phi$$
$$=\int_0^{\frac{\pi}{2}}(k'^2+\sin^2\phi)^{-\frac{1}{2}}d\phi \text{ approximately.}$$

Since k' is small, the quantity under the integral becomes large when $\phi=0$. We shall therefore split up the range of

integration into two parts, namely 0 to α and α to $\frac{1}{2}\pi$, where α, though small, is large compared with k'. Thus

$$K = \int_0^{\alpha} (k'^2 + \sin^2\phi)^{-\frac{1}{2}}\, d\phi + \int_{\alpha}^{\frac{\pi}{2}} (k'^2 + \sin^2\phi)^{-\frac{1}{2}}\, d\phi$$

$$= \int_0^{\alpha} (k'^2 + \phi^2)^{-\frac{1}{2}}\, d\phi + \int_{\alpha}^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi}$$

$$= \sinh^{-1}(\alpha/k') - \log\tan\tfrac{1}{2}\alpha$$

$$= \log(2\alpha/k') - \log(\tfrac{1}{2}\alpha)$$

$$= \log(4/k') = \log(8a/r).$$

Hence $$M = 4\pi a\,(K - 2E) = 4\pi a\left(\log_e \frac{8a}{r} - 2\right),$$

and thus $$L = 4\pi a\left(\log_e \frac{8a}{r} - \frac{7}{4}\right).$$

Putting this in a form suitable for numerical calculation, it becomes

$$L = a\left(28{\cdot}93 \log_{10} \frac{8a}{r} - 21{\cdot}99\right) \ldots\ldots\ldots\ldots (4).$$

For example, a circular ring of radius 2 cm., of wire 2 mm. in diameter, would have by the formula a self-inductance of 83·5 electromagnetic units; and though the formula would not be very exact in this case it gives us an idea of the order of magnitude of the self-inductance of small loops (10^{-7} henry). The effect of thinning down the wire is very noticeable: thus if the wire was a mere filament of diameter 1/10 mm., the self-inductance of the same loop would be 158·8 absolute E.M.U.

123. Growth of current on closing a circuit. Let the self-inductance and resistance of the circuit be L and R respectively, and suppose that an E.M.F. of constant magnitude V is applied at time $t = 0$. Then the equation for the current at any subsequent time is

$$L\frac{di}{dt} + Ri = V \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The solution of this is

$$ie^{Rt/L} = A + \frac{V}{R}e^{Rt/L},$$

where A is a constant. Since $i = 0$ when $t = 0$, we have $A = -V/R$, and

$$i = \frac{V}{R}(1 - e^{-Rt/L}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Thus the current creeps up to its final value V/R gradually, and the larger the ratio R/L the greater will be the rapidity of approach to this value.

Fig. 163 shows the effect of applying 5 volts to a circuit of

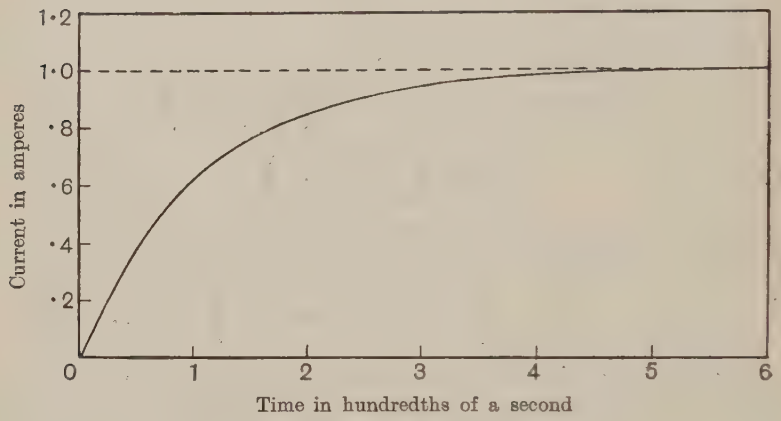


Fig. 163

resistance 5 ohms and inductance 1/20 henry. It will be noticed that the final value is almost exactly attained in one-twentieth of a second.

The time of attaining the steady state may be much greater if the coil is wound on an iron core or ring, as the changing induction in the core is added to that due to the current in the circuit. The effect is roughly equivalent to an increase in the self-inductance of the coil, but as the flux of induction is no longer proportional to the current we cannot assign any particular self-inductance to the coil. The current in an electro-magnet may take several seconds to attain its final value.

In order to avoid induction effects in resistance coils it is usual to bend the wires back on themselves before winding them on the bobbins (see Art. 63). This is called double winding.

The effect which occurs on breaking a circuit must be carefully

distinguished from what happens on closing it. A spark generally passes on opening a circuit containing an electro-magnet or a large inductance, and a shock can also be felt if the terminals of the coil are touched with the hands. To explain this, we notice that in the absence of inductance the current would fall exceedingly rapidly to zero. This however tends to give rise to a large E.M.F. of electromagnetic induction in the circuit, and it is easy to see that this acts in a direction favouring the continuance of the current. If the self-inductance is great enough the insulation of the air may break down, and the current continue for a short time after the key is pulled up.

124. Measurement of mutual and self-inductance. The following methods are suited for the measurement of comparatively large inductances (from 1/100 henry upwards).

(1) *Direct measurement of the mutual inductance of two coils.*

The primary circuit contains, in addition to the cell and coil, a reversing key, a resistance box and an accurate small resistance X (1 ohm or 1/10 ohm).

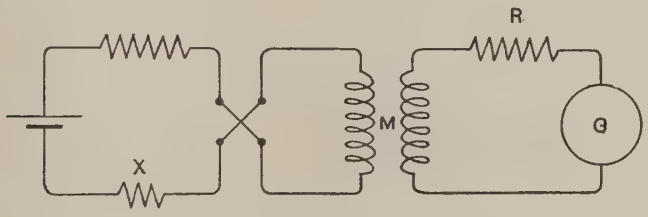


Fig. 164

Consider the general case in which currents i_1, i_2 flow in the two circuits. Then since the secondary contains no battery, we have

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + (R+G)\,i_2 = 0,$$

where G is the resistance of the galvanometer. Applying the equation to the short interval of time occupied in reversal of the primary current from i to $-i$, and integrating with respect to t over that interval, we have

$$2Mi + [L_2 i_2]_{\text{limits}} + (R+G)\int i_2 dt = 0.$$

But i_2 vanishes both initially and finally, and $\int i_2 dt$ is equal to e

the total charge sent through the galvanometer. Hence we have from the theory of the ballistic galvanometer

$$2Mi = (R + G)\frac{T}{2\pi}k\tau,$$

where τ is the corrected throw (Art. 83) and T the time of swing of the moving coil.

A second experiment consists in disconnecting the galvanometer from the secondary and shunting it across the small resistance X, adjusting the resistance S in series with it till there is a steady deflexion τ equal to the preceding ballistic throw. Then

$$Xi = (S + G)\,k\tau,$$

approximately. Dividing the last two equations, we have

$$M = \frac{X}{2}\frac{R+G}{S+G}\frac{T}{2\pi} \quad ..(7).$$

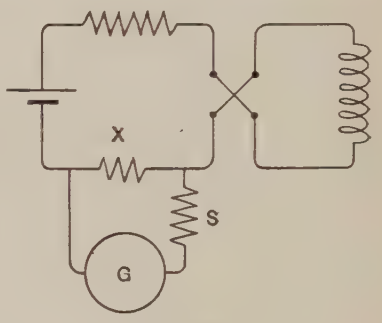


Fig. 165

Variable standards of mutual inductance can be obtained consisting of one fixed and one moveable coil, the mutual inductance being known in each position.

(2) *Direct measurement of self-inductance.*

The coil whose self-inductance is to be measured is placed in the arm AB of a Wheatstone's bridge in series with a box whose resistance can be increased by a small amount X (say 1/10 ohm), the other arms being adjustable and non-inductive (Fig. 166).

First balance the bridge accurately for steady currents. Let τ be the ballistic throw of the galvanometer on reversing the current, θ the steady deflexion caused by making the small increase X in the resistance of AB. Then

$$L = \frac{X}{2}\frac{\tau}{\theta}\frac{T}{2\pi} \quad .....(8).$$

To prove this, suppose that an E.M.F. V introduced into the arm AB of the bridge gives rise to a current λV in the galvanometer, where λ is some constant. Then if i_1 is the current in AB at

any time $V = Ldi_1/dt$, and a current $\lambda L di_1/dt$ flows through the galvanometer. The total charge passing through the galvano-

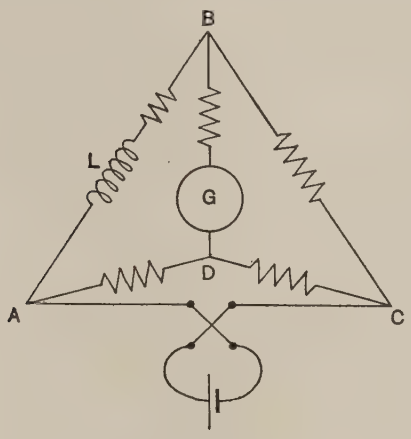


Fig. 166

meter during reversal is therefore $e = 2\lambda Li$, where the current in AB changes from i to $-i$ on reversal. Hence

$$2\lambda Li = \frac{T}{2\pi} k\tau.$$

A small increase X in the resistance of AB gives rise to an E.M.F. Xi in the arm AB and to a current λXi through the galvanometer. Thus

$$\lambda Xi = k\theta,$$

and (8) follows by division of the last two equations.

(3) *Comparison of the self-inductance of a coil with the capacity of a condenser.*

The self-inductance L is set up in one arm AD of a Wheatstone's bridge, and the capacity C (of the order of a microfarad) placed as a shunt across the opposite arm BC (Fig. 167). The bridge is first balanced for steady currents, and the balance must be made very exactly, if necessary by shunting round about 5 ohms of one of the resistances with an auxiliary box. On reversing there will in general be a ballistic throw, but this can be made to disappear by suitable choice of the absolute values of the resistances, always keeping their ratios so that the bridge is balanced for steady currents.

Let us find the condition that no current should pass through the galvanometer whatever occurs in the battery circuit. The

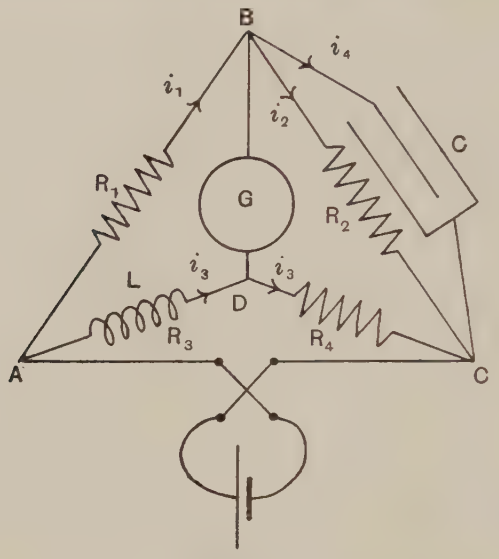


Fig. 167

currents in the arms being as shown in the figure, we have

$$L\frac{di_3}{dt} + R_3 i_3 = V_A - V_D = V_A - V_B$$
$$= R_1 i_1,$$
$$V_B - V_C = R_2 i_2 = R_4 i_3, \quad i_4 = C\frac{d}{dt}(V_B - V_C) = CR_4\frac{di_3}{dt}.$$

Substituting in the first equation, we have

$$Li_4 = CR_4(R_1 i_1 - R_3 i_3) = CR_1 R_4 i_1 - CR_2 R_3 i_2$$
$$= CR_1 R_4 (i_1 - i_2).$$

But since $i_4 = i_1 - i_2$ we have

$$L = CR_1 R_4 = CR_2 R_3 \ldots\ldots\ldots\ldots\ldots (9).$$

This gives L when C is known.

125. Measurement of magnetic fields with the ballistic galvanometer. The arrangements are somewhat different according as we wish to measure (1) fields of the order 5000 units, such as those occurring between the poles of an electro-magnet, (2) weak fields such as that of the earth. Fig. 168 shows the

connexions for performing both measurements in one circuit. MM represents the electro-magnet, through which a known current can be sent. A small coil of about 100 turns of thin wire lies between its poles, held loosely in a clamp so that it can be knocked out of the region of strong magnetic force. There is also in circuit an *earth inductor*, or large coil capable of being turned through two right angles round an axis. The coil should have about 1000 turns, and a diameter of at least 20 cm. G is a

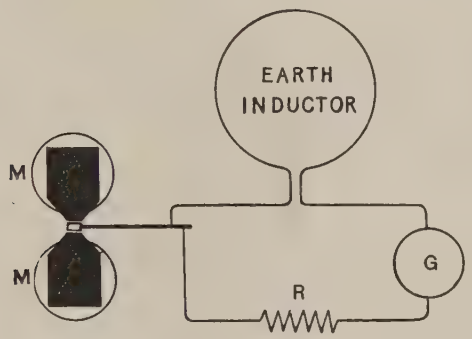


Fig. 168

sensitive ballistic galvanometer and R a resistance sufficiently large to render the damping of the galvanometer small (cf. Art. 128).

The electro-magnet, galvanometer and ammeter should be at some distance from the earth inductor in order to avoid disturbing the earth's field more than necessary. With the electro-magnet cut off first set the plane of the earth inductor magnetically east and west by the aid of a compass-needle. Then turn the coil of the inductor quickly through two right angles, producing a ballistic throw τ_1. Let A be the mean area of the coil, m the number of turns, and R the total resistance in the circuit, inclusive of the two coils and the galvanometer. If N is the flux of magnetic force through the circuit at any time and L its self-inductance, the current i is given by

$$L\frac{di}{dt} + Ri = -\frac{dN}{dt}.$$

Initially $N = mAH$, where H is the earth's horizontal component,

and finally $N = -mAH$. Hence we have by integration with respect to t

$$2mAH = R\frac{T}{2\pi}k\tau_1 \quad \ldots\ldots\ldots\ldots\ldots(10),$$

where T is the time of swing and k the galvanometer constant. A similar observation of the throw τ_2 when the earth inductor is supported horizontally gives the vertical component V, by the formula

$$2mAV = R\frac{T}{2\pi}k\tau_2 \quad \ldots\ldots\ldots\ldots\ldots(11).$$

Obtain a throw τ_3 by knocking out the small coil quickly from between the poles of the electro-magnet. If B is the mean area of the small coil, n the number of its turns, the field H' between the poles is given by

$$nBH' = R\frac{T}{2\pi}k\tau_3 \quad \ldots\ldots\ldots\ldots\ldots(12).$$

The galvanometer constant k being determined by standardising (Art. 68), H, V and H' are known from the last three equations. If everything is measured, as usual, in the derived units, the field-strengths will have to be multiplied by 10^8 to bring them to true electromagnetic units.

There is a general theorem covering the use of the ballistic galvanometer in all these cases, namely that if the flux of magnetic induction through the circuit due to external causes changes from N_1 to N_2, the total charge passing round the circuit is given by

$$e = (N_1 - N_2)/R \quad \ldots\ldots\ldots\ldots\ldots(13).$$

For we have $Ldi/dt + dN/dt + Ri = 0$, and i vanishes both initially and finally. Hence on integration we find

$$(N_1 - N_2) + R\int i\,dt = 0,$$

which is the above result.

Curie made an interesting application of these principles in determining dH/dx in his experiments (Art. 111 and Fig. 152). He placed a small coil at right angles to the axis of x in the position of the body C, and measured the ballistic throw caused by moving it a short distance h in the direction of the axis of y. Here

$$N_1 - N_2 = nAh\frac{\partial H_x}{\partial y},$$

where n is the number of turns of the coil and A its mean area. Since $\partial H_x/\partial y = \partial H_y/\partial x = dH/dx$, the total charge passing is $\frac{nAh}{R}\frac{dH}{dx}$, from which dH/dx can be found.

Wild's *induction inclinometer* is an example of the application of electromagnetic induction to the measurement of magnetic dip. It consists of a coil rotating about an axis whose direction can be varied until no currents flow on rotating the coil. This direction is then that of the resultant magnetic field of the earth.

126. Induction with permanent magnets. The law of electromagnetic induction enables us to find out how the flux of induction through a small coil varies as the coil is moved about, and hence to find the value of B at various points in the field of a magnet. We shall consider the variation of B along the axis of a cylindrical bar-magnet. A small coil of n turns and resistance R is wound on a bobbin sliding over the magnet, and connected in series with a resistance box and ballistic galvanometer so that the total resistance of the circuit is $R + G$. Let A be the area of the cross-section of the magnet, and let a ballistic throw τ occur when the coil is suddenly moved along from a place where the magnetic induction has the value B_0 to one where it has the value B_1. Then

$$nA\,(B_1 - B_0) = \frac{T}{2\pi}\,(R+G)\,k\tau \ldots\ldots\ldots (14).$$

The magnet to be tested is held in a clamp with the exploring coil slipped over it, and a wooden rod of the same diameter is held in the same line with the magnet with another clamp. In this way the measurement of B (which then becomes H) can be extended beyond the end of the magnet until its further changes are inappreciable. When the galvanometer is once standardised, equation (14) gives the difference in the values of B at any two points; and since B vanishes at a great distance from the magnet it is known at all points along the magnet. In practice, the values thus obtained are not quite accurate since the coil is slightly larger than the magnet and also has an appreciable length. But they indicate the continuous falling off of the flux of induction as we move away from the centre of the magnet that we should expect

from our knowledge of the distribution of the lines of magnetic induction (Art. 114). The following table gives the result of an experiment on a cast steel magnet (length 20 cm., diameter 6·5 mm.). If derived units are used throughout the measurements, the values of B obtained from equation (14) must be multiplied by 10^8 to reduce them to true electromagnetic units, in which they are usually expressed.

Distance from centre of magnet (centimetres)	0	2	4	6	8	10	12
B	6570	6360	5750	4640	2910	440	30

If $B_1 - B_0$ is known beforehand, an experiment of this kind gives the constant $\frac{T}{2\pi} k$ characteristic of the ballistic galvanometer. The principle of Hibbert's *magnetic standard* for this purpose is shown in Fig. 169. The outer shell is a piece of cast iron CC in cup form. A steel rod R is firmly screwed into this frame, and carries a cast iron plate PP separated from the shell only by a narrow annular gap. The whole is magnetised by a current sent round the steel rod, so that lines of magnetic induction run up the rod and across the narrow gap, returning through the outer shell. A sliding coil is then fitted so that it can be let fall through the gap after having first been raised to a definite position, so that the value of $B_1 - B_0$ is always the same for a particular instrument.

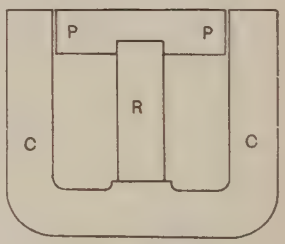


Fig. 169

Ballistic galvanometers can also be standardised by means of two coils of known mutual inductance, the current in one being reversed. In this case, however, an observation of the initial current is required as well. The method of the magnetic standard is the simplest both in theory and practice.

127. Measurement of magnetic induction. (H, B) curve. It is now evident that the value of B in any given case can be ascertained experimentally without previous knowledge of that of I. We proceed to show how to take off a hysteresis curve for a specimen of iron, showing the relation of B to the magnetising force H. The specimen to be examined is best made in the form of a ring so that there are no ends to exercise a disturbing influence, the magnetising coil consisting of a solenoid wound completely over it. Let m be the number of turns, r the mean radius of the ring. Then the magnetising force for a current of i amperes or $i/10$ electromagnetic units is given by $H = \frac{2m}{r}\frac{i}{10}$, or

$$H = mi/5r \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

It is advisable to use comparatively thin rings in order that the magnetic force may not vary very much over the cross-section.

The primary circuit contains, in addition to an ammeter and rough adjustable resistance, a resistance box S for high currents, shunted across a key. The secondary consists of n turns of wire

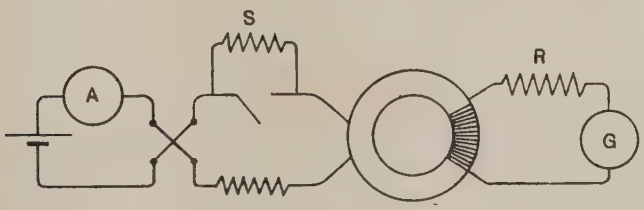


Fig. 170

wound over the ring, in series with a resistance box R and ballistic galvanometer. Let A be the cross-section of the core, τ the ballistic throw consequent on a sudden change of induction from B_0 to B_1. Then the change of induction in absolute units is given by

$$nA\,(B_1 - B_0) = 10^8 \frac{T}{2\pi}(R + G)\,k\tau \ldots\ldots\ldots (16).$$

The method of obtaining the points on the curve is best understood by reference to the complete curve, supposed already drawn (Fig. 171). We can tell whether H is positive or negative by

looking at the reversing key, while the direction of the ballistic throw tells us whether B is increasing or decreasing. Send first of all enough current through the solenoid to saturate the iron (point P). Then the throw produced by reversal (from P to Q)

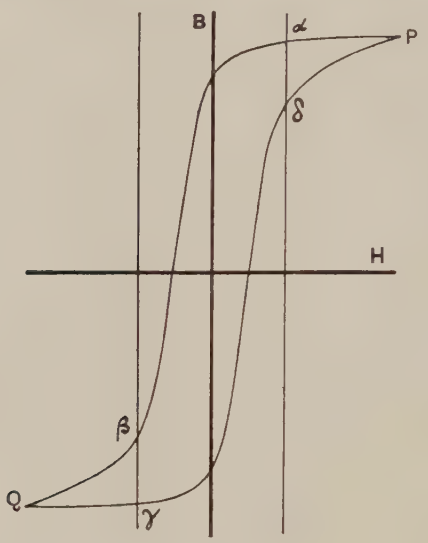
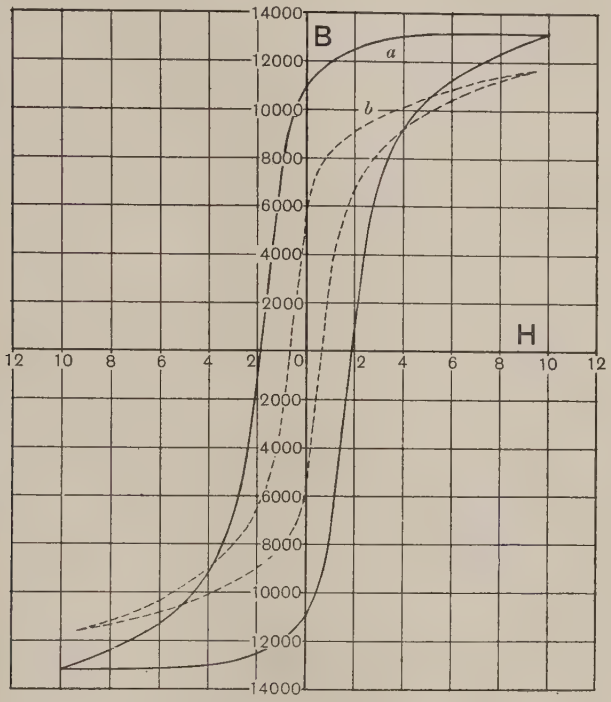


Fig. 171

gives $2B_P$ if we suppose the curve symmetrical. Starting from P we can find the change of induction produced by a sudden diminution of the current from its saturation value. Place a resistance in the shunt box S (Fig. 170) and knock out the key. Read the throw and the ammeter (point α). Since $B_\alpha - B_P$ is known and also B_P, we can find B_α. Reverse the current with the key, and again observe the throw. Since $B_\beta - B_\alpha$ is known and also B_α, B_β is known. Proceed similarly to the point Q by pressing down the shunt key, then to γ by knocking it out, to δ by reversal, and back to P. This gives a complete set of four points corresponding to a particular current, and any other set can be found by taking other resistances in the shunt circuit. Check the accuracy of the experiment by adding up the throws from P to Q and Q to P, which should give the same throw as that obtained by direct reversal of the saturation current.

Fig. 172 shows the hysteresis curves for specimens of soft iron and stalloy* (silicon transformer iron of low coercive force).



(*a*) soft iron; (*b*) stalloy.

Fig. 172

Numerical data as to the maximum values of B and the remanent magnetism of various materials have already been given (Art. 108).

128. Damping of galvanometers by induction. Suppose that a ballistic moving-coil galvanometer is contained in a circuit of total resistance R. Then the swinging of the coil in the magnetic field of the fixed magnets induces currents in the circuit, and the action of the field on these currents tends to bring the coil to rest. Consider the case of a moving coil of n turns on a core of cross-section A, making small swings in a uniform magnetic field H. Let I be the moment of inertia of the coil about the suspending

* I am indebted to Mr F. Stroud for this curve.

fibre, θ the small angle which the plane of the coil makes with H at time t, $\mu\theta$ the torsional couple due to the fibre, $f d\theta/dt$ a retarding couple due to air friction. If N is the total flux of magnetic force through the coils, the couple on the coil is $i dN/d\theta$ due to the magnetic field, and its equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\mu\theta - f\frac{d\theta}{dt} + i\frac{dN}{d\theta}.$$

The effect of self-induction in the coil is generally negligible, and the law of electromagnetic induction becomes

$$Ri = -dN/dt.$$

Since N is approximately equal to $nAH\theta$, we have

$$I\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + \mu\theta = inAH = -\frac{n^2A^2H^2}{R}\frac{d\theta}{dt}.$$

Hence the effect is the same as if f was increased to $f + \dfrac{n^2A^2H^2}{R}$.

We have thus a ready means of controlling the damping of a ballistic galvanometer by altering the resistance in circuit. The damping is very great when the galvanometer is short-circuited, and the spot of light comes to rest almost immediately.

Fig. 173 shows one way of connecting a ballistic galvanometer to a circuit, by means of a two-way key which allows B to be joined to A or C at will. When BC is joined the galvanometer is short-circuited for reading the zero, and when AB is joined it is placed in circuit. The larger the resistance in circuit the less the damping, and thus it is often necessary to have a considerable resistance in order to make the damping small enough for an accurate application of the damping correction (Art. 83). In measuring the time of swing of a galvanometer it is advantageous to set the coil moving through a large arc and then to disconnect it altogether, leaving the oscillations to go on on open circuit.

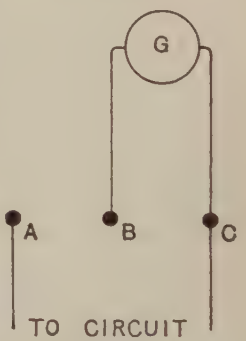


Fig. 173

An interesting application of the principle of electromagnetic damping occurs in the Grassot Fluxmeter, which is essentially

a moving-coil galvanometer with a very weak restraining fibre, so that the predominant force on the coil is that due to the fixed magnets. Any change in the flux of magnetic induction N through the external circuit sets the coil in motion, and the electromagnetic damping subsequently brings it to rest in a new position. It can be shown that the angle turned through by the coil is approximately proportional to the total change of flux. The equation of motion of the coil, in fact, is now $Id^2\theta/dt^2 = inAH$. On integration with respect to t we have $\int i dt = 0$, since $d\theta/dt$ vanishes both initially and finally. The law of electromagnetic induction gives

$$\frac{dN}{dt} + L\frac{di}{dt} + nAH\frac{d\theta}{dt} + Ri = 0,$$

and integrating once more with respect to t we have

$$N_2 - N_1 = nAH\,(\theta_1 - \theta_2),$$

where θ_1 is the initial position of the coil and θ_2 its final position.

129. Alternating currents and potentials. The behaviour of coils with self and mutual induction is conveniently studied with currents varying periodically with the time, on account of the ease with which these currents can be produced. Such currents generally alternate in direction during the period, and are therefore called alternating currents. The simplest alternating current is given by an expression of the form $i = I\cos(pt - \alpha)$, I being called the *amplitude*, and similarly the simplest alternating potential is given by

$$v = V\cos(pt - \alpha).$$

The principle of the generation of alternating potentials will be understood from Fig. 174. A coil of wire, capable of rotating about an axis, is rigidly connected to two *slip-rings* A, B on the same axle. These are solid rings of copper fastened to the ends of the coil and insulated from one another.

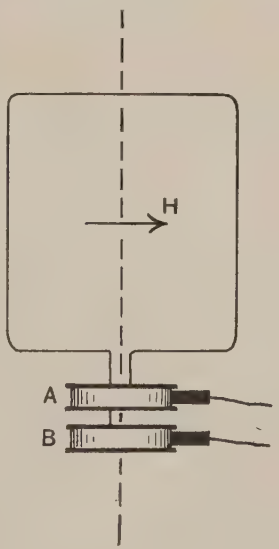


Fig. 174

Suppose now that the whole rotates with uniform angular velocity ω in a uniform transverse field H. If the plane of the coil makes an angle ωt with the direction of H at time t, the flux of force through it is $N = HA \sin \omega t$, where A is the area of the loop, and the E.M.F. of electromagnetic induction $-\omega HA \cos \omega t$. This is therefore the difference of potential between A and B on open circuit. Strips of carbon (called brushes) can be pressed against the slip-rings and attached to wires for utilising the alternating potential generated in this way.

If the field is not uniform the potential difference remains periodic, but is no longer given by a simple formula. We shall define an important quantity in connexion with periodic currents and potentials not necessarily of simple harmonic type. An alternating current is said to have the *effective value* I when its heating effect in a given coil is the same as that of a steady current I. If i is the value of the alternating current at time t, the heating effect in a coil of resistance R in time dt is Ri^2dt, and in a whole period $R\int_0^T i^2dt$. This must however be equal to RI^2T by definition. Hence the effective current I is given by

$$I^2 = \frac{1}{T}\int_0^T i^2dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

and for this reason is often called the root-mean-square (R.M.S.) value of the current. Similarly the effective value of an alternating E.M.F. is given by

$$V^2 = \frac{1}{T}\int_0^T v^2dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

We shall see shortly that the effective values of alternating currents and voltages can be read off directly in a large class of alternate current measuring instruments. It follows easily from the definition that the effective value of a simple-harmonic current of amplitude I is $I/\sqrt{2}$, and similarly for the voltage. Hence an alternating current voltmeter applied to a simple-harmonic E.M.F. of amplitude 100 would only indicate 70·7.

130. Examples of alternating current circuits.

(1) *Circuit containing self-inductance and resistance.* Let an E.M.F. $V \cos pt$ be applied to a circuit of self-inductance

L and resistance R. Then the current i is given by the equation

$$L\frac{di}{dt} + Ri = V\cos pt \quad \ldots\ldots\ldots\ldots\ldots (19).$$

The complementary function of this equation is $Ae^{-Rt/L}$, and diminishes rapidly with increase of t. Hence after some time the current will be given entirely by the particular integral

$$i = V\frac{R\cos pt + Lp\sin pt}{L^2p^2 + R^2} \quad \ldots\ldots\ldots\ldots (20).$$

This may be written

$$i = I\cos(pt - \alpha) \quad \ldots\ldots\ldots\ldots\ldots (21),$$

where

$$I = \frac{V}{(L^2p^2 + R^2)^{\frac{1}{2}}}, \quad \tan\alpha = \frac{Lp}{R} \quad \ldots\ldots\ldots (22).$$

Any given stage in the current i occurs later than the corresponding stage in the applied voltage by a time t given by $pt = \alpha$: the angle α is called the *lag* of the current behind the voltage.

The work expended in a complete period T is $\int_0^T vi\,dt$, where v is the instantaneous value of the applied voltage. Hence the work performed per second by the applied voltage, or the *power*, is

$$W = \frac{1}{T}\int_0^T vi\,dt \quad \ldots\ldots\ldots\ldots\ldots (23).$$

In the present case $v = V\cos pt$ and i is given by (21), so that

$$W = VI\int_0^T \cos pt\cos(pt - \alpha)\,dt$$
$$= \tfrac{1}{2}VI\cos\alpha.$$

The effective values of voltage and current are $V/\sqrt{2}$ and $I/\sqrt{2}$ respectively: thus the last equation may be written

$$W = VI\cos\alpha \quad \ldots\ldots\ldots\ldots\ldots (24),$$

where V and I are understood to mean the root-mean-square values. It is easy to see that the work done is entirely expended in heating the circuit, and that none is done in the long run against electromagnetic induction. We notice further an important property of inductive circuits, namely that of preventing the passage of current to a greater extent than non-inductive circuits

of equal resistance. Thus equation (22) shows that coils of great self-inductance let through very little current, particularly at high frequencies, even though their resistance is small. The quantity $(L^2p^2 + R^2)^{\frac{1}{2}}$, which determines the amplitude of current for a given voltage, is called the *impedance* of the circuit.

(2) *Circuit containing capacity and resistance.* Let C be the capacity, R the resistance. Then if e is the charge on the outer plate of the condenser in the figure, the current is $i = -de/dt$ and the difference of potential between the plates e/C. Thus

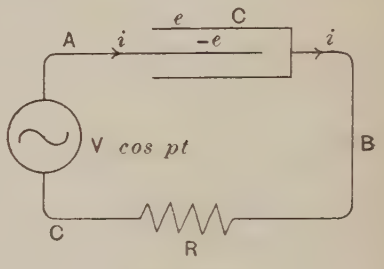


Fig. 175

$$V_A - V_B = -\frac{e}{C},$$

$$V_B - V_C = Ri = -R\frac{de}{dt},$$

and $$V_A - V_C = V \cos pt.$$

Hence we find on addition

$$RC\frac{de}{dt} + e = -VC \cos pt \ldots\ldots\ldots\ldots (25).$$

The particular integral of this equation is

$$e = -\frac{VC}{R^2C^2p^2 + 1}(\cos pt + RCp \sin pt),$$

so that $$i = \frac{VCp}{R^2C^2p^2 + 1}(RCp \cos pt - \sin pt).$$

This may be written in the form $i = I \cos(pt - \alpha)$, where

$$I = \frac{VCp}{(R^2C^2p^2 + 1)^{\frac{1}{2}}}, \quad \tan\alpha = -\frac{1}{RCp} \ldots\ldots\ldots (26).$$

The angle α is essentially negative, so that the current *leads* the applied E.M.F.

(3) *Circuit containing capacity, self-inductance and resistance.* This is the most general case. If the coil in Fig. 175 has self-inductance L as well as resistance R, the equation $V_B - V_C = Ri$ is replaced by $V_B - V_C = Ldi/dt + Ri$, and (25) by

$$LC\frac{d^2e}{dt^2} + RC\frac{de}{dt} + e = -VC \cos pt \ldots\ldots\ldots (27).$$

The particular integral is $i = I \cos(pt - \alpha)$ as before, where

$$I = \frac{V}{\left\{\left(Lp - \frac{1}{Cp}\right)^2 + R^2\right\}^{\frac{1}{2}}}, \quad \tan\alpha = \frac{Lp - \frac{1}{Cp}}{R} \quad \ldots(28).$$

The discussion of this solution, and of the complementary function of (27), is a very important matter which will occupy us later in Ch. XI.

131. Alternating current ammeters, voltmeters and wattmeters. Alternating current ammeters can be constructed on the principle of the electrodynamometer (Art. 88). Suppose that we have an electrodynamometer consisting of a moving-coil galvanometer in which the permanent magnets are replaced by a coil, and suppose that a current i flows through the fixed and moving coils in series. Let I be the moment of inertia of the moving coil about its axis, $\mu\theta$ the restraining couple due to the suspension for a deflexion θ, λi^2 the couple due to the current i passing through the coils. Then the equation of motion of the coil is

$$I\frac{d^2\theta}{dt^2} = \lambda i^2 - \mu\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(29).$$

Integrate the last equation from $t = 0$ to $t = T$. Then the left-hand side vanishes since the motion is periodic, and after division by T we have

$$\frac{\lambda}{T}\int_0^T i^2 dt = \frac{\mu}{T}\int_0^T \theta\, dt.$$

But the coefficient of λ is I^2, where I is the effective value of the alternating current; and the coefficient of μ is $\bar{\theta}$ the mean deflexion of the coil. Hence $\lambda I^2 = \mu\bar{\theta}$. In practice the moment of inertia is so great that the small oscillations about the mean position are inappreciable. Hence we may replace $\bar{\theta}$ by the observed deflexion θ and write

$$\lambda I^2 = \mu\theta.$$

Equation (29) however shows that we should obtain the same equation if a *steady* current I passed through the coils. It follows that *if an electrodynamometric ammeter is graduated with direct*

current, it will read effective values when used on alternating current circuits.

The alternating current voltmeter often consists of an electrodynamometer in series with a large non-inductive resistance, and the preceding remarks apply without modification. It is obvious from the definition that all thermal ammeters and voltmeters read effective values on alternating current circuits; and since these two types represent almost all the accurate kinds of alternate current instruments, we may say briefly that all such instruments read effective values directly. It should be noticed that an ordinary moving-coil galvanometer cannot be used with alternating currents: for the mean deflexion is proportional to $\int_0^T i\,dt$, which vanishes in nearly all practical cases. These instruments would then give no reading whatever.

The *wattmeter* is an instrument giving the power consumed in any part of a circuit, for example a coil, by a single reading. If v is the difference of potential at time t between the ends of the coil and i the current in the circuit, this power is $\frac{1}{T}\int_0^T vi\,dt$ watts.

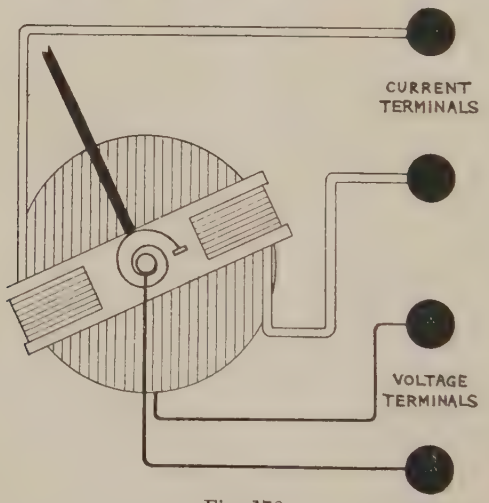


Fig. 176

The ends of the coil are joined to the "voltage terminals" of the wattmeter (Fig. 176), which go to a moving coil mounted between

bearings like that of a millivoltmeter, and the "current terminals" are placed in the main circuit so that the current i flows through a fixed coil taking the place of the permanent magnets. The current in the moving coil is proportional to v at any time, and the couple between the two coils proportional to vi. A simple extension of the preceding theory shows that the deflexion of the moving coil is determined entirely by the quantity $\frac{1}{T}\int_0^T vi\,dt$; that is, by the power consumed in the coil. The instrument is graduated with direct current, using steady voltages v and currents i, applied from independent sources. For the practical use of the wattmeter see Arts. 156, 157.

Among the sensitive instruments used in the laboratory for detecting, ~~but~~ not measuring, alternating currents we may mention the *telephone receiver* and the *vibration galvanometer*. The principle of the former, which was invented by Graham Bell in 1877, is shown in Fig. 177. The current from the terminals T, T passes through two coils wound over a horse-shoe magnet MM. Just in front of the magnet is a diaphragm D of sheet iron. A changing current in the coils causes a fluctuation in the force with which the diaphragm is attracted, so that the latter is set into vibration. The natural period is made to be somewhere in the range of audition, and the instrument is at once most accurate and most pleasant to use when the frequency of the impressed current is that of some ordinary musical note.

Fig. 177

The vibration galvanometer has many forms, one of which consists of a ballistic moving-coil galvanometer of high natural period, which responds readily to currents of its own frequency. The natural period of oscillation of the coil is adjusted to be equal to that of the source of alternating current used.

A lamp and scale is used, and a band of light is seen on the

scale, whose width corresponds to a greater or less amplitude of vibration of the moving coil. When the width is least we conclude that the current is zero, or at any rate a minimum. The vibration galvanometer has an important advantage over the telephone for measuring purposes in that it is selective, and can be used when the applied alternating E.M.F. is far from being of pure sine-form.

132. Measurement of inductance by alternating currents. A convenient source of alternating potential is a small rotary converter (Art. 153), which gives a nearly simple harmonic E.M.F.

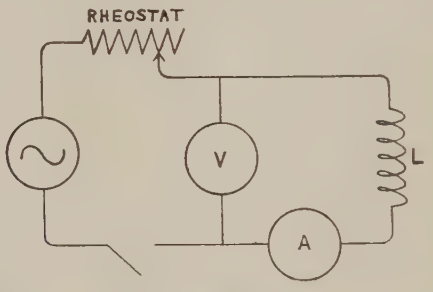


Fig. 178

The connexions are as shown. The self-inductance of the alternate-current ammeter is usually small in comparison with that of the coil L, and may be neglected. Let R be the sum of the resistances of the coil and ammeter. Then if V, I are the readings of the voltmeter and ammeter respectively, and $2\pi/p$ the periodic time of the simple harmonic E.M.F., we have

$$V^2 = I^2 (L^2 p^2 + R^2),$$

for it is easily seen that equation (22) still applies when effective values are used instead of amplitudes. Next replace the source of alternating potential by an accumulator, and adjust the rheostat till the ammeter reading is the same as before. If V_0 is the voltmeter reading,

$$V_0^2 = R^2 I^2,$$

whence

$$L^2 p^2 I^2 = V^2 - V_0^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The frequency $p/2\pi$ of the alternating current is simply related to the number of revolutions of the rotary converter per second; or it may be measured on a frequency-meter (Art. 153), and p is therefore known. Hence L is known from equation (30).

Mutual inductances may also be measured in this way. Suppose that we have two coils of known self-inductance L_1, L_2 joined in series. Then the compound coil has self-inductance $L_1 + L_2 + 2M$, as is easily seen by calculating the total flux of magnetic force for unit current. If one of the coils is turned over without changing the connexions, the self-inductance becomes $L_1 + L_2 - 2M$. Hence two measurements with the compound coil give M and $L_1 + L_2$, and the agreement of the latter number with that obtained from the separate coils affords a check on the accuracy of the experiment.

Alternate current measurements are not so accurate as those with direct current, for several reasons. In the first place, there is nothing in alternating current work comparable in steadiness with the current given out and the E.M.F. maintained by an accumulator. Alternating currents always vary slightly in frequency or amplitude and the readings of the instruments are not entirely steady. Again, direct current galvanometers are exceedingly sensitive, so that they take very little current when used as voltmeters and have very low resistance when shunted as ammeters. Electrodynamometric instruments require more current, and it often happens that low-reading A.C. voltmeters take an appreciable fraction of the current in the circuit. Low-reading wattmeters also absorb appreciable power, which makes the power actually consumed in a coil somewhat less than that indicated by the wattmeter. Electrodynamometers are by no means free from self-induction, and for high frequencies the readings of thermal instruments are generally more reliable.

133. The generalised Wheatstone's bridge for alternating currents. We shall here consider a generalisation of the Wheatstone's bridge (Fig. 179) which covers all cases of experimental application. The detector across BD is a telephone or vibration galvanometer. Let the arm AB contain a condenser of capacity C_1 in series with a coil of self-inductance L_1 and resistance R_1, and further let the condenser have a finite internal resistance ρ_1, as it would if filled with a conducting liquid. If no current flows in BD, $i_1 = i_2 = I$ (say), $i_3 = i_4 = J$. Let all the variable quantities be proportional to e^{ipt}, where the real part of the expressions may be taken if desired for physical interpretation.

Further, let A, B, C, D denote the potentials at the four corners of the bridge, $\pm e_1$, $\pm e_2$, $\pm e_3$, $\pm e_4$ the charges on the plates of the condensers at time t.

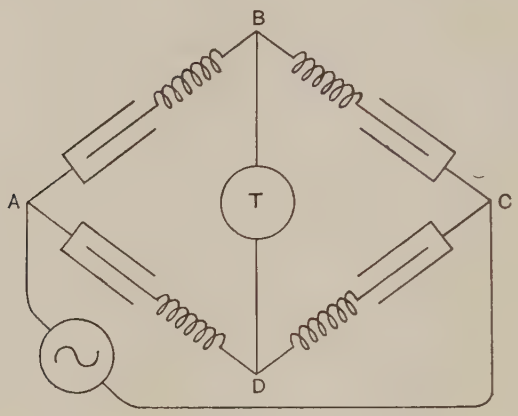


Fig. 179

The difference of potential between the plates of C_1 being e_1/C_1, the internal current in the condenser is $e_1/\rho_1 C_1$. Hence

$$I = de_1/dt + e_1/\rho_1 C_1,$$

and

$$A - B = L_1 \frac{dI}{dt} + R_1 I + \frac{e_1}{C_1}.$$

Substituting ip for the operator d/dt, we have

$$e_1 = \frac{I}{\dfrac{1}{\rho_1 C_1} + ip},$$

so that

$$A - B = I\left[iL_1 p + R_1 + \frac{1}{\dfrac{1}{\rho_1} + iC_1 p}\right] \quad \ldots\ldots(31).$$

The three similar equations can easily be written down. Remembering that $B = D$ we therefore have

$$\frac{iL_1 p + R_1 + \dfrac{1}{\dfrac{1}{\rho_1} + iC_1 p}}{iL_2 p + R_2 + \dfrac{1}{\dfrac{1}{\rho_2} + iC_2 p}} = \frac{iL_3 p + R_3 + \dfrac{1}{\dfrac{1}{\rho_3} + iC_3 p}}{iL_4 p + R_4 + \dfrac{1}{\dfrac{1}{\rho_4} + iC_4 p}}.$$

The conditions of balance* are obtained by equating real and imaginary parts of this equation, after cross-multiplication, to zero.

The results in particular cases are of course simpler. Thus the case in which there are no condensers can be treated by putting $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$, which gives

$$\left.\begin{aligned} R_1R_4 - R_2R_3 &= p^2(L_1L_4 - L_2L_3) \\ R_1L_4 + R_4L_1 &= R_2L_3 + R_3L_2 \end{aligned}\right\}.$$

These conditions will not be satisfied for all frequencies unless either $L_1/R_1 = L_2/R_2$, $L_3/R_3 = L_4/R_4$, or $L_1/R_1 = L_3/R_3$, $L_2/R_2 = L_4/R_4$, so that the bridge cannot be balanced by adjusting the resistances alone. In general all that can be obtained by a single adjustment is a *minimum* sound in the telephone.

An examination of the order of magnitude of the terms in equation (31) will enable us to judge exactly what is being measured on the Wheatstone's bridge in the presence of various disturbing causes. For example, in measuring the resistance of electrolytes (Art. 169) the liquid is enclosed in what is really a parallel-plate condenser, so that we wish the term $1/\rho_1$ to be large in comparison with C_1p, and at the same time ρ_1 to be large in comparison with both L_1p and R_1, so that $A - B$ shall be approximately equal to $\rho_1 I$. If, on the contrary, $1/\rho_1$ was small in comparison with C_1p, we should be measuring the capacity of the condenser when filled with the liquid, that is the dielectric constant of the liquid, instead of its resistance. An example will make this clear. Let C_1 consist of two parallel plates 5 cm. in diameter, distant 2 cm. apart, and let $L_1 = 10^{-7}$ henry, $R_1 = 10^{-2}$ ohm (the order of the irreducible inductance and resistance of the connecting leads). If the condenser is filled with distilled water of conductivity 10^{-6} (cf. Art. 170) we have $\rho_1 = 1{\cdot}02 \times 10^5$ ohms, and allowing for the dielectric constant of water C_1 is found to be 7×10^{-11} farads. For $p = 500$, $1/\rho_1$ is still large compared with

* The reader who is unaccustomed to the use of imaginaries should notice why there are *two* conditions. If the calculation of the current in BD had been performed quite generally by the ordinary methods, it would have appeared in the form $P \cos pt + Q \sin pt$. The conditions $P = Q = 0$ are therefore necessary if it is to vanish. The amplitude of current in all these cases is of the form $(P^2 + Q^2)^{\frac{1}{2}}$.

$C_1 p$, and experiments on the Wheatstone's bridge would measure ρ_1. As p rises the two terms become of the same order of magnitude, and finally for high frequencies of the order $p = 10^7$ (electric oscillations) the capacity term predominates.

134. The alternating current transformer. The simplest type of alternating current transformer is Faraday's ring (Fig. 157) consisting of two coils wound on the same iron core. An alternating current flowing in the primary will induce currents in the secondary and give rise to an E.M.F. which in general differs from that applied to the primary. The presence of iron makes practically all the lines of induction due to the primary current pass through the secondary, and *vice versâ*; but as the theory would be complicated by hysteresis we may represent all the principal features of the phenomenon by the following ideal arrangement. Let us have two endless solenoids closely interwound on the same ring so that the lines of force circulate only in the interior. Let n_1, n_2 be the number of turns of the two solenoids, r the mean radius of the ring. When unit current flows in the first solenoid the magnetic force inside the core is $2n_1/r$, so that the fluxes of induction through the two solenoids are $2n_1^2 A/r$ and $2n_1 n_2 A/r$ respectively, where A is the area of the cross-section. Hence $L_1 = 2n_1^2 A/r$, $M = 2n_1 n_2 A/r$, and similarly $L_2 = 2n_2^2 A/r$, so that L_1, L_2 and M satisfy the condition

$$L_1 L_2 = M^2 \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

This may fairly be taken as the condition that no lines of force escape between the two coils, or the condition of no magnetic leakage, as it is called.

Let an E.M.F. $V \cos pt$ be applied to the primary as shown

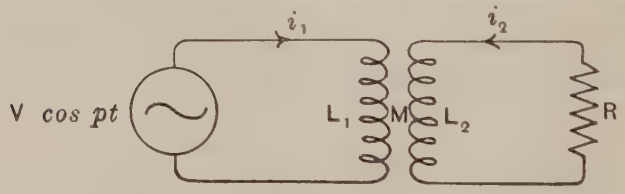


Fig. 180

in the figure. The effect of self-inductance in transformers generally predominates over that of resistance, so that in this

representation we may neglect the resistance of the coils altogether. Let the secondary circuit be completed by a non-inductive resistance R. Then the equations for the currents are

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} &= V \cos pt \\ M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} &= -Ri_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(33).$$

Multiply the first equation by M and the second by L_1. The left-hand sides become equal by virtue of (32), and we have

$$Ri_2 = -\frac{M}{L_1} V \cos pt \quad \ldots\ldots\ldots\ldots\ldots(34).$$

Now Ri_2 is the difference of potential between the terminals of the secondary coil, and $M/L_1 = n_2/n_1$. Hence the E.M.F. developed in the secondary is the fraction n_2/n_1 of that applied to the primary. If the secondary has many turns the E.M.F. developed in it may greatly exceed that applied; if few turns it will be greatly less. We have thus a simple means of *transforming* alternating E.M.F.'s by electromagnetic induction, and this is the chief reason for the extensive use of alternating currents in modern electrical practice.

The current in the secondary differs by a phase of 180° from the E.M.F. applied to the primary for all values of R. The primary current is easily calculated by substituting from (34) in the first of the equations (33) and integrating. Hence we find

$$i_1 = \frac{V}{L_1 Rp} (L_2 p \cos pt + R \sin pt) \quad \ldots\ldots\ldots(35).$$

For small values of R the primary current lags nearly a quarter of a period behind the voltage, but the phase-difference diminishes as more and more current is taken off the secondary. This may be expressed, if desired, by saying that the effective inductance of the primary is diminished by the presence of the secondary.

135. The induction coil. The problem of obtaining high-voltage electricity from an ordinary battery was first successfully solved by Ruhmkorff, who succeeded in 1851 in producing sparks exceeding those of the influence machine in length. His apparatus, which is the same in principle as that previously used by du Bois-Reymond, is called the *induction coil*. It depends on the fact that

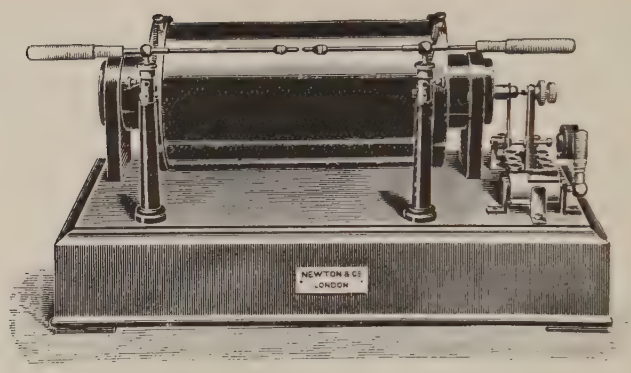


Fig. 181

a large E.M.F. is induced in a secondary coil of many windings when the current in a neighbouring primary coil is suddenly cut off. Fig. 181 shows an induction coil as now manufactured for sparks of medium length, and Fig. 182 the internal connexions, with the exception of the secondary coil. The battery is connected to two terminals generally marked *P* and *N*, and the instrument also contains a commutator *R* which is the same in principle as

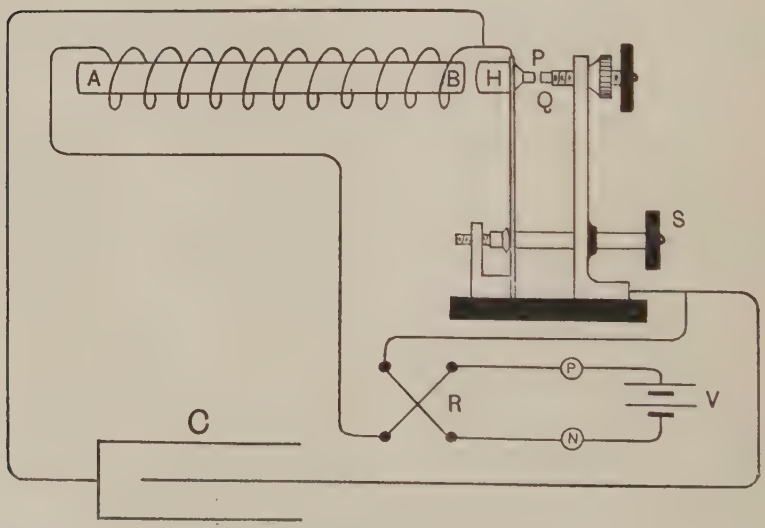


Fig. 182

an ordinary reversing key. The primary coil consists of about 400 turns of thick copper wire wound on an iron core AB composed of a bundle of soft iron wires, the resistance of the coil being of the order of $\frac{1}{3}$ ohm. The hammer H is a short iron cylinder attached to a stiff spring, carrying on the remote side a stout piece of platinum P. A similar piece Q is fixed on the end of a screw so that its distance from P is adjustable between certain limits, and the position of the hammer can also be adjusted by a screw S which pulls it away from the core. A condenser C, of the order of 1 microfarad, is built into the base of the instrument and connected as a shunt across the platinum "break" PQ. The secondary coil consists of about 15 miles of thin copper wire having a resistance of from ten to fifteen thousand ohms. The different layers of the coil must be most carefully insulated from one another, which can be done by interposing a paper disc covered with shellac between each successive pair of layers. The secondary is enclosed in a thin ebonite cylinder and its ends brought out to two terminals to which pointed conductors are attached for testing the length of spark produced.

In using an induction coil it is best to begin with the screw S as far back as possible so that the hammer is near the core of the primary. The platinum point Q is screwed up till it is about 2 millimetres from P, and the hammer then brought forward slowly by means of the screw S. The coil will begin to act as soon as P and Q touch, and slight sparking will take place at the break. Excessive sparking is undesirable as it destroys the platinum contacts, and usually occurs when the initial distance between P and Q is too great. The platinum contacts will become rough and uneven with use, and should be smoothed at intervals with a fine file. The E.M.F. applied depends on the use to which the coil is put. Thus 6–8 volts suffice for vacuum tube discharges and soft Röntgen ray bulbs, more for spark gaps at atmospheric pressure, particularly with condensers in parallel: but 16 volts is the most that should be used if no additional resistances are inserted in series with the battery.

The action of the coil is as follows. When P and Q touch, the circuit is completed and a current flows through the primary, rising to a maximum somewhat as shown in Fig. 163. The iron

core AB becomes magnetised and attracts the hammer H, thereby breaking the circuit. The attraction disappears with the current, the hammer springs back and again makes contact with Q, and so on.

When the circuit is broken part of the current continues to flow for a short time across PQ as a spark, and a momentary current also flows into the condenser C, charging it up. The subsequent current in the condenser circuit is probably oscillatory in nature (cf. Art. 176), dying out before the circuit is closed again. In this way the core is effectively demagnetised in a very short time after the breaking of the circuit. The change of induction on "break" is thus much more rapid than that on "make," resulting in a much greater E.M.F. in the secondary.

If the secondary coil is closed by a circuit obeying Ohm's law, alternating currents flow in the circuit. For if R is the total resistance of the secondary and N the flux of magnetic induction through it due to all causes, the current at any time is given by $dN/dt + Ri = 0$. On integration between limits 0 and T, where T is the time of oscillation of the hammer spring, we find $\int i\,dt = 0$, since N resumes its original value. Hence the total charge passing any point of the secondary in a period vanishes, so that the current must alternate in direction. This however is not necessarily the case when the coil is used to give spark discharges or to work Röntgen ray bulbs. In these cases the E.M.F. induced on "make" is not sufficient to break down the insulation of the air, but a current flows only on "break." The current is then practically unidirectional, and can be reversed by means of the commutator in the primary circuit.

The hammer interrupter is not suitable for working large coils, and the making and breaking of the current has to be done by other means. The Wehnelt interrupter (Fig. 183) is essentially a large electrolytic cell containing dilute sulphuric acid. The cathode is a large lead plate, the anode a platinum

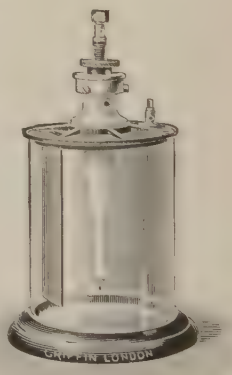


Fig. 183

wire projecting slightly from a porcelain tube, the length of free wire being adjustable with a screw. When working the cell makes a somewhat disagreeable noise and the anode is surrounded by a luminous glow. The current under these conditions is intermittent, the number of interruptions per second being of the order of 1000.

The most satisfactory interrupter of all is the mercury interrupter, which takes various forms, one of which is shown in Fig. 184. A cylinder C has a nozzle at the lower end dipping into mercury, and also a side-tube T some way up. The nozzle is so shaped that when C is rotated rapidly the mercury rises and is ejected at T. The stream falls on a metal plate P bent in the form of a sector of a circle, so that the current between C and P is interrupted regularly whenever the cylinder reaches a certain position. The whole cylinder and reservoir is enclosed in an air-tight case and filled with coal-gas.

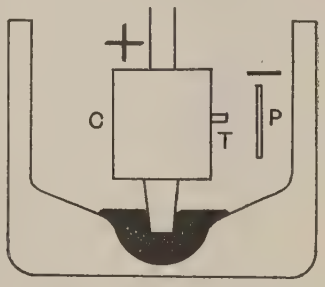


Fig. 184

The two preceding breaks can be worked off the electric mains, in series with a high-current rheostat.

136. Elihu Thomson's experiments. Let an iron rod, made up of a bundle of iron wires bound together, be inserted in a hollow solenoid so as to protrude from one end, and a powerful alternating current sent through the solenoid. If the rod is fixed vertically, an aluminium ring sliding over it will suddenly rise on applying the current and hang freely in the air without being suspended. The position of equilibrium will be stable, for the ring will return to its position when either lifted or raised. Further, if the ring is forcibly held down by hand it will become in a very few seconds too hot to hold.

These facts are all in accordance with theory and afford an interesting example of electromagnetic induction. On account of the spreading out of lines of magnetic force sideways from the coil and magnet, there will be a radial component of magnetic

force at all points of the ring, which we may write $H \cos pt$ (Fig. 186). In addition there will be a periodic flux of induction

Fig. 185

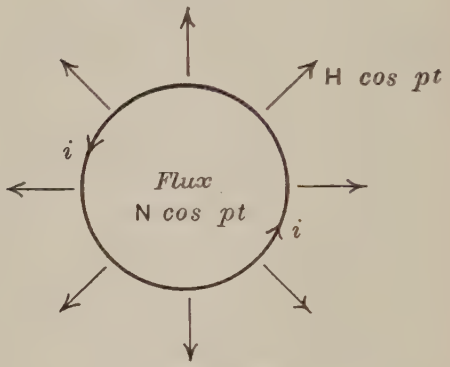


Fig. 186

through the ring, say $N \cos pt$. If L, R are the self-inductance and resistance of the ring and i the induced current, we have

$$L\frac{di}{dt} + Ri = Np \sin pt.$$

The solution is $i = X \cos pt + Y \sin pt$, where

$$X = -\frac{LNp^2}{L^2p^2 + R^2}, \quad Y = \frac{RNp}{L^2p^2 + R^2}.$$

The radial component $H \cos pt$ acts on this current and produces a vertical force on the ring which at time t has the value

$$-2\pi iaH \cos pt = -2\pi aH \cos pt\,(X \cos pt + Y \sin pt).$$

The force varies within a period (and its fluctuations can be distinctly felt while holding the ring). Its mean value is

$$-\pi aHX = \pi aHLNp^2/(L^2p^2 + R^2).$$

The mean rate of development of heat is easily seen to be

$$\tfrac{1}{2}R\,(X^2 + Y^2) = RN^2p^2/2\,(L^2p^2 + R^2).$$

As an example consider an aluminium ring of 2 cm. radius, the diameter of the cross-section being 2 mm. Then $L = 83{\cdot}5$ true E.M. units, and $R = 1{\cdot}2 \times 10^{-3}$ ohm $= 1{\cdot}2 \times 10^{6}$ E.M. units. If the core is of soft iron, fully magnetised for the maximum

current and of diameter 3 cm., $N = 100{,}000$. The maximum radial field H is probably at least 100, which for $p = 150$ gives a lifting force of 82 dynes. This is about one-twelfth of the weight of the ring, showing that the two forces are of the same order of magnitude. The rate of development of heat is found to be $9{\cdot}4 \times 10^7$ ergs or 2·2 calories per second. Since the specific heat of aluminium is 0·21, the temperature of the ring would rise about 10° per second until checked by radiation.

137. Currents induced in solid masses of metal. The electron theory throws much light on induction phenomena, particularly those caused by the motion of conductors in magnetic fields. Such effects, in fact, are covered by the expression already found for the force experienced by a charge moving in a magnetic field (Art. 95).

The electrons in a piece of metal at rest are supposed to be in a state of irregular motion, colliding at intervals with the molecules of the metal. When the metal moves, however, its velocity is superposed on the irregular movements of the electrons. The electrons are therefore deflected by the magnetic field, while the molecules are unable to yield to the forces exerted on them as they are more or less securely anchored in position. We shall calculate the electromotive forces in the particular case of Faraday's rotating disc (see Fig. 158).

Let ω be the angular velocity of the disc, H the magnetic field at distance r from the axis, supposed perpendicular to the plane of the disc. The field is not necessarily uniform, but will be assumed to be symmetrical about the axis, so that H is a function of r only. The average velocity of a group of electrons distant r from the axis is $r\omega$ in a direction perpendicular to the radius vector, and hence the average mechanical force exerted by the field on an electron of the group is $He r\omega/c$ radially outwards. In other words, there is an apparent electric force, or force per unit charge, of magnitude $Hr\omega/c$ in electrostatic units, or $Hr\omega$ electromagnetic units. If the disc is insulated there will be an accumulation of electrons near the axis, until the radial electric field is exactly compensated at every point by the electrostatic field due to the accumulated charges. On connecting the rim

and axle by a wire electrons tend to flow back again to the rim, so that the direction of the current is correctly accounted for by the theory.

The work done in moving a unit charge from the axle to the rim, or the difference of potential between the brushes in Fig. 158, is $\omega \int_c^b Hrdr$ in electromagnetic units, where b is the radius of the rim and c that of the axle. Now the flux of magnetic force through the space between the rim and axle is $N = 2\pi \int_c^b Hrdr$, and $\omega = 2\pi n$, where n is the number of revolutions of the disc per second. Hence the induced E.M.F. is

$$V = nN \text{ electromagnetic units} \ldots\ldots\ldots (36).$$

The effect of magnets in retarding the motion of neighbouring metallic masses, or of having their own motion retarded by their presence, was examined by several experimenters prior to 1830. Thus Arago found in 1825 that a magnet suspended over a metal disc tends to follow the motion of the disc when the latter is rotated. Babbage and Herschel, who repeated Arago's experiments, found that the magnet ultimately came to rest making an angle θ with the meridian, where $\sin\theta$ was proportional to the angular velocity of the disc. The order of magnitude of the effect in different metals was also found to be that of the conductivities of the metals. Moreover, when the continuity of the disc was destroyed by radial cuts extending nearly to the centre, the deflexions were very much reduced.

These effects were first adequately explained by Faraday in the two papers already cited, in which it was shown conclusively that they were due to currents induced in the metals by the relative motion of the disc and magnet. The electric forces generated are entirely determined by the strength of the magnet and the speed of rotation; but the currents are smallest in discs of low conductivity, and their circulation is also hindered by cuts which destroy the continuity of the metal.

The heating effect of currents in metallic masses was first observed by Foucault in 1855, whence the name Foucault currents often applied to them. Consider the case of a solid cylinder

rotating in a field *perpendicular* to the axis (Fig. 187). The velocity at a point P (r, θ) of a cross-section is $r\omega$ at right angles to the radius vector, so that an electron at P would experience a force $-er\omega H \sin(\frac{1}{2}\pi + \theta)$ parallel to the z-axis. Thus the

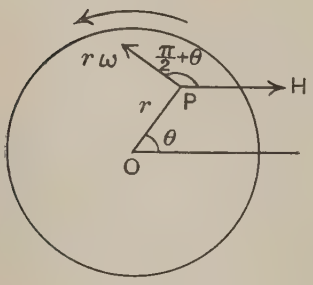


Fig. 187

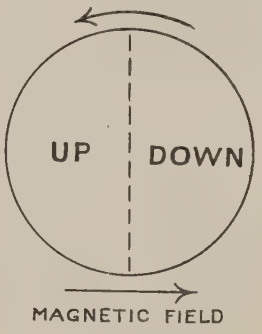


Fig. 188

induced electric force is of magnitude $-r\omega H \cos\theta$, measured vertically upwards. Fig. 188 gives roughly the way in which the currents would move in the central parts of the cylinder. These currents would be largely prevented in a thin disc since there would now be but little opportunity for continuous circulation, the electric force simply tending to drive electrons towards the face of the disc. Hence if a metal cylinder is built up of discs parallel to the axis and insulated from each other, it can be rotated in a transverse magnetic field without giving rise to heavy Foucault currents. This arrangement is adopted in the armatures of electric generators, where it is necessary to take some steps to prevent the loss of energy that would otherwise occur.

138. Determination of the ohm in absolute measure. The absolute value of the resistance of a given coil can be accurately determined by a method due to L. Lorenz (1873), depending on the use of Faraday's rotating disc. The following arrangement may be adopted in the laboratory if very great accuracy is not required (Fig. 189).

C is a circular coil of a large number m of turns concentric with the rotating disc, B_1 and B_2 brass brushes making contact with the rim and axle of the disc respectively. R is a standard

coil whose resistance is required in absolute electromagnetic units, S and T other resistances whose magnitudes are known relatively to that of R. The disc can be driven at a rapid and uniform speed of n revolutions per second by a belt from a small electric motor, and is fitted with a counting device for registering the revolutions.

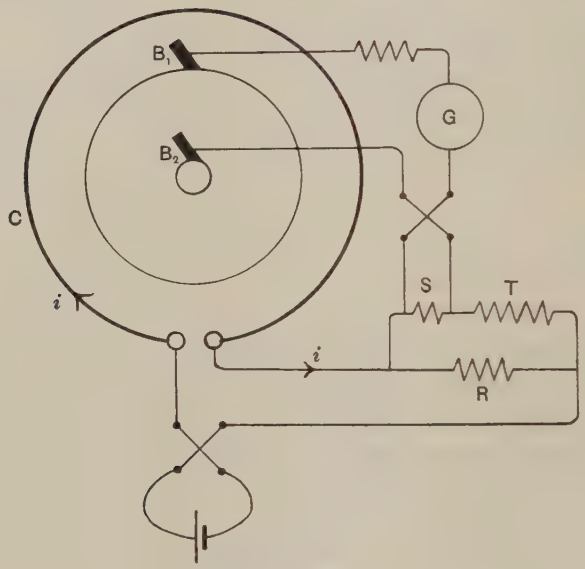


Fig. 189

A deflexion of the galvanometer will generally occur on rotation even when no current flows in the coil C, on account of the earth's magnetic field and other stray fields causing a flux of force through the disc. This deflexion is taken as the zero and T adjusted till the deflexion does not alter on reversing the current in the battery circuit. The fall of potential down S due to the external current is then equal to that generated by the rotation of the disc in the field of the coil C, so that equation (36) gives

$$\frac{RSi}{R+S+T} = nN.$$

Let M_1 be the mutual inductance of the coil C and the rim of the disc, calculated for a single turn in the mean position of the

turns of C, M_2 the same calculated for C and the axle of the disc. Then $N = mi\,(M_1 - M_2)$, so that

$$R\frac{S}{R+S+T} = mn\,(M_1 - M_2) \ldots\ldots\ldots (37).$$

M_1 and M_2 can be found from formulae already given (cf. Art. 94, and Table, p. 197); and since the ratios of R, S, T are known R is given by this equation in absolute electromagnetic units.

As far as the principle of the method is concerned, we might suppress the shunt circuit $S + T$ and shunt the Lorenz disc directly across the resistance R, in which case equation (37) becomes simply $R = mn\,(M_1 - M_2)$. But with an apparatus of ordinary size the requisite value of R would be very small (of the order 1/1000 ohm), and also the only means of obtaining the balance would be to adjust the speed of rotation, which is not very practicable.

Other methods have been used to determine resistances in absolute measure, all depending on the principle of electromagnetic induction. Thus the ordinary method of measuring a mutual inductance consists in comparing it with a resistance X, in terms of which other resistances R, S, G are supposed to be known (cf. Art. 124, equation (7)). But whereas in the laboratory X would be taken as known in ohms, we might just as well calculate the mutual inductance theoretically from Neumann's formula or otherwise, and use the experiment to determine X in absolute measure.

The older experiments seemed to show that the ohm is the resistance, at 0° C., of a column of mercury of cross-section 1 sq. mm. and of length nearly equal to 106·3 centimetres. In practice the diameter of the containing tube of the mercury resistance would itself be determined by weighing a column of mercury, so that it is expedient to define by weight instead of by cross-section. The international conference held in London in 1908 defined the *international ohm* as the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 grams in mass, of a constant cross-sectional area and of a length 106·300 centimetres. It had

previously been found that resistance standards consisting of mercury columns in glass were far more reliable than wire coils.

Of all these methods that of Lorenz is generally considered the best. Nevertheless, until quite recently the results of the most accurate absolute measurements of resistance were uncertain to about one part in 2000. The chief difficulties that have to be overcome are as follows.

(1) Secular changes in wire standards of resistance. While carefully constructed mercury standards can be relied on to about two parts in 100,000, the best resistance coils are liable to alter in time, the change in some cases exceeding one part in 5000. This seems to be chiefly due to the effect of moisture on the shellac used to cover the wire. Recently hermetically sealed coils have been introduced, and are found much more reliable. All coils should however be frequently compared with mercury standards.

(2) Effects due to rapid variations of terrestrial magnetism.

(3) Thermoelectric effects at the sliding contacts of the brushes. Such effects are liable to sudden changes which, like the foregoing, render it difficult to obtain an accurate balance.

(4) The difficulty of keeping the angular velocity of the disc sufficiently uniform.

A very careful determination of the ohm has recently been carried out by F. E. Smith at the National Physical Laboratory. In these experiments two discs were used, the currents in the two sets of coils circulating in opposite directions, so that the effect of terrestrial magnetism is eliminated. Thermoelectric effects were much reduced by this arrangement and by the use of a special form of brush resembling a violin bow, lubricated with petrol. Further, it was arranged that the edge of the discs should be in a region of zero magnetic force, so that a slight uncertainty as to their diameters should not affect the mutual inductance between them and the coils. The axle carried with it a commutator which charged and discharged a condenser in one arm of a Wheatstone's bridge (cf. Art. 140); and by keeping the bridge balanced during a run the speed of rotation could be kept very accurately uniform.

Smith found that the international ohm is equal to 1·00052 absolute ohms, with an estimated probable error of 4 in the last

place. It thus appears that the true ohm may be defined, to an accuracy of about one part in 25,000, as the resistance of a mercury column of mass 14·4446 grams and length 106·245 cm., the area of cross-section being nearly 1 sq. mm. as before.

139. International and true electrical units. E.M.F. in absolute measure. The true electrical units are defined in a general way without reference to the feasibility of any tests that may be implied. Thus the ampere is specified by its magnetic effect. In this case the definition is practically useful, as it really amounts to supposing that all currents are measured with an electrodynamometer of known dimensions. The case of potential is however quite different, the true volt being taken as the difference of potential between two points A, B such that 10^7 ergs of work are required to move one coulomb from A to B. The definition by energy is unsuited for practical realisation, and in practice we are driven to take the ohm as the second fundamental unit, defining the volt as the difference of potential between A and B when the resistance is one ohm and a current of one ampere is flowing. Fortunately, the ohm can be determined in absolute measure independently of all other electrical units, as we have shown in the last article.

The *international* system of units recommended by the London Conference (1908) aims at being easily realisable in practice, and at the same time sufficiently near to the true units for all but the most accurate experiments. It has already been mentioned that the international ampere is too small by about one part in 4000, while the international ohm is too large by about one part in 2000. These errors are too small to affect ordinary experiments, and the international units have very properly been adopted for legal purposes connected with the sale of electricity and electric power. The important point to notice is that the international units themselves can be reproduced with great accuracy. For example, the researches of Smith, Mather and Lowry have shown that the amounts of silver deposited by the same current in different silver voltameters agree to about one part in 100,000 if proper precautions are taken, so that a current can be measured in international amperes to this order of accuracy. A slightly

smaller accuracy is attained in the construction of mercury standards of resistance.

The London Conference defined the *international volt* as the difference of potential which, when applied to a conductor whose resistance is one international ohm, would produce a current of one international ampere.

We have now to complete the account of standard cells given in Art. 59, by showing how their E.M.F. may be determined in absolute measure. A circuit is set up containing a silver voltameter and a standard resistance of (say) one international ohm, together with a battery and a regulating resistance. A standard cell is placed in series with a key and a sensitive galvanometer so as to form a potentiometer circuit connected to the terminals of the standard resistance. The current is adjusted till the galvanometer shows no deflexion, and is then measured by means of the silver voltameter. If its magnitude is V international amperes, then the E.M.F. of the standard cell is V international volts.

The London Conference recommended the following formula for the E.M.F. of the Weston normal cell at temperatures between 0° and 40° C. in terms of its E.M.F. E_{20} at 20° C.:

$$E_t = E_{20} - 0{\cdot}0000406\,(t-20)^2 - 0{\cdot}00000095\,(t-20)^2 + 0{\cdot}00000001\,(t-20)^3 \ldots (38).$$

The value of E_{20} was taken provisionally as 1·0184 international volts, but this number was not insisted on and a committee was appointed to ascertain the most probable value to be assigned. Experiments were accordingly carried out at Washington in 1910 by representatives of the four principal standardising laboratories of the world, and gave the value 1·0183 international volts for the E.M.F. of the Weston normal cell at 20°. This value is probably correct to about one part in 50,000, and is now universally adopted for standardising purposes. In the case of a cell whose E.M.F. at 20° differs from the normal value, formula (38) may still be used with the proper value of E_{20}.

Using the most recent value for the international ohm and ampere in terms of the true units, the E.M.F. of the Weston cell at 20° may be taken as 1·0188 true volts.

140. Measurement of the ratio of the electromagnetic to the electrostatic unit of charge. The question of the absolute magnitude of the electromagnetic units having been placed on a firm basis, we are now in a position to consider the relation of these units to the electrostatic system.

An approximate determination of the ratio of the units may be made in the laboratory as follows. T_1 is a tuning fork of

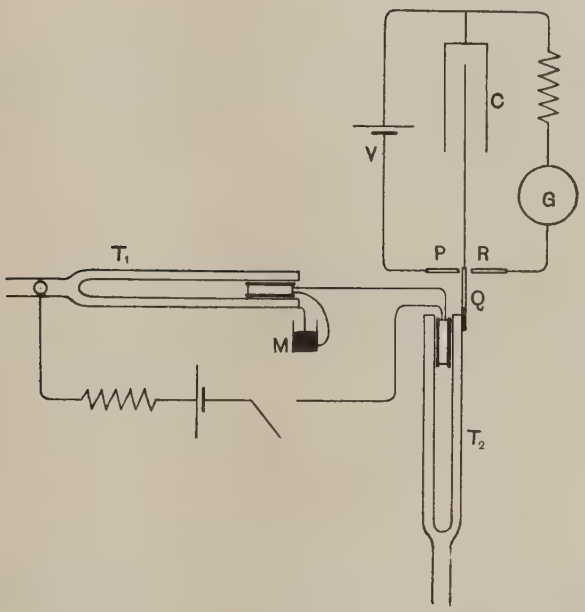


Fig. 190

known frequency (n per second), one of whose prongs carries a projection which dips into a trough M containing mercury covered with a layer of alcohol. T_2 is a fork of similar dimensions, and both forks are fitted with small flat electro-magnet coils between their prongs, attracting the prongs whenever the circuit is completed. The whole arrangement serves to maintain the vibrations of the tuning fork T_1, in just the same way that the vibrations of the hammer of an induction coil are maintained. The current through both coils is broken regularly n times per second, and the tuning fork T_2 is set into forced vibration, which may be

made as vigorous as desired by adjusting a small weight on one of the prongs. An insulated piece Q vibrates between two fixed pieces P, R fitted with springs, P, Q and R being connected to an electrical circuit as shown. In this way the condenser C is discharged n times per second through the dead-beat galvanometer G after having been charged up to potential V, producing a steady deflexion θ. Hence

$$nCV = k\theta.$$

The experiment is performed in succession with two parallel plate condensers of equal perimeter and separation. Subtracting eliminates the edge correction and the capacity of the leads in the experiment, and then

$$n\,(C_1 - C_2)\,V = k\,(\theta_1 - \theta_2),$$

where $C_1 - C_2$ can be calculated in electrostatic units from the dimensions of the condensers.

The galvanometer is then standardised as in Art. 68, using the same cell V and adjusting the resistances for a steady deflexion $\theta_1 - \theta_2$. Then

$$VX = R\,(S + G)\,k\,(\theta_1 - \theta_2).$$

The last two equations give on division

$$C_1 - C_2 = \frac{X}{nR\,(S + G)} \quad \ldots\ldots\ldots\ldots (39).$$

If derived units are used throughout, this equation gives $C_1 - C_2$ in farads. By reduction $C_1 - C_2$ can be found in true electromagnetic units, and since it is known in electrostatic units the quantity

$$c^2 = \frac{\text{electromagnetic unit of capacity}}{\text{electrostatic unit of capacity}}$$

is found. Hence c is known.

Maxwell devised a most accurate null method for measuring the capacity in electromagnetic units, which does away with the standardisation of the galvanometer. The condenser is placed in the arm BC of a Wheatstone's bridge (Fig. 191). The terminals P, Q, R represent a vibrating tuning fork, so that Q makes contact alternately with P and R. The phenomena occurring

during a single period of vibration may be divided into two stages:

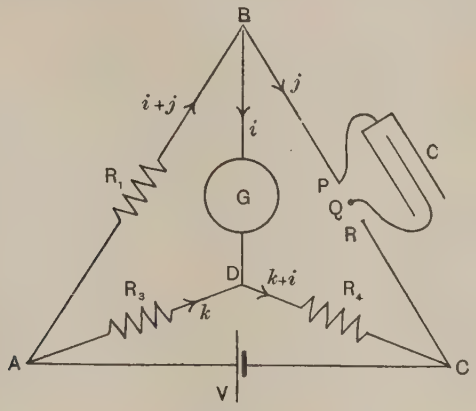


Fig. 191

(1) When QR is joined the condenser begins to be charged up, and rapidly acquires its final charge corresponding to the difference of potential v_0 which would exist in the absence of the arm BC of the bridge.

(2) When PQ is joined the condenser is discharged and steady currents flow through the arms of the bridge just as if BC was absent.

The stage (1) differs from (2) in superposing a sudden discharge of electricity on the steady current that would flow in the absence of the condenser. We shall find the condition that the total charge passing through the galvanometer in a single period is zero, i.e. that the galvanometer should show no deflexion.

Let the period (1) last from time $t = 0$ to $t = T$, the currents at any intermediate time t being as shown in the figure. Let v be the difference of potential between B and C at time t. Then

$$\left.\begin{aligned} j &= C\frac{dv}{dt} \\ Gi + R_4(k+i) &= v \\ R_3k + R_4(k+i) &= V \\ R_1(i+j) &= R_3k - Gi \end{aligned}\right\}.$$

Writing I, J, K for $\int_0^T i\,dt$, $\int_0^T j\,dt$, $\int_0^T k\,dt$ respectively, we have on integration

$$\left.\begin{aligned} J &= Cv_0 \\ R_3K + R_4(K+I) &= VT \\ R_1(I+J) &= R_3K - GI \end{aligned}\right\}.$$

Hence
$$I = \frac{R_3VT - (R_3+R_4)R_1Cv_0}{(R_1+G)(R_3+R_4)+R_3R_4} \quad \ldots\ldots\ldots(40).$$

Let the stage (2) last from time $t = T$ to $t = 1/n$, where n is the frequency of the tuning fork. The currents in this stage are given by $j = 0$ and

$$\left.\begin{aligned} R_3k &= (R_1+G)\,i \\ R_3k + R_4(k+i) &= V \\ Gi + R_4(k+i) &= v_0 \end{aligned}\right\}.$$

Hence
$$i = \frac{R_3V}{(R_1+G)(R_3+R_4)+R_3R_4} \quad \ldots\ldots\ldots(41),$$

and
$$\frac{v_0}{V} = \frac{R_3(R_4+G)+R_4(R_1+G)}{(R_1+G)(R_3+R_4)+R_3R_4} \quad \ldots\ldots\ldots(42).$$

It follows from (40) and (41) that the total charge passing through the galvanometer in the period $1/n$ is

$$I + i\left(\frac{1}{n} - T\right) = \frac{R_3V/n - (R_3+R_4)R_1Cv_0}{(R_1+G)(R_3+R_4)+R_3R_4}.$$

Hence the galvanometer will show no deflexion if

$$R_3V = n(R_3+R_4)R_1Cv_0,$$

or, using (42), if

$$C = \frac{R_3}{nR_1(R_3+R_4)}\,\frac{(R_1+G)(R_3+R_4)+R_3R_4}{R_3(R_4+G)+R_4(R_1+G)} \quad \ldots(43).$$

Thus the capacity is measured in electromagnetic units in terms of the resistances.

In the experiments of Thomson and Searle the tuning fork was replaced by a rotating commutator running at a known speed, and the method was modified so as to permit of the use of a condenser with a guard-ring, whose electrostatic capacity could be ascertained by direct measurement.

A beautiful method of determining c by measurement of electromotive force was used by Fabry and Perot. By means of

their sensitive absolute electrometer (Art. 43) they found the E.M.F. of the Clark cell at 0° C. to be $4{\cdot}8443 \times 10^{-3}$ electrostatic units; and taking the accepted value in electromagnetic units ($1{\cdot}4491$ volts $= 1{\cdot}4491 \times 10^7$ E.M.U.) the quantity

$$c = \frac{\text{electrostatic unit of potential}}{\text{electromagnetic unit of potential}}$$

is found to be $2{\cdot}9913 \times 10^{10}$.

The following table gives some of the more recent determinations of the ratio of the units:

Rowland (1889)	$2{\cdot}982 \times 10^{10}$
J. J. Thomson and Searle (1890)	$2{\cdot}996$
Abraham (1892)	$2{\cdot}991$
Hurmuzescu (1896)	$3{\cdot}001$
Fabry and Perot (1898)	$2{\cdot}991$
Rosa and Dorsey (1907)	$2{\cdot}997$

Hence although it is probable that c is less than 3×10^{10}, the difference is certainly small and not yet known with sufficient accuracy to make it worth while to take it into account.

141. Localisation of the induced E.M.F. in a moving circuit. The law of electromagnetic induction tells us the total E.M.F. produced in any circuit by electromagnetic induction, that is the line-integral of electric force round the circuit. It is of some theoretical interest to find what the electric force is at every point. We can do this for a circuit moving in a fixed magnetic field by means of the electron theory, as has already been done in the case of Faraday's rotating disc. Let $PQ = ds$ be an element of an open or closed wire moving in a magnetic field, the field-strength in the vicinity being H (H_x, H_y, H_z) and the velocity of the element u (u_x, u_y, u_z). In addition to the impressed velocity u, the electrons in the element of wire have also a certain velocity v in virtue of the electric current; that is, the component velocities of an electron altogether are $u_x + v_x$, $u_y + v_y$, $u_z + v_z$. Hence the magnetic field exerts on the group of electrons a mechanical force per unit of charge which consists of two parts: one, the vector product of H/c and v which would exist already in a circuit at rest; and the other which we are now seeking, the vector product of H/c and u. The resultant of

these is the *apparent electric force* in the position of the element as far as it is due to electromagnetic induction, the electrostatic effect of any charges present in the wire or surrounding field being added to obtain the total electric force. We are not however concerned with the resultant force but only with its component in the direction of PQ; for a transverse force only tends to drive electrons to one side of the wire, and this effect is unimportant* when the wire is thin. Now of the two forces mentioned above, the first is entirely transverse and may be neglected; the components of the second are

$$\frac{1}{c}\begin{vmatrix} u_x, & u_y, & u_z \\ H_x, & H_y, & H_z \end{vmatrix}.$$

Since the direction-cosines of PQ are $(dx/ds,\ dy/ds,\ dz/ds)$ the apparent electric force in question may be taken as

$$\frac{1}{c}\begin{vmatrix} \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds} \\ u_x, & u_y, & u_z \\ H_x, & H_y, & H_z \end{vmatrix}.$$

The difference of potential between P and Q, as far as it is due to electromagnetic induction, is therefore

$$-\begin{vmatrix} H_x, & H_y, & H_z \\ dx, & dy, & dz \\ u_x, & u_y, & u_z \end{vmatrix}$$

in electromagnetic units. Now suppose that PQ moves in time dt into the position $P'Q'$, and let dN be the flux of magnetic force through the small parallelogram $PQQ'P'$. The direction-cosines of the normal to the plane of the parallelogram are (l, m, n), where

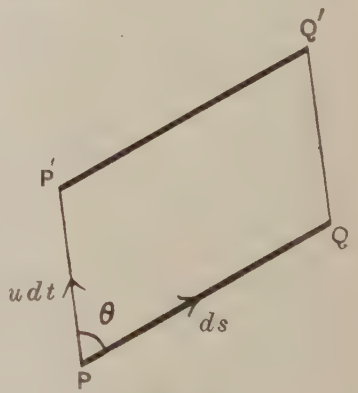


Fig. 192

$$l = \frac{dy \,.\, u_z - dz \,.\, u_y}{ds \,.\, u \sin\theta}.$$

* That is, electrically. The transference of this transverse mechanical action on the electrons to the molecules of the metal is of course the cause of the mechanical action of magnetic fields on currents.

Hence
$$dN = ds\,u\,dt \sin\theta\,(lH_x + mH_y + nH_z)$$
$$= dt \begin{vmatrix} H_x, & H_y, & H_z \\ dx, & dy, & dz \\ u_x, & u_y, & u_z \end{vmatrix}.$$

The difference of potential between P and Q is therefore equal to $-dN/dt$. Evidently the result may be extended at once to finite portions of the wire, and we have the following enunciation:

Let P, Q be any two points on a wire moving in a magnetic field, and let the ends of the wire move into the position P', Q' in the time dt. Then if dN is the flux of magnetic force through the band-like space contained between PP', QQ' and the two positions of the wire, the difference of potential between P and Q caused by electromagnetic induction is $-dN/dt$.

When the wire forms a closed loop this reduces to the ordinary form of the law of induction, since what we have called difference of potential is the line-integral of electric force along the wire. Moreover, there is no appreciable accumulation of electric charge in an ordinary closed circuit, so that the only potential differences are those due to electromagnetic induction. But it need not be thought that the theory of this article applies only to closed circuits, because it also gives the electric force at all points of a limited piece of wire in motion in a magnetic field. Suppose for example that we have a straight piece of wire PQ moving uniformly across lines of magnetic force at right angles to its length. Here there will be an apparent electric force, say, from P to Q. But this will pull electrons down to the P end of the wire, where they will accumulate until the electrostatic effect of the separated charges exactly neutralises the effect of electromagnetic induction, because there can be no steady current along the incomplete wire. It is, perhaps, of some interest to notice that in this and similar cases there are charges spread through the volume of the wire in addition to possible surface-charges on the outside.

When the wire (open or closed) remains stationary and the magnetic field changes, we could not explain the observed E.M.F. of electromagnetic induction on the present lines. It is, in fact, necessary to suppose that a changing magnetic field is

accompanied by an electric field, which in general has a tangential component at points on the wire. The point of view has thus changed somewhat, and we fix attention rather on the medium in which the interaction of magnetic and electric force occurs than in any peculiarity in the action of the field on the electrons in the wire. This widening of our views is of the utmost importance, and it is the properties of the electromagnetic medium (the ether) that will occupy our attention later (cf. Ch. XI).

CHAPTER IX

APPLIED ELECTRICITY

142. Mechanical generation of electricity. The use of primary batteries on a large scale is, for various reasons, quite out of the question; and recourse is had to the generation of currents by electromagnetic induction. The simplest apparatus for generating direct currents is shown diagrammatically in Fig. 193.

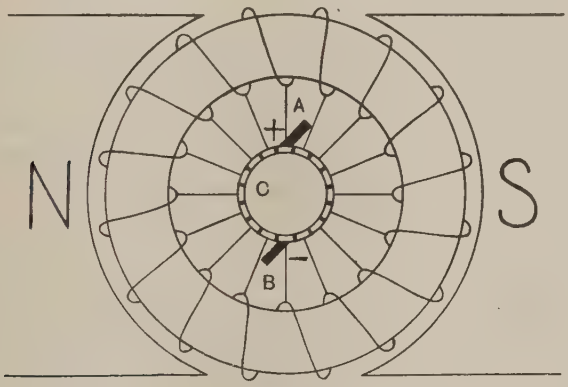


Fig. 193

An iron ring lies between the poles of a magnet, and is fixed on an axle so that it can be made to revolve at any desired speed. The axle also carries a *commutator C*, consisting of a number of copper strips parallel to the axle and insulated from it and from each other. The ring is wound uniformly with a continuous length of wire, and contacts are taken off at regular intervals (in Fig. 193 at every single turn) to the separate segments of the commutator. The whole system of ring, wires and commutator

is rigidly attached to the axle and rotates with it. The *brushes* A, B are two pieces of carbon pressed against the commutator at opposite points of its circumference, but fixed in space so that when the machine is running the commutator slides under the brushes. Wires lead from the brushes to carry off the currents generated.

Fig. 194 shows roughly the way in which the lines of magnetic induction pass through the ring from left to right. We may, if

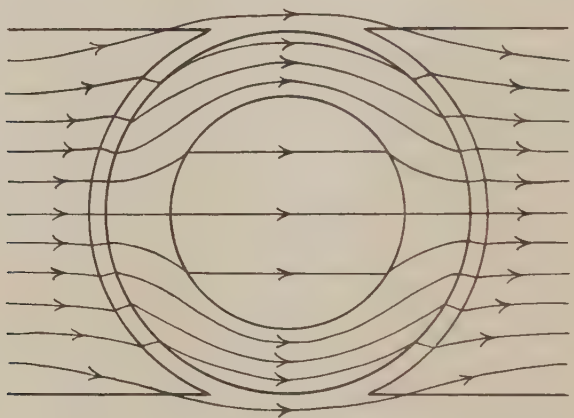


Fig. 194

we wish, suppose that they pass straight across; but this is unnecessary, as the E.M.F. generated can easily be found for any given distribution of the lines. Such a distribution is specified in terms of the flux of magnetic induction across an assigned cross-section of the ring: thus if N is the flux across a section making an angle θ with the horizontal, we shall take $N = f(\theta)$, a given function of θ. The flux evidently changes sign twice as we pass round the ring: the sign may be fixed by supposing N positive when θ lies between 0 and π, and negative when θ lies between π and 2π.

Let P, Q (Fig. 195) represent points on two adjacent segments of the commutator, joined to the windings on the ring by wires PP', QQ' respectively. The angle which PP' makes with the horizontal at time t will be denoted by θ or ωt according as it is

desired to call attention to the actual position of P in space or the position of a particular commutator segment at time t.

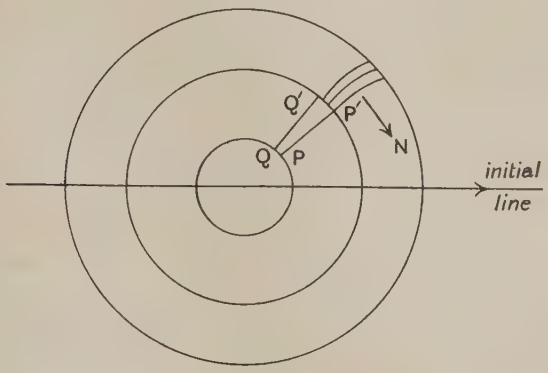


Fig. 195

The number of commutator segments is usually considerable (at least 20), and the theory is simplified by supposing the number to be indefinitely great. We shall further assume that the windings on the ring are very close together like those on a solenoid. Writing $d\theta$ for the infinitesimal angle between PP' and QQ', P' and Q' may be regarded as the ends of a flat and nearly closed coil of $md\theta/2\pi$ turns, where m is the number of turns of wire on the whole ring. The flux of magnetic induction through this small coil at time t is of the same sign as N if the coil is supposed to begin at Q' and end at P', the magnitude of the flux being

$$\nu = \frac{Nmd\theta}{2\pi} = \frac{md\theta}{2\pi} f(\omega t),$$

since $N = f(\theta)$.

Inside the ring electromagnetic induction causes an E.M.F. which drives the electric charges on the ring into certain definite positions, as in an electrostatic problem. But at a stationary point outside the ring (such as one of the brushes) there is no electromagnetic induction, but only the effect of the accumulated charges. Hence what we are really interested in is the potential at any point of the ring *due to the charges on the ring*. Let this

be V at the point θ. Then applying the law of electromagnetic induction to the small coil $Q'P'$, we have

$$dV - \frac{dv}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

since no current flows in the coil. Hence

$$dV = \frac{m\omega\, d\theta}{2\pi} f'(\omega t) = \frac{m\omega\, d\theta}{2\pi} f'(\theta).$$

Integrating, we have

$$V = \text{const.} + \frac{m\omega f(\theta)}{2\pi}$$

$$= \text{const.} + \frac{mN\omega}{2\pi}.$$

The arbitrary constant is unimportant, so that the distribution of potential is completely determined. If n is the number of revolutions of the ring per second, $\omega = 2\pi n$, and

$$V = \text{const.} + mnN \quad \ldots\ldots\ldots\ldots\ldots (2).$$

In Fig. 193 the potential is greatest at the top, opposite the brush A, and least opposite the brush B.

The general formula (2) is easily adapted to the ideal case, not realisable with an iron ring, in which the lines of magnetic induction pass straight across from left to right, so that the ring is situated in a region of constant B. If A is the area of cross-section of the ring, we have $f(\theta) = AB \sin\theta$, and therefore

$$V = \text{const.} + mnAB \sin\theta \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Since $\theta = \omega t$ this may equally well be written

$$V = \text{const.} + mnAB \sin\omega t \ldots\ldots\ldots\ldots\ldots (4).$$

A study of the last two formulae reveals the part played by the commutator of the machine. If we fix our attention on a particular segment of the commutator and follow it round as it moves, the potential will alternate in time in accordance with the law (4). A fixed brush however does not follow the segment round in this way, but picks out a definite position in space corresponding to a particular value of θ, and takes up the potential of the segments as they pass it. The formula (3), or more generally (2), therefore gives the *steady* potential acquired by a brush placed in the position θ.

The brushes are naturally placed in such a position that their difference of potential is as great as possible. If N_{max} is the greatest value of N for any cross-section, the opposite point on the ring will have the value $-N_{max}$ of N, and if the brushes are placed at these points the E.M.F. developed is

$$V_0 = 2mnN_{max} \text{ true electromagnetic units}$$
$$= \frac{2mnN_{max}}{10^8} \text{ volts} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

To see the order of magnitude of the potentials generated, consider the case of a ring of cross-section 150 sq. cm., lying in a strong field so that the greatest value of B in the ring is 15,000, and performing 25 revolutions a second. If there are 100 turns of wire on the ring,

$$m = 100, \quad n = 25, \quad \text{and} \quad N = 2{\cdot}25 \times 10^6.$$

Hence
$$V_0 = 112 \text{ volts}.$$

The positive sign in equation (5) means that the brush A in Fig. 193 is the positive brush. This would of course cease to be true if the north and south poles of the permanent magnet were interchanged, or if the rotation was in the clockwise direction. A reversal of polarity would also be produced if the ring was wound in the opposite direction to that shown in the figure. For the sake of uniformity we shall always suppose the ring wound as shown.

143. The drum armature. The term *armature* is used generally for the part of a direct-current generator from which the currents are drawn, in contrast with the *field magnets* which furnish the field of force. The ring armature already described has the advantage of being easy to understand: but it is nevertheless rarely used, as it does not advantageously utilise the space between the field magnets. The effective flux of induction is confined to the comparatively small cross-section of the ring itself, whereas it is desirable to use the much greater flux obtainable from a solid core lying between the poles of the field magnets. The latter object is attained in the *drum armature* introduced by von Hefner-Alteneck in 1872. The principle of the drum

armature, illustrated diagrammatically in Fig. 196, is to wind a solid cylindrical core uniformly with wire somewhat after the

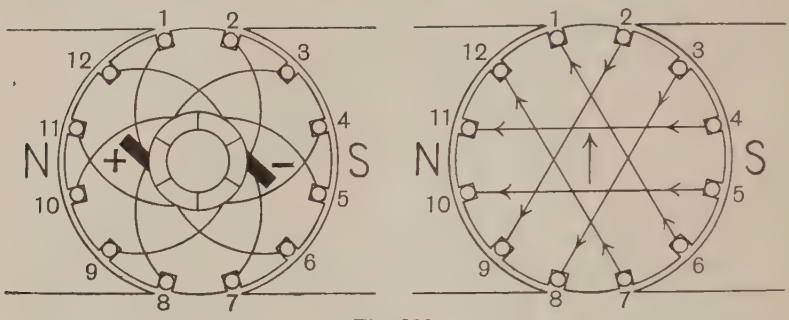


Fig. 196

manner in which wool is wound into a ball. The small circles marked 1 to 12 represent copper wires or bars lying parallel to the axle, and let into slots in the core. The commutator, here shown as having six segments only, is rigidly fixed to the axle as before, and is joined to the numbered wires by connexions shown in the left-hand figure. The right-hand figure shows the way in which the wires are joined up at the back, and represents a view from the same end as before, i.e. it is what we should see at the back if the armature core was transparent.

The drum winding gives a closed circuit returning on itself. The wire may be supposed to begin at the nearest end of the bar 1, to proceed to the commutator and to the bar 8, thence to the back and to 3. Coming to the front it proceeds to 10 *viâ* a commutator segment, thence to 5, and so on. The complete cycle is

where the top line includes bars traversed from back to front and the bottom line those traversed from front to back*.

The essential identity of ring and drum armatures may be

* The arrows on the right-hand figure have a special significance. See Art. 147.

seen by considering the case in which the number of turns of wire is very great. The section 1 8 3 of the drum winding, for example, may be considered as a single nearly closed flat coil, since the ends 1, 3 are now very near together. This is connected to a commutator segment like the top turn of the ring winding in Fig. 193, the only difference being that the corresponding segment of the drum is displaced through an angle of 90° in the counter-clockwise direction with reference to that of the ring. Hence, with drum armatures, the line of brushes is nearly parallel to the direction of the magnetic field.

The formulae of the last article therefore apply to the drum winding also, N now denoting the maximum flux of induction through any cross-section of the whole core, which is much greater than that for a ring occupying the same space.

If the iron armature was solid its motion in a magnetic field would give rise to Foucault currents in the mass of metal, which would heat the armature considerably and also absorb a great deal of power which might otherwise be saved. For this reason the cores of armatures are *laminated*; that is, formed of thin iron discs insulated from one another.

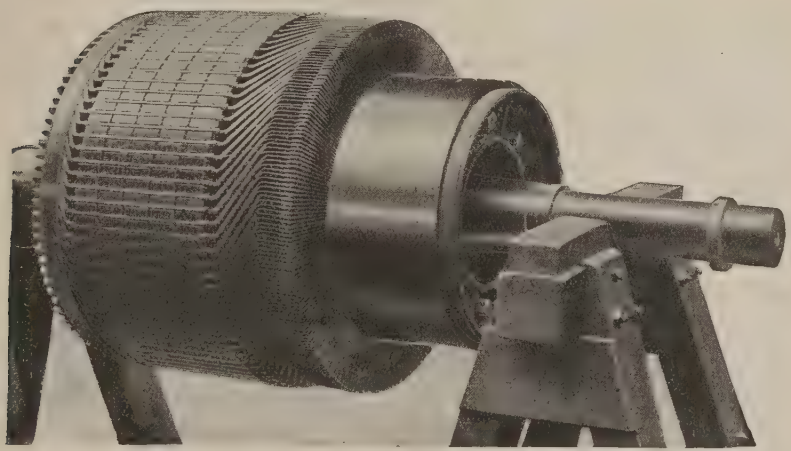


Fig. 197

Fig. 197 shows a drum armature completely wound and with commutator attached.

144. Field-magnets. Saturation curve. The power of the magneto-electric machine is considerably increased by discarding the permanent steel field-magnets in favour of electromagnets excited by external current. A fundamental experiment is to find out how the difference of potential between the brushes when no current is taken off (called the brush voltage on open circuit) varies with the current used to excite the field-magnets. Fig. 198 gives a diagrammatic representation of the generator and

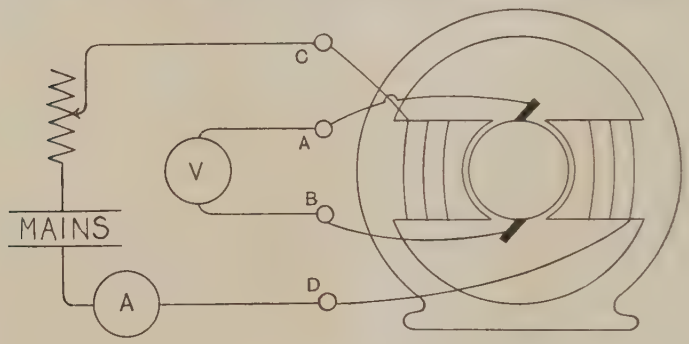


Fig. 198

of the connexions for this experiment. It is convenient to have wires from the brushes brought out to two insulated screw terminals A, B on the case of the machine, while the terminals C, D are joined inside to the magnetising coils, which are themselves connected across by a wire not shown in the figure. The magnetising current is supplied from the electric mains or from accumulators, and regulated by a rheostat capable of carrying the required current at every stage. The terminals A, B are joined to a suitable voltmeter, and readings of the ammeter and voltmeter are taken with the machine running at constant speed. Fig. 199 shows a curve taken off from a certain small generator. Its general shape is readily predicted: for the field magnets and the armature core constitute a nearly continuous iron or steel bar wound with wire somewhat after the manner of a solenoid. The "magnetising force" is therefore roughly proportional to the current in the coils, while equation (5) shows that the brush voltage is proportional to the flux of induction across the armature core. The (i, V) curve of Fig. 199 should therefore resemble

the (H, B) curve of a specimen of iron: the bend in the curve is however generally found to be less pronounced. In practice, there will always be enough residual magnetism left over from

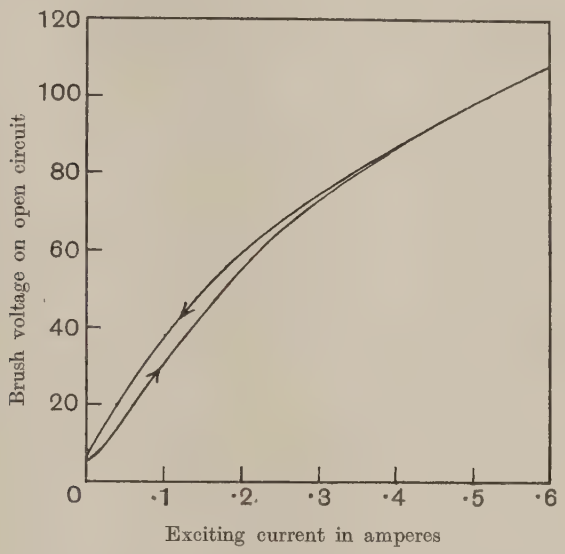


Fig. 199

previous experiments to prevent the curve from passing through the origin. The curve with decreasing current, it will be noticed, is but slightly higher than that obtained on ascending, as the complete hysteresis cycle is not described.

Although Fig. 198 is entirely diagrammatic, it does in fact represent a favourable shape for the field-magnets and shell of the machine. The field-magnet windings should be near the armature, so that the flux of magnetic induction through the latter should be as great as possible. Some old patterns of generator are very unfavourable in this respect, as many lines of magnetic induction which might have passed through the core are allowed to crowd into other masses of iron in the vicinity.

The (i, V) curve of the present article is known as the *saturation curve* (or *magnetic characteristic*) of the machine, and is useful in helping us to predict its performance under various conditions. In making the experiment it is convenient to drive the machine

by means of an electric motor, the speed of which can be kept constant or varied at will. The fact that V is proportional to the speed of rotation can be verified in this way.

145. Shunt, series, and compound generators. Machines which have their field-magnets excited electrically are known as *generators* (or *dynamos*) to distinguish them from magneto-electric machines with permanent magnets. In the early stages of electrical engineering the exciting current was supplied by a few primary batteries. An improvement on this practice was to use a small auxiliary magneto-electric machine to excite the field-magnets; but the really important advance was that made by S. Hjorth about 1851, and by W. von Siemens in 1867, who suggested the use of the currents produced by the generator for the excitation of its own field-magnets.

The two principal kinds of self-exciting generators are the *shunt-wound* and *series-wound* types. These are shown diagrammatically in Fig. 200, where the field-magnets are drawn as a

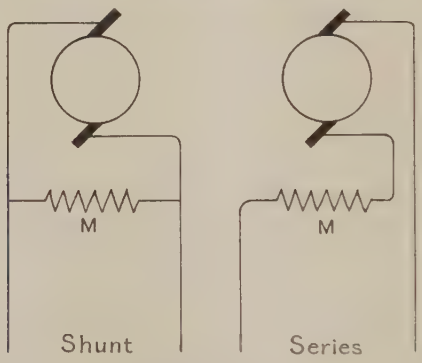


Fig. 200

separate coil M for the sake of clearness. The wires at the bottom lead to the external circuit, and the names of the machines are derived from the fact, which is obvious from the diagrams, that the field-magnets are shunted across the armature in one case and are in series with it in the other. If the generator is provided with four separate terminals as shown in Fig. 198 it may be arranged as a shunt or series generator at will: thus if AC and BD are joined and the current taken from A and B the machine is

shunt-wound, while a series winding is obtained by joining A to C and using B and D as the terminals for the external circuit.

These generators depend in the first place on residual magnetism to start them. Consider the case of a shunt-wound machine on open circuit. The residual magnetisation of the field-magnets gives rise to a small E.M.F. on rotation of the armature. This results in a small current being sent round the field-magnets, which may reinforce or destroy the original magnetisation. If the former, the increased field causes a rise in the brush voltage, which again strengthens the field-magnets, and so on. The result is that after a few revolutions of the machine the field-magnets are saturated and the generator is developing its full brush voltage. If on the other hand the first weak current sent through the field-magnets is in the direction of demagnetisation, the process will go on until the magnetisation is all removed, and the generator will not work at all. The reader can verify this latter fact himself, by taking a shunt-wound generator and reversing the connexions of the field-magnets, when the generator will be found to yield no brush voltage.

The *compound* generator has both shunt and series windings on the field-magnets, thus combining the two previous types.

146. Characteristic curves of generators. We have hitherto supposed that no current is taken off the generator to any external circuit. But if the armature terminals are connected through an external resistance, a current flows round the circuit, and this in general will give rise to a change in the brush voltage. Consider for example the case of the magneto-electric machine or separately excited generator, and assume for the present that the currents in the armature do not disturb the impressed magnetic field. If i is the current from A to B in the external circuit, a current i flows inside the armature from B to A, and this is divided so that a current $\frac{1}{2}i$ flows down the elementary coil $P'Q'$ considered in Art. 142. If dR is the resistance of $P'Q'$, equation (1) now becomes

$$dV - \frac{dv}{dt} = -\tfrac{1}{2}i\,dR,$$

or

$$dV = \frac{m\omega f'(\theta)\,d\theta}{2\pi} - \tfrac{1}{2}i\,dR.$$

This may be written $dV = dV_0 - \frac{1}{2}idR$, where V_0 corresponds to the case of zero current. Hence if V is the actual brush voltage and V_0 the brush voltage on open circuit, or the total E.M.F. developed, we have

$$V = V_0 - Ri \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where $R = \frac{1}{2}\int dR$ is the resistance of the armature considered as a divided circuit connecting B and A.

The behaviour of a generator under load, i.e. when supplying current, is shown by a curve connecting the external current i with the difference of potential V between the brushes. This curve is called the *external characteristic*, or simply the *characteristic* of the generator, and it is usually understood that the speed is maintained at the same value for all the currents. It follows from what has been said that the characteristic of the separately excited generator would be a straight line if it were not for the effect of the currents in the armature in disturbing the impressed magnetic field (armature reaction). In practice, the brush voltage is slightly lower than that calculated from the formula (6), particularly for the higher currents. The calculated and observed

Separately excited generator
Exciting current ·54 ampere, resistance of armature 2·15 ohms

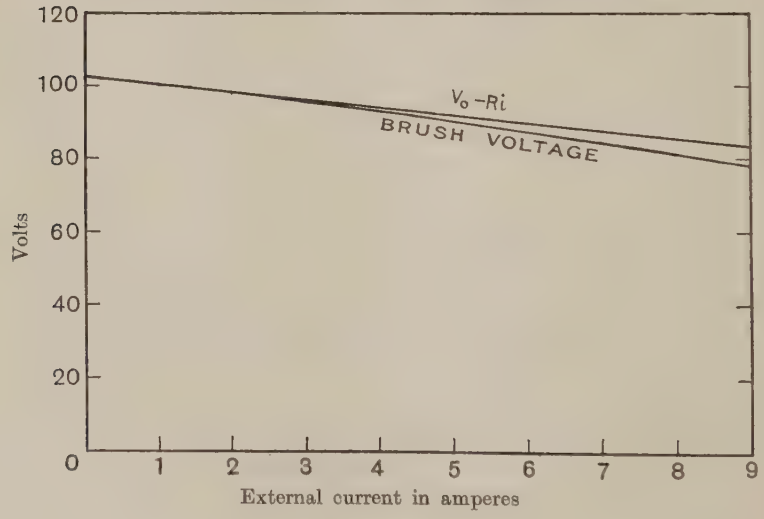


Fig. 201

voltages for a certain small generator are given in Fig. 201, the difference between the two curves showing the effect of armature reaction.

Fig. 202 shows the connexions for taking off the characteristic of a shunt-wound generator, *A*, *B* being the terminals of the armature and *C*, *D* of the field-magnets. In this case any fall of brush voltage diminishes the current in the field-magnets in the same ratio, so that the brush voltage falls more rapidly than before as the load increases. If the armature resistance is sufficiently high it may even happen that the generator will not yield more than a certain current, and if the external resistance is gradually decreased the current will ultimately fall as well as the brush voltage (Fig. 203). Peculiar results are also found for the return half of the curve obtained by increasing the external resistance from a small value to infinity, the effect of hysteresis

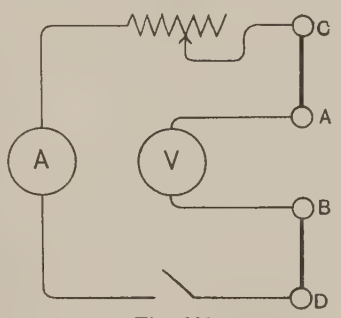


Fig. 202

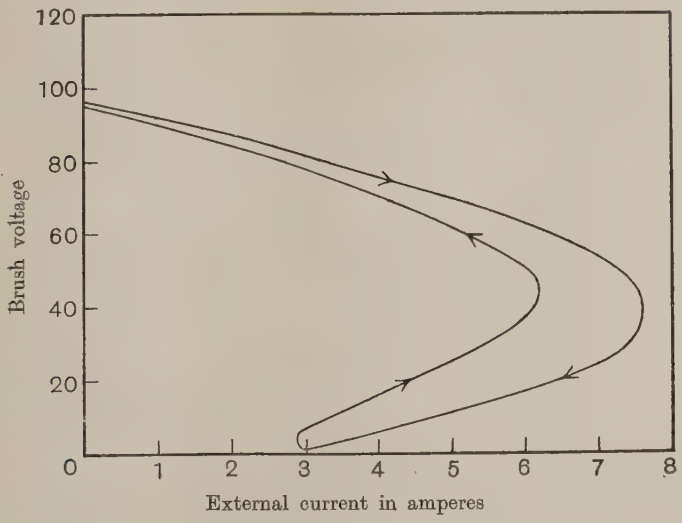


Fig. 203

being particularly noticeable here. These curves give insight into the working of generators (cf. Art. 147), but it is only fair to say that they do not represent the facts with large modern machines. The armature resistance is generally so low that the voltage falls only a few per cent. up to full load, the theoretical maximum current being far greater than the windings of the generator would carry.

In the series generator the whole current passes through the field-magnets. It follows that there is no excitation on open circuit, so that the terminal voltage has the low value resulting from remanent magnetism. As more and more current is taken off the excitation rises, and with it the terminal voltage: but the increase cannot go on indefinitely since the field-magnets must ultimately become saturated. The terminal voltage thus rises to a maximum and then decreases somewhat like that of a separately excited generator. In the compound generator the initial rise of excitation due to the series winding tends to prevent the drop of terminal voltage that would occur with the simple shunt machine, and by suitable design the terminal voltage may be made nearly independent of the current over a large range.

The compound generator is useful for small lighting installations, where constancy of voltage is very desirable. But it is by no means generally employed in large generating stations, because it is found quite convenient to regulate the voltage of simple shunt machines with an adjustable resistance in the field-magnet circuit. The series generator, on account of the great variation of terminal voltage with current, is but rarely used.

147. Theory of direct-current generators. We have seen that the variation of terminal voltage with current depends on three factors, namely

(1) loss of potential due to the resistance of armature, or armature and field-magnets;

(2) change of potential due to alteration in the excitation of the field-magnets;

(3) armature reaction.

The effect of the first two causes may be determined graphically from a knowledge of the saturation curve and the resistances of

the armature and field-magnets. We shall neglect hysteresis and consider first the case of the shunt generator.

Let R be the resistance of the armature, S the resistance of the field-magnets and j the current through them, i the current in the external circuit. The curve V_0 in Fig. 204 is the mean of the ascending and descending curves in Fig. 199. The brush voltage V is evidently given by $V = Sj$, and is therefore represented by a straight line in the diagram. Now the current in the armature

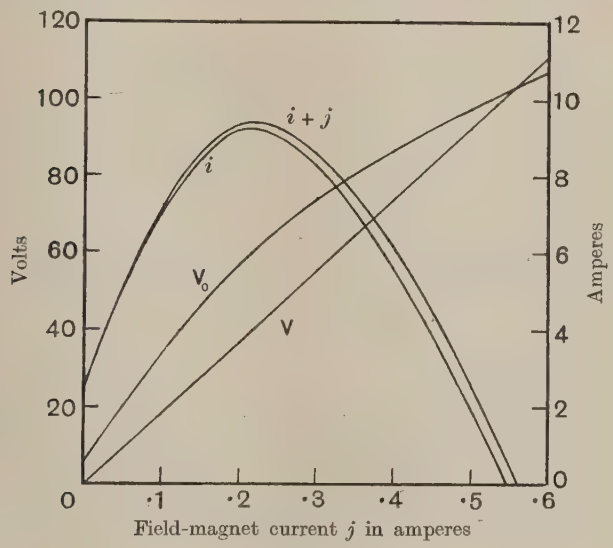


Fig. 204

is $i + j$ since the armature supplies current to the field-magnets as well as to the external circuit, so that $V = V_0 - R(i + j)$. Hence $i + j = (V_0 - V)/R$ and may be plotted on the figure for various values of j, and the curve representing i as a function of j may also be drawn. If we now pick off values of i and plot them against the corresponding values of V we have the theoretical form of the shunt characteristic (Fig. 205). This curve may be compared with Fig. 203, and we see that the chief features of the characteristic are correctly represented. We should not expect absolute numerical agreement, firstly because we have neglected armature reaction and secondly because it is not always easy to

ascertain accurately the resistance of the armature under the conditions of running. An appreciable part of the resistance always occurs where the carbon brushes press on the commutator.

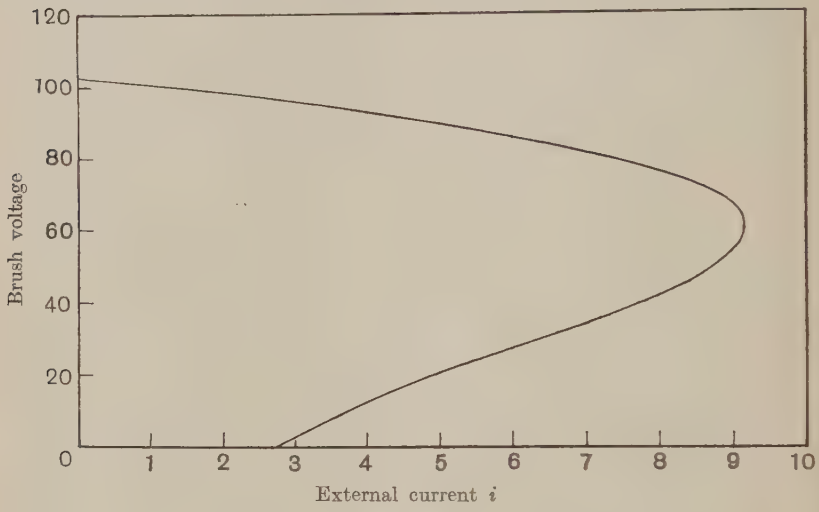


Fig. 205

The theory of the series generator is simpler, because the external current i is the same as that through the field-magnets. If V_0 is the voltage corresponding to the current i in the saturation curve and V the terminal voltage (difference of potential between the ends of the external circuit, not that between the brushes), we have $V = V_0 - (R + S)\, i$.

As regards armature reaction, it is easy to see in a general way what effects it will produce. Consider for example the drum armature represented in Fig. 196. If a current is taken from the brushes, the complete cycle of the armature winding is represented by the diagram

where we have shown the position of the brushes at the time being and the direction of the currents in the armature coils.

These directions have already been shown on the right-hand part of Fig. 196. Remembering that this diagram represents the coils at the back of the armature, it is easy to see that the magnetic effect of the armature currents is to give a transverse field pointing from below to above as shown by the central arrow. The field is thus twisted round in a counter-clockwise direction inside the armature, and if the brushes are left in the same position and various currents taken from the generator the E.M.F. developed will fall on this account alone, quite apart from the loss of voltage due to resistance.

Since the brushes always short-circuit a part of the armature when passing from one segment of the commutator to the next (cf. Fig. 193), sparking will occur if the difference of potential is too great. For this reason it is usual to *advance* the brushes of a generator slightly, i.e. to move them round in the direction of rotation. The field due to the armature currents in Fig. 196 is now turned slightly in the counter-clockwise direction, and the reader can satisfy himself that it is possible to arrange matters so that the resultant field for any particular current is parallel to the line of brushes. It is not necessary, or even desirable, that the adjustment should be exact, and in practice it is usual to give the brushes a *fixed* advance corresponding to moderate armature currents.

148. The direct-current motor. Suppose that the armature of a magneto-electric machine (Fig. 206) is held fast, and a

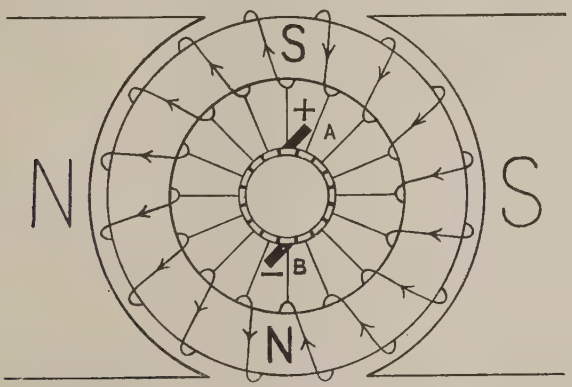


Fig. 206

current sent through the armature from A to B from an external battery, A being attached to the positive pole of the battery. The two halves of a ring armature may be considered as two solenoids bent over until their ends meet, so that the armature coils may for all intents and purposes be replaced by two magnetic poles SN at the ends of the solenoids (cf. Art. 85). The N pole of the field-magnets repels the N pole in the armature and attracts the S pole. Similarly the S pole of the field-magnets exerts attractions and repulsions on the armature poles, and the result of all four forces in the present case is to produce a couple tending to turn the armature in a counter-clockwise direction. If the armature is free to move it will do so and may be made to perform mechanical work. This is the principle of the electric motor.

It is clear from what has been said that motors and generators are essentially the same in construction, and indeed every generator can be made to run as a motor without structural alteration. We proceed to find an expression for the couple exerted on the armature, or the *torque* as it is called, when a current i flows through it. Consider the elementary coil in Fig. 195, in which current $\frac{1}{2}i$ now flows from Q' to P'. The potential energy of this coil in the magnetic field is $-\frac{1}{2}iN$ per turn, in the sense that the couple on it is $\frac{1}{2}i\dfrac{dN}{d\theta}$ per turn. Since there are $m\dfrac{d\theta}{2\pi}$ turns, the couple on the elementary coil is $\dfrac{im}{4\pi}dN$.

When the brushes are placed in the position of maximum flux the total couple on the coils of the right-hand half of the armature is $imN_{\max}/2\pi$; and since there is an equal couple on the left-hand half the torque T_0 is given by

$$T_0 = imN_{\max}/\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If i is measured in true electromagnetic units T_0 comes out in absolute c.g.s. units; but it is more instructive to measure i in amperes and T in kilogram-metres (moment due to the weight of 1 kilogram at a distance of 1 metre). Then equation (7) is replaced by

$$T_0 = \frac{3{\cdot}24}{10^{10}}imN_{\max} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

For example, if a current of 50 amperes is sent through the

armature of the magneto-electric machine considered in Art. 143, $T_0 = 3{\cdot}6$ kilogram-metres. It should be noticed that in these formulae N_{max} is the maximum flux due to the field-magnets only, neglecting any change in the magnetism of the ring caused by the armature current and that the formulae apply as well to the drum as to the ring armature.

The electric motor has the remarkable property of self-regulation; that is, it takes only just enough current from the mains to perform the work on hand. In order to obtain insight into the process of regulation, let us suppose that the motor is running steadily at a speed of n revolutions per second, and that its rotation is resisted by a couple T arising from external braking and from friction or other dissipative causes in the motor itself. Since there is no angular acceleration T must be equal to the couple T_0 arising from the mechanical action of the magnetic field on the current i in the armature, so that

$$T = imN_{max}/\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

If A and B are connected to the terminals of a battery of potential V_0, and R is the resistance of the armature, we cannot conclude that $i = V_0/R$, because the mere rotation of the armature makes it develop an E.M.F. just as if it was the armature of a generator. This E.M.F. is already known to be given by

$$V = 2mnN_{max} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

and tends to drive a current through the armature from B to A, i.e. to oppose the applied potential V_0. Hence

$$Ri = V_0 - V = V_0 - 2mnN_{max} \quad \ldots\ldots\ldots (11).$$

The E.M.F. V is called the back E.M.F. of the motor, and it is necessary to take it into consideration in any discussion of the action of motors.

Neglecting for the present the effect of armature currents on N (motor armature reaction), we have the two equations (9) and (11) to determine i and n when the applied voltage and the resisting couple are given; that is, current and speed regulate themselves after the applied load. As T increases, equation (9) shows that i rises in proportion. Equation (11) then shows that n must

decrease, and with it the back E.M.F. Hence with a motor whose field-magnets have constant excitation, an increase in the load causes a rise in the armature current and a drop in the speed: this drop will be smaller the smaller the resistance of the armature. In modern machines it does not amount to more than a few per cent. at full load: in fact, the limit to the load is determined rather by the heating of the armature coils with increasing current than by any marked slowing down of the motor.

Returning to the example of the magneto-electric machine already considered, suppose that the armature resistance is 1 ohm and the applied potential 100 volts. The following table gives the current and speed for various resisting couples:

Couple in kilogram-metres	·1	·5	1	2
Armature current in amperes	1·4	7	14	28
Back E.M.F. in volts	98·6	93	86	72
Speed, revolutions per second	22	21	19	16

For a given direction of field and rotation, the direction of the armature current is different in a motor and a generator. Hence the effect of armature reaction is reversed: and for this reason the brushes of a motor have to be displaced *backwards* along the commutator in order to ensure sparkless commutation.

Motors may be either shunt or series-wound like generators. The two types, which have important and very distinct properties, are considered separately in the two following articles. Compound motors are seldom used.

149. The shunt motor. In the shunt-wound motor running normally, the armature AB is joined in parallel with the field-magnet coil CD. Thus both armature and field-magnets are connected directly to the mains. But it would not be safe to start the motor by simply putting on the mains, because the armature resistance is generally so low that an excessive current

would have passed before the motor acquired enough speed to develop sufficient back E.M.F. The proper method of starting a shunt motor is by means of a starting switch, one form of which is shown diagrammatically in Fig. 207. The switch carries three

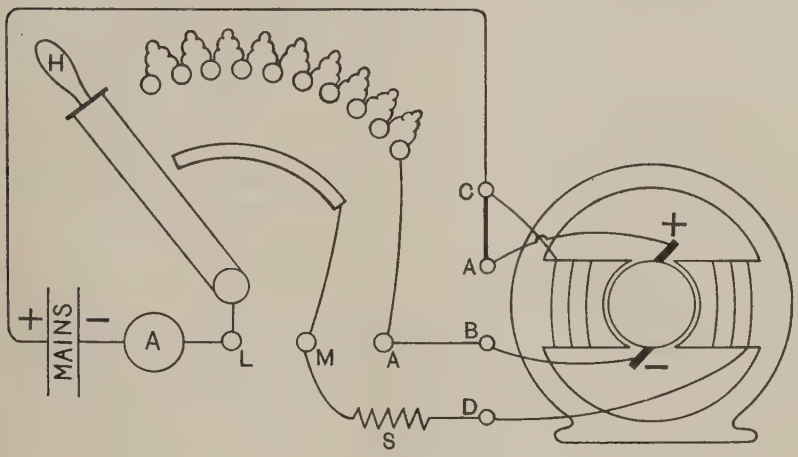


Fig. 207

terminals usually marked *L* (line), *M* (magnets), *A* (armature) respectively. *L* is joined inside to a lever moved by a handle *H*, and carrying two metallic spring contacts. One of these contacts moves over a solid metal bar, in the form of a sector of a circle, connected to the field-magnets; the other over a row of metallic studs, between each of which is a resistance coil so that the whole forms a rheostat, the further end being connected to one end of the armature. The other ends of the armature and field-magnets are connected together and joined directly to the mains. On putting on the switch connexion is first made with the metal sector, thus exciting the field-magnets fully while the motor is still at rest. When connexion is made with the first stud of the rheostat a weak current is sent through the armature, sufficient to set it rotating. A back E.M.F. is thus developed tending to reduce the armature current. In this way, by cutting out the resistance in the armature circuit gradually, the motor can be got going without any excessive current passing at any stage;

and when the switch is pushed right over the mains are applied directly to both armature and field-magnets.

The shunt-wound motor, having constant excitation, has thus the properties mentioned in the last article: that is, it will work at a variety of loads at nearly constant speed. The speed can, however, be regulated by means of a rheostat S in the field-magnet circuit, being greater the greater the resistance. We shall consider this shortly in detail, as it affords an interesting illustration of the principles underlying the action of motors.

When the motor is running unloaded a certain power W_1 will be required to overcome friction, hysteresis, and Foucault currents. If T_1 is the corresponding torque that has to be exerted by the motor at speed n, $W_1 = 2\pi n T_1$. Now the power required to overcome friction is probably proportional to the speed: that required to overcome hysteresis varies as the speed and also depends on the excitation. The strength of the Foucault currents at any point of the armature is proportional to B and to the speed: hence the power lost in Foucault currents is proportional to n^2 and also depends on the excitation. Hence T_1 is of the form $A + Bn$, where A and B depend on the excitation: from which it is not difficult to devise experiments to determine T_1 for all speeds and exciting currents. But for our present purpose, since T_1 is generally small in comparison with the external couple T required to be overcome, it may be regarded as independent of the speed or excitation, particularly as the variation of these magnitudes with a shunt motor are not great. Moreover, T_1 will be taken to be independent of the load: in short, an absolute constant.

With this simplifying assumption, which is not claimed as representing the actual facts, we can examine the behaviour of motors theoretically, provided that we also neglect armature reaction as hitherto. As in the case of the generator, we require a knowledge of the saturation curve: this can be found by running the motor as a generator from an external source of power. Writing N for N_{max}, the brush voltage at constant speed n_0 is given by $V = 2mn_0N$; hence the saturation curve gives us the value of mN for any value of the field-magnet current j. As an example of these methods we may consider the way in which the speed

of a motor depends on the total resistance S in the field-magnet circuit, when yielding a constant torque T (inclusive of that required to overcome friction, hysteresis and Foucault currents). If V_0 is the applied voltage, the field-magnet current $j = V/S$ is known in terms of S, and therefore also mN is known. The equation $T = imN/\pi$ then gives the armature current i, and the back E.M.F. of the motor is $V = V_0 - Ri$. But if n is the speed $V = 2mnN$; and thus n can be calculated for all values of S. The following table gives some figures for the generator considered in Art. 147, when run as a motor off a 100 volt circuit. The saturation curve V_0 of Fig. 204, which is the mean of the ascending and descending curves of Fig. 199, is taken as a basis, and the resisting torque is taken as ·3 kilogram-metres.

Total resistance S in field-magnet circuit	Field-magnet current j (amperes)	mN in absolute units $10^8 \times$	Armature current i (amperes)	Back E.M.F. V (volts)	Speed (revolutions per second)
185	·54	1·65	5·62	87·9	26·6
200	·50	1·59	5·82	87·5	27·5
220	·46	1·51	6·14	86·8	28·7
240	·42	1·43	6·48	86·1	30·1
260	·39	1·37	6·76	85·5	31·2
280	·36	1·31	7·06	84·8	32·4
300	·33	1·26	7·35	84·2	33·4
320	·31	1·22	7·60	83·7	34·3
340	·29	1·18	7·85	83·1	35·2

It is easy to see from the general formulae how it is that the speed rises with decreasing excitation. The speed is proportional to V/N; but if R is small V does not vary very much, so that n varies inversely as the flux. But the armature current is accurately proportional to $1/N$ on the present theory, so that it rises even more rapidly than the speed. Hence the resistance of the field-magnet circuit cannot be altered within very wide limits without causing a dangerous speed of the motor, or an excessive armature current, or both. For this reason it is important to make sure before starting a motor that the field-magnet circuit is not disconnected.

An examination of the diagrams already given in this chapter will show that if A is the positive brush of a shunt generator, it will run as a motor in the *same direction* when A is connected with the positive pole of the external supply.

150. The series motor. In the series motor, as in the series generator, the same current flows through both armature and field-magnets. The connexions are shown in Fig. 208: the

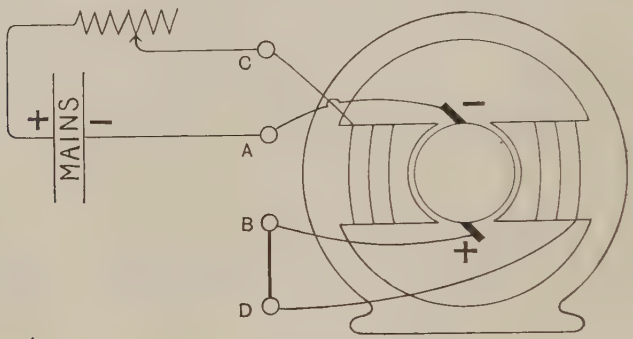


Fig. 208

starting arrangements are simpler than for the shunt motor, a simple resistance that can be cut out in stages being all that is required.

The series motor has the following distinguishing advantages over the shunt motor:

(1) It has a high "initial torque"; that is, when started against a load it exerts a large couple to begin with, so that it comes quickly to its full speed.

(2) As the load increases the consumption of current increases, but not so rapidly. This, of course, necessitates a slowing down of the motor under heavy load.

These two properties render the series motor invaluable for electric traction. By their aid electric trains are enabled to get up speed rapidly: less time is wasted in starting and stopping at stations. As regards (2), it is highly desirable in the interests of uniform consumption of power that excessive current should not be taken, for example, by an electric tram mounting an incline, even though it results in a decrease of speed.

Against these advantages is the great drawback that, if the load of a series motor is suddenly removed the motor will generally run at a dangerously high speed. This prevents its use in laboratory work and generally in cases of sudden fluctuation of load; e.g. in driving by a belt, which may suddenly come off.

In order to see clearly how these effects are produced, let us make the approximate assumption of the last article, namely that it requires a certain constant torque T_1 to overcome friction, hysteresis and Foucault currents. Let T be the torque exerted by the motor at speed n when carrying a current i, and let R be the total resistance in the motor circuit, inclusive of any starting resistance that may be present. Then $T = imN/\pi$ and

$$Ri = V_0 - 2mnN.$$

As before, a knowledge of the saturation curve gives mN as a function of the magnetising current, which in this case is i. Since mN increases with i, the torque increases more rapidly than the current, which explains the above-mentioned property (2). Moreover, when the motor is starting there is a small back E.M.F. and a large momentary current. As a result the field-magnets are highly saturated and a great torque is produced.

In general, the speed is given by

$$n = \frac{V_0 - Ri}{2mN},$$

and in normal running, with the starting resistance short-circuited, R is small. Hence the speed is not far from being inversely proportional to the flux N. If the load is suddenly removed T sinks immediately to T_1, the current falls to a low value and the speed becomes very high.

151. Brake test of a motor. In order to test a motor in the laboratory we require a means of applying a known retarding couple to it. One method is afforded by the *rope brake*, shown in Fig. 209. A represents an end view of

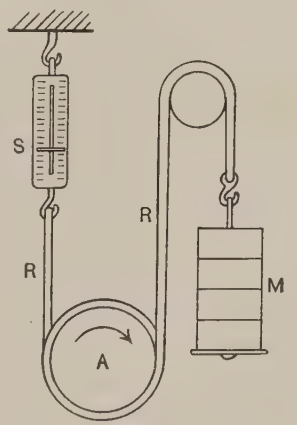


Fig. 209

a drum fixed on one end of the armature shaft. A rope R is wound once or twice over the drum, and after passing over a pulley supports a mass M whose magnitude can be varied. The other end of the rope is attached to a fixed beam, a spring balance S being placed between the rope and the beam to register the pull on the "light" side of the rope, equivalent say to the weight of a mass M_1. If M and M_1 are measured in grams and a is the distance of the centre-line of the rope from the axis of the motor, measured in centimetres, the external couple is $(M - M_1)\frac{a}{10^5}$ kilogram-metres. If n is the number of revolutions of the motor per second, the rate at which the motor is performing useful work is easily seen to be

$$2\pi n\,(M - M_1)\,ga \text{ ergs per second} = \frac{6{\cdot}16}{10^4}\,(M - M_1)\,na \text{ watts.}$$

The test of a shunt-wound motor is advantageously carried out at constant speed and with a constant E.M.F. applied to the brushes. The voltage of the mains will tend to drop as more current is taken by the motor, so that it is necessary to have a resistance in the main circuit, which can be adjusted until a voltmeter attached to the brushes gives a certain specified reading V_0. The speed is maintained constant by means of a regulating resistance in the field-magnet circuit.

Let i be the main current, j the current in the field-magnets at any stage. Then the electrical power supplied is $V_0 i$ watts. If W is the useful power, measured as explained above, the fraction of the power supplied which is really utilised is given by $\eta = W/V_0 i$. This fraction is called the *efficiency* of the motor, and is obviously of great importance in judging the economy of motors under various circumstances.

It is interesting to discover where the energy which is not utilised goes to, and to do this we require to know the resistances of the armature and field-magnets of the motor. The field-magnet resistance can be measured on the Wheatstone's bridge. The armature resistance is often very low and can be measured by passing various currents through the armature and observing the corresponding difference of potential between the brushes,

the motor of course remaining at rest during the process. It is found that the resistance is less for large than for small currents, because it partly consists in contact resistance between the brushes and the commutator. No great error is committed in taking the value corresponding to the normal working current, as has already been done in theoretical discussions. If R is the armature resistance, $R\,(i-j)^2$ watts are lost in heating the armature conductors; and $V_0 j$ watts go to heat the field-magnet windings. The remainder, say W_0, is expended in friction, hysteresis and Foucault currents. We have thus a means of estimating the relative magnitudes of the three losses; and it may be mentioned that methods have also been devised for further analysing the loss W_0. Fig. 210 shows the result of an experiment on a 4 horse-power shunt motor running at speed 1150 per minute and brush voltage 98.

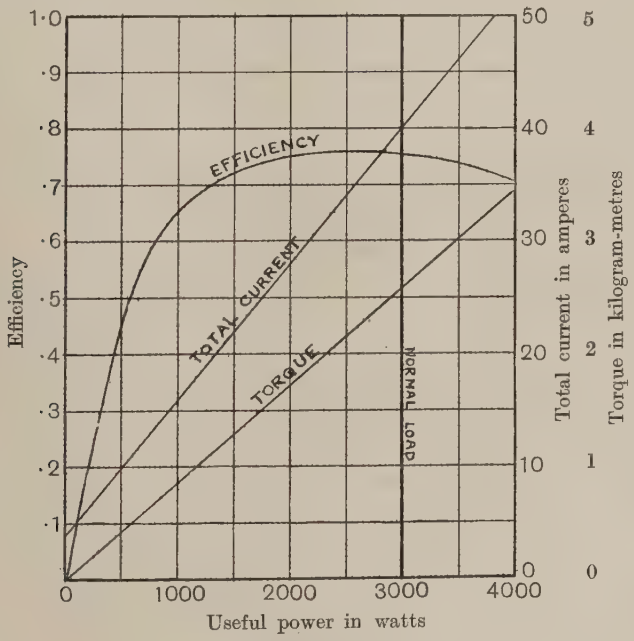


Fig. 210

The loss W_0 is found to rise, sometimes very considerably, as the load increases. This is largely due to the increased friction

of the bearings caused by the weight on the rope brake. Of recent years another brake method has been introduced for use with small motors, which does away with this difficulty. A copper disc is fixed to the axle of the motor, and lies partly between the poles of an electro-magnet, or better between the poles of two similar electro-magnets placed symmetrically on either side of the axle. As the motor rotates Foucault currents are developed in the disc, which react and produce a couple on the electro-magnets. This couple can be altered by adjusting the exciting current, and can be counterbalanced by a sliding weight on the frame holding the electro-magnets, which is made moveable about an axis. The retarding couple on the motor, which is equal and opposite to that on the electro-magnets, is therefore known.

The testing of series motors is similar to that of shunt motors, except that care has to be taken not to reduce the load too much, otherwise the speed of the motor may become excessive. Fig. 211

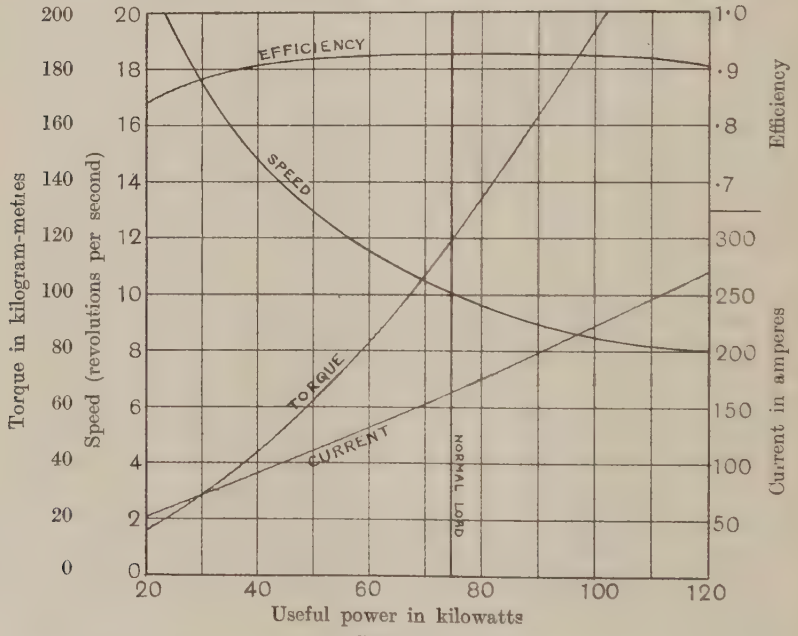


Fig. 211

shows the result of a test on a 100 horse-power series motor for use in electric traction. The applied potential is 500 volts.

The attainment of high efficiency in motors of course depends on careful construction and design. In this connexion it may be noted that the two-pole construction is unusual except for small motors, as it does not advantageously utilise the space surrounding the armature. Large motors are always built with four or more poles.

152. Testing and efficiency of generators. If it requires W watts to drive a generator which is only delivering W' watts in electrical power, the ratio $\eta = W'/W$ is called the *efficiency* of the generator. Theoretically, the best way to find the efficiency of a generator for various currents is to drive it with an auxiliary motor which has already had a brake test performed on it, so that the power W supplied to the generator is known in terms of the main motor current. In practice, however, there is always some uncertainty as to whether the friction of the bearings is the same in the two cases, and for laboratory purposes the method described below is found more convenient and of equal or greater accuracy.

Let R be the resistance of the armature and S that of the field-magnets of a shunt generator delivering a current i at voltage V. The useful electrical power of the generator is Vi watts; but it requires more power than this to drive the generator on account of the energy lost (1) in heating the armature coils, (2) in heating the field-magnet coils, (3) in hysteresis, Foucault currents and friction. The loss under (1) is $R(i+j)^2$ watts, where $j = V/S$ is the field-magnet current, and the loss under (2) is Sj^2 watts. As regards (3), we notice that the drag of the generator on the motor, with direct coupling, does not tend to increase the friction of the bearings, and therefore the loss under (3) will certainly not increase to the same extent as under a rope brake test. This loss will be provisionally assumed to have the constant value W_1. Then the efficiency of the generator is given by

$$\eta = \frac{Vi}{Vi + R(i+j)^2 + Sj^2 + W_1}.$$

It follows that η can be found for all currents if W_1 is known. To find it, first run the driving motor alone at the assigned speed. The electrical power consumed, less that lost in heating the armature and field-magnet coils, is the W_0 required to overcome

hysteresis, Foucault currents and friction in the motor. Next couple the generator to the motor and let it run with shunt winding on open circuit. Subtracting always the power lost in heating the various coils, we can find the power $W_0 + W_1$ absorbed in hysteresis, Foucault currents and friction in motor and generator together, and hence W_1 is known.

There is a simple means of checking the assumption that W_0 and W_1 are constant, because the total loss of energy at any stage, less that lost in heating the conductors, gives the loss of energy in hysteresis, Foucault currents and friction in both motor and generator. This will usually be found not to differ very much from $W_0 + W_1$; but if it increases we may divide the loss among the two machines in the ratio of W_0 to W_1. This is all the more reasonable in that we have no means of judging exactly where the frictional losses arise. Moreover, when the efficiency is high a slight error in estimating the losses is of no importance.

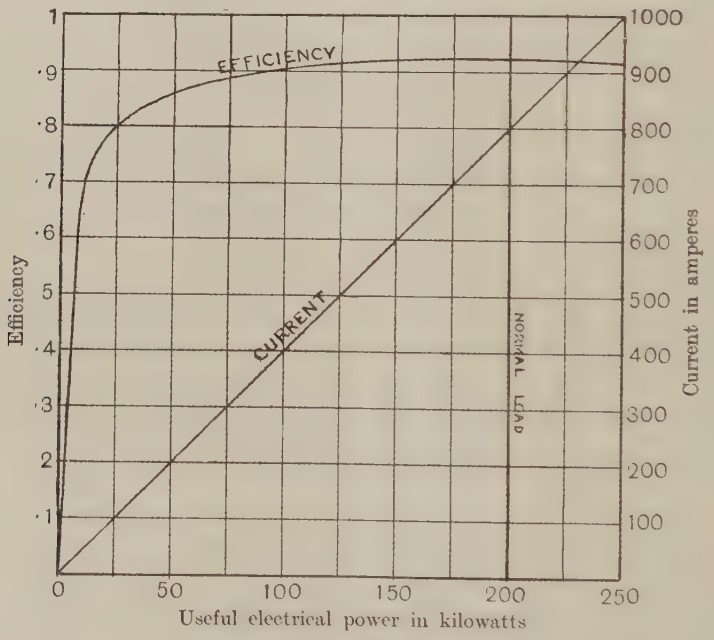


Fig. 212

Fig. 212, which refers to a 200 kilowatt compound generator

delivering current at 250 volts, will give the reader an idea of the efficiency attainable with large machines under favourable conditions.

153. Generation of alternating currents. The principle of the generation of alternating currents has already been given (Art. 129). There are, however, practical reasons which make it inadvisable to construct large generators in the simple way there described. It may be said that the chief advantage of alternating over direct current occurs when high voltages are used (cf. Art. 157). In this case it is desirable that the slip-rings should be abolished and the currents collected from stationary windings: the induced currents may be obtained by making the field-magnets move instead of the armature because the requisite magnetising current can always be obtained with quite low voltages. The principle on which modern alternators are constructed will be understood from Fig. 213. The field-

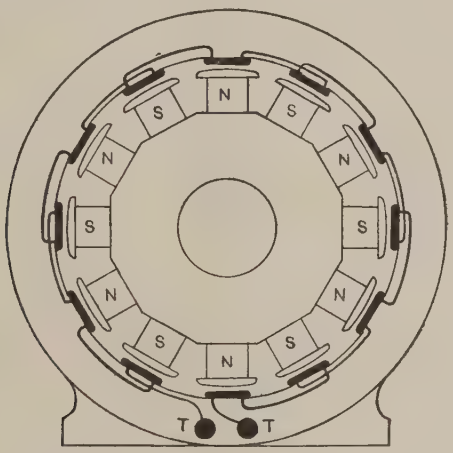


Fig. 213

magnet frame, which carries an even number of coils, forms the rotating part of the machine. Current is furnished to the coils, which are not shown in the figure, by slip-rings attached to the axle, and the windings are arranged so as to produce positive and negative magnetic poles alternately along the circumference. The armature coils consist of one or more turns of stout wire

or of copper strip, and are wound alternately in opposite directions, being finally connected to two fixed terminals TT on the frame of the machine. In the position shown each coil is pierced by the maximum magnetic flux: the sign of the flux is the same for all the coils since the direction of winding alternates to keep pace with the alternations of the field-poles. When the axle has moved round so that a north pole is in the place originally occupied by a south pole, the sign of all the fluxes has changed, and this corresponds to one-half the period of alternation of the current. It is thus easy to see that if the speed of revolution is n per second, and if there are m poles on the machine, the number of complete periods of the alternating current is $\frac{1}{2}mn$ per second.

A convenient source of alternating current for laboratory purposes is the *rotary converter* (Fig. 214). It consists essentially

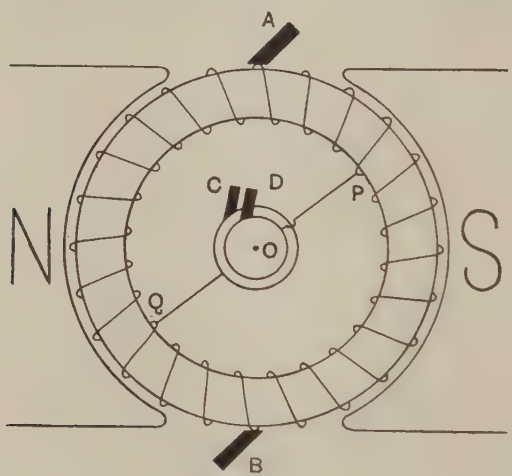


Fig. 214

of a direct-current shunt motor with certain additions made to the armature. In order to avoid confusing the figure the commutator has been suppressed and the brushes A, B allowed to slide directly over the armature windings. Connexions are taken from two opposite points P, Q on the armature to two slip-rings sliding under two brushes C, D. Let us first calculate the difference of potential between C and D as a function of the

time, supposing that no current is taken from CD and that the lines of magnetic induction run straight across from left to right. If OP makes an angle ωt with the initial line at time t, it follows from equation (4) of Art. 142 that the potential of P may be taken to be equal to $mn\ AB \sin \omega t$.

The potential of Q is $mn\ AB \sin(\omega t + \pi) = -mn\ AB \sin \omega t$, so that the difference of potential between the brushes C, D is given by $V = 2mn\ AB \sin \omega t$. Now since the rotary converter is running as a motor with A, B connected directly to the mains, this potential-difference must become equal to V_0, the E.M.F. of the mains, when ωt becomes equal to $\frac{1}{2}\pi$. Hence $2mn\ AB = V_0$, so that

$$V = V_0 \sin \omega t.$$

Thus alternating currents can be taken from the brushes C, D; but the amplitude of the E.M.F. developed is not adjustable at will, being equal to the E.M.F. of the mains for all speeds of the converter. However, smaller amplitudes can be obtained by connecting different points of the armature to slip-rings. Suppose for example that the angle POQ, instead of being two right angles, is equal to α. Then the potentials at P and Q are $\frac{1}{2}V_0 \sin \omega t$ and $\frac{1}{2}V_0 \sin(\omega t + \alpha)$ respectively, so that we should have

$$V = V_0 \sin \tfrac{1}{2}\alpha \cos(\omega t + \tfrac{1}{2}\alpha) \ldots\ldots\ldots\ldots (12).$$

This is the most general case. The amplitude is proportional to $\sin \frac{1}{2}\alpha$, the "effective value" of the voltage being $\frac{V_0}{\sqrt{2}} \sin \frac{1}{2}\alpha$. It is thus possible to ascertain the angle α in any given case by measurements with an alternate-current voltmeter.

The rotary converter, with drum armature, generally gives a good approximation to a sine-curve of E.M.F.

The frequency of alternating currents can be determined experimentally by an ingenious method depending on the principle of resonance. A number of steel springs are arranged side by side, and near each one is placed a small electro-magnet coil, all the coils being placed in parallel across the source of alternating potential. The alternating currents through the coils give rise to a forced vibration of the steel springs, which is most pronounced for that spring whose natural frequency of vibration most nearly

coincides with the frequency of the alternating supply. This is the principle of the *resonance frequency-meter* of Hartmann-Kempf.

High-frequency alternators (400 to 2000 complete periods per second) are now supplied for laboratory purposes. One pattern (Fig. 215) consists of a small motor carrying on the end of the axle a corrugated wheel made up of steel stampings. A small electro-magnet *AA* with wedge-shaped poles is placed near the wheel and arranged so that the poles are opposite projections on the corrugated wheel at the same time. When this occurs the lines of magnetic induction find an easy path through the wheel, and the flux of magnetic induction through the core has its greatest value. Half-way between the teeth the induction has its least value. The coils *AA* are excited with direct current and the change of induction in the coil *B* wound over the core gives rise to an alternating electromotive force. The frequency is mn, where m is the number of teeth on the wheel and n the number of revolutions per second.

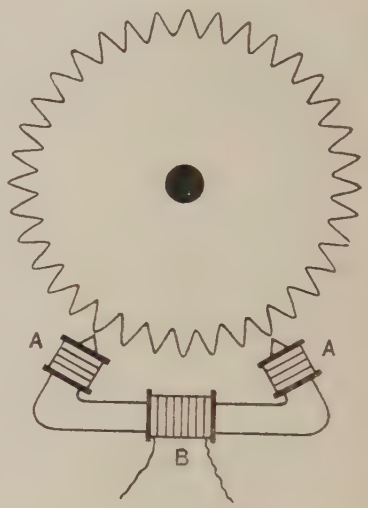


Fig. 215

154. Delineation of alternating current curves. The oscillograph. The form of the curve connecting the E.M.F. of an alternator with the time can be determined experimentally by a method due to Joubert. On the axle *C* of the alternator (Fig. 216) is fixed a thick disc of ebonite or insulating fibre, interrupted at one point of its circumference by a brass strip *EE* let into it. Two copper brushes *FF*, carrying screw terminals, press on the disc, and are mounted on a holder so that they can be turned round into any position by the handle *G*. The angular position of the brushes *FF* is indicated by means of a pointer

fixed opposite a scale on a plate K attached to the brush-holder. To determine the E.M.F. between the terminals A, B of the alternator a circuit is set up containing the terminals AB, the brushes FF, a suitable resistance and a galvanometer as shown. The galvanometer should give readings on both sides of the zero.

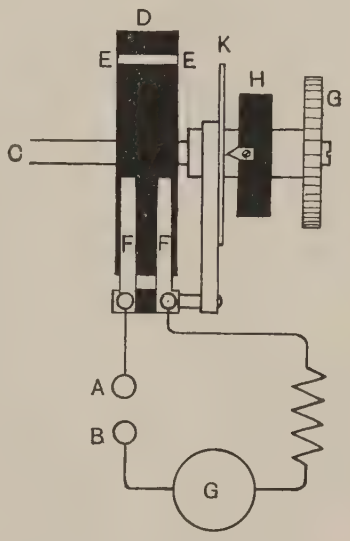


Fig. 216

It is obvious that a current can only flow in the galvanometer when the brass piece EE completes the circuit, i.e. when the armature occupies a definite position in space. In this position the difference of potential between A and B will have a certain value, and an instantaneous current will be sent round the circuit in a direction depending on whether the potential of A is higher than that of B or not. The result of these instantaneous currents occurring once in every revolution will be to give a steady deflexion of the galvanometer proportional to the difference of potential between A and B at the time being. By moving the brushes round the E.M.F. is found corresponding to all positions of the armature. Absolute values are obtained by replacing AB by an accumulator of known E.M.F. and observing the deflexion of the galvanometer.

This method is not restricted to measuring the E.M.F. at the terminals of an alternator, but can also be used to find the time-variation of any current or potential having the same frequency as the alternator. Thus AB may be replaced by the ends of a 1 ohm coil placed in any of the circuits used: then the method will measure the fall of potential down the coil, i.e. the current in the circuit. In this way we can put in evidence the difference of phase between the current and E.M.F. in a circuit containing self-inductance (Art. 130).

Alternate current and potential curves can be exhibited to the eye, and also photographed, by means of the *oscillograph*. The principle of the instrument is shown in Fig. 217. A piece of phosphor-bronze is bent over a small pulley P so as to form two thin parallel strips *ss* lying between the wedge-shaped poles of a powerful electro-magnet. The strips are kept tight by a spring attached to P, so that the restraining tension is high and the inertia low. Hence the natural period of vibration of the strips, even with the attached mirror M, is very small (of the order 1/10,000 second). When a current flows through the strip the two sides are urged in opposite directions like the fibre of a string galvanometer, and the small mirror M shows a deflexion. With an alternating current the mirror oscillates backwards and forwards, but this oscillation has nothing to do with the natural vibration of the strips and corresponds merely to a succession of equilibrium positions. The oil-bath KL, in fact, makes the movement of the strips so dead-beat that they respond completely even to rapidly changing currents.

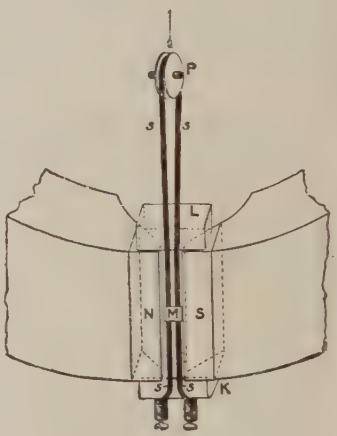


Fig. 217

In order to obtain a record of the motion of the mirror we may receive the spot of light on a photographic plate which is made to move at right angles to the direction of vibration. In this way the curve connecting the current with the time is obtained directly on the plate. Oscillographs are usually fitted with two separate strips for simultaneous observation of the current and E.M.F. in a circuit, the current strip being shunted and the voltage strip in series with a fairly high resistance.

The chief difficulty with oscillographs is to obtain sufficient sensitiveness together with a small natural period of vibration. Instruments on the principle of the string galvanometer, with perforated magnetic poles, have been devised and are found very satisfactory.

155. Harmonic analysis of curves. According to Fourier's theorem* an arbitrary function $f(x)$ can be expanded, in the range $0 \leqq x \leqq 2\pi$, in a series of the form

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \ldots + a_n \cos nx + b_n \sin nx + \ldots \quad \ldots\ldots(13),$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx$$

and the other coefficients are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$$

In the case with which we are concerned $f(x)$ is a periodic function of x, of period 2π, and the expansion (13) holds for all values of x.

The curve $y = f(x)$ is said to be harmonically analysed when the coefficients a and b are known. The constant term a_0 represents the mean value of $f(x)$ over a complete period, and is generally zero in electrical problems. The most important terms are usually the terms $a_1 \cos x + b_1 \sin x$ of the first order; and if the curve is a pure sine-curve these are the only terms occurring. The higher terms are called *harmonics* by analogy with the modes of vibration of a stretched string, and the magnitude of the harmonics shows the extent to which the curve deviates from the sine-form.

It is interesting to examine the effect of inductance on the form of the current curve when harmonics are present in the alternating E.M.F. applied to a circuit. Let $p/2\pi$ be the frequency of the applied E.M.F. V, and let V be expanded in the Fourier series

$$V = a_0 + a_1 \cos pt + b_1 \sin pt + \ldots + a_n \cos npt + b_n \sin npt + \ldots.$$

Suppose that $a_0 = 0$ and that the resistance in the circuit is negligible. Then the equation $L\, di/dt = V$ gives on integration

$$i = \frac{a_1 \sin pt - b_1 \cos pt}{Lp} + \ldots + \frac{a_n \sin npt - b_n \cos npt}{nLp} + \ldots.$$

* The conditions of validity are the reverse of stringent. See Carslaw, *Fourier Series and Integrals*, pp. 126–7

Since the denominators increase steadily with n, it is obvious that the effect of harmonics in the E.M.F. curve is diminished: hence the effect of self-inductance is to make the current curve approximate more nearly to a curve of sines. The effect of capacity in the circuit, on the other hand, is to exaggerate the harmonics and may cause a considerable deviation from the sine-form even when the E.M.F. curve does not deviate much from it. For in this case we have

$$i = C\frac{dV}{dt} = C\sum_{n=1}^{\infty} np\,(b_n \cos npt - a_n \sin npt),$$

and the higher harmonics are multiplied by larger and larger factors.

The curves considered in electrical problems nearly always satisfy the condition $f(x+\pi) = -f(x)$, i.e. they repeat themselves after a half-wave on the opposite side of the axis of x. In this case the only admissible terms are those which change sign when x is increased by π, so that only *odd* harmonics can occur. In practical harmonic analysis we may take as given $2n$ ordinates on a half-wave corresponding to equidistant values of x and determine the coefficients in the equation

$$y = a_1 \cos x + a_3 \cos 3x + \ldots + a_{2n-1} \cos (2n-1)\,x + b_1 \sin x + b_3 \sin 3x + \ldots + b_{2n-1} \sin (2n-1)\,x \quad \ldots(14),$$

so that y may have assigned values for the values

$$0, \quad \frac{\pi}{2n}, \quad \frac{2\pi}{2n}, \quad \ldots \quad \frac{(2n-1)\,\pi}{2n}$$

of x. If these values are $y_0, y_1, \ldots y_{2n-1}$, we have then $2n$ equations to determine the coefficients, namely those of the type

$$y_r = a_1 \cos\frac{r\pi}{2n} + a_3 \cos\frac{3r\pi}{2n} + \ldots + a_{2n-1} \cos\frac{(2n-1)\,r\pi}{2n} + b_1 \sin\frac{r\pi}{2n} + b_3 \sin\frac{3r\pi}{2n} + \ldots + b_{2n-1} \sin\frac{(2n-1)\,r\pi}{2n} \quad \ldots(15).$$

To solve the equations (15), multiply the rth equation by $\cos\dfrac{(2m-1)\,r\pi}{n}$ and sum from $r=0$ to $r=2n-1$. The left-

hand side becomes $\sum_{r=0}^{2n-1} y_r \cos \frac{(2m-1)\,r\pi}{n}$. On the right-hand side all the terms vanish except that containing a_{2m-1}. Hence

$$\sum_{r=0}^{2n-1} y_r \cos \frac{(2m-1)\,r\pi}{2n} = a_{2m-1} \sum_{r=0}^{2n-1} \cos^2 \frac{(2m-1)\,r\pi}{2n},$$

so that
$$na_{2m-1} = \sum_{r=0}^{2n-1} y_r \cos \frac{(2m-1)\,r\pi}{2n}.$$

Similarly
$$nb_{2m-1} = \sum_{r=0}^{2n-1} y_r \sin \frac{(2m-1)\,r\pi}{2n}. \qquad \ldots\ldots(16)$$

Thus the coefficients can be found. The following scheme, due essentially to Runge, gives a convenient method of carrying out the calculation for 12 equidistant ordinates $y_0, y_1, \ldots y_{11}$ on the half-wave, corresponding to the values 0°, 15°, ... 165° of x. It is understood that the ordinates corresponding to $x = 180°$, 195°, ... 345° are equal and opposite to the above. The higher harmonics are not calculated.

Scheme for twelve ordinates.

	y_0	y_1	y_2	y_3	y_4	y_5	y_6
		y_{11}	y_{10}	y_9	y_8	y_7	
Sums		s_1	s_2	s_3	s_4	s_5	s_6
Differences	d_0	d_1	d_2	d_3	d_4	d_5	

$$e_1 = d_1 - d_3 - d_5$$
$$e_2 = d_0 - d_4$$
$$e_3 = s_1 + s_3 - s_5$$
$$e_4 = s_2 - s_6$$

	$6a_1$	$6b_1$	$6a_3$	$6b_3$	$6a_5$	$6b_5$
sin 15° = ·259	d_5	s_1	—	—	d_1	s_5
sin 30° = ·500	d_4	s_2	—	—	d_4	s_2
sin 45° = ·707	d_3	s_3	e_1	e_3	$-d_3$	$-s_3$
sin 60° = ·866	d_2	s_4	—	—	$-d_2$	$-s_4$
sin 75° = ·966	d_1	s_5	—	—	d_5	s_1
sin 90° = 1	d_0	s_6	e_2	e_4	d_0	s_6

The meaning of the last table is that the rows are first to be multiplied by the sines on the left, and the columns are then to be added. Thus

$$6a_5 = d_1 \sin 15° + d_4 \sin 30° - d_3 \sin 45° - d_2 \sin 60° + d_5 \sin 75° + d_0 \sin 90° \quad \ldots(17).$$

The accuracy of this table may be verified from the formulae (16). Consider for example the last equation. Here $m = 3$ and $n = 6$, so that (16) gives

$$\begin{aligned} 6a_5 = {} & y_0 \cos 0° + y_1 \cos 75° + y_2 \cos 150° + y_3 \cos 225° \\ & + y_4 \cos 300° + y_5 \cos 375° + y_6 \cos 450° + y_7 \cos 525° \\ & + y_8 \cos 600° + y_9 \cos 675° + y_{10} \cos 750° + y_{11} \cos 825°. \end{aligned}$$

The trigonometric functions can clearly be expressed in terms of $\sin 15°$, $\sin 30°$, ... $\sin 90°$, and we find

$$\begin{aligned} 6a_5 = {} & (y_1 - y_{11}) \sin 15° + (y_4 - y_8) \sin 30° + (y_9 - y_3) \sin 45° \\ & + (y_{10} - y_2) \sin 60° + (y_5 - y_7) \sin 75° + y_0 \sin 90°, \end{aligned}$$

which is identical with (17).

156. Investigation of a coil with an iron core. When a simple harmonic E.M.F. is applied to the terminals of a coil without iron, the current curve is also simple harmonic, but

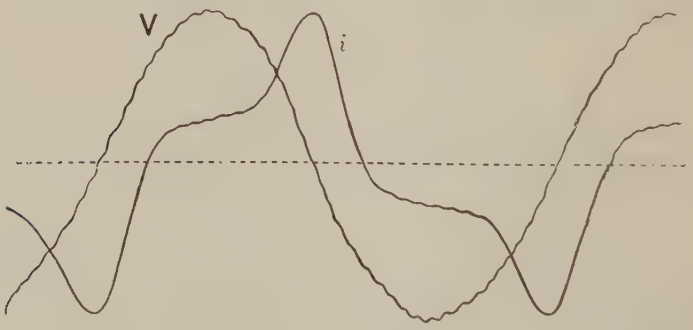


Fig. 218

displaced from the E.M.F. curve by the amount $\tan^{-1}(Lp/R)$, reckoning a whole wave as 2π. This ceases to be true when the

coil has an iron core, the current curve being in general distorted as well as displaced. Fig. 218 is a copy of an oscillograph curve of current and potential in this case.

The important things to find out about a given coil are the current I that it takes for an applied voltage V (effective values) and the power W absorbed by it. The connexions for this are

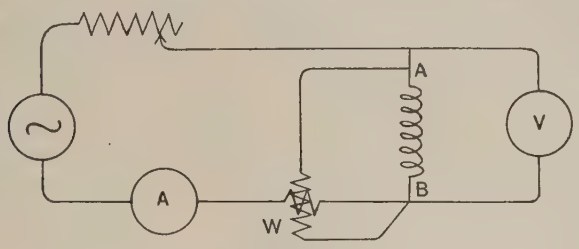


Fig. 219

shown in Fig. 219, where the crossed coils represent symbolically the current and voltage coils of the wattmeter W. The current coil is placed in the main circuit and the voltage coil applied to the coil AB as a shunt, so that the wattmeter measures the power W directly.

Strictly speaking, the coil cannot be said to have a self-inductance at all, since the flux through it is not proportional to the current. But there must be something which corresponds to inductance, and we should define it in such a way that the commonest electrical formulae still hold good.

The *apparent resistance* of the coil is defined so that the power W is equal to RI^2. Here R is no longer the true resistance of the coil, but considerably greater, since power is absorbed not only in heating the windings but also in hysteresis and Foucault currents in the core. We next define a *phase angle* α so that $W = VI \cos \alpha$ (cf. Art. 130); and finally the *self-inductance* L so that $\tan \alpha = Lp/R$, where $p/2\pi$ is the frequency of the E.M.F. applied.

Since the last three equations have been used as definitions, it follows that they always hold. The apparent resistance and the self-inductance, of course, are not constants, but vary with

the frequency and voltage. The extent to which these quantities may vary in practice will be seen from Fig. 220.

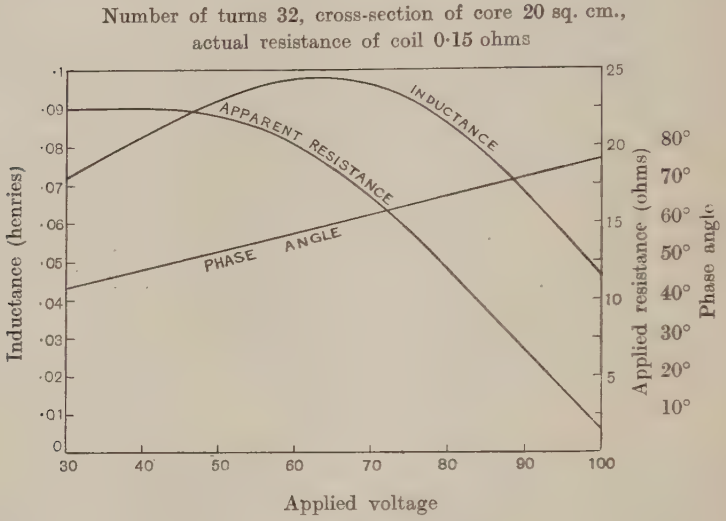


Fig. 220

157. The transformer and the transmission of power. The principle of the alternating current transformer has already been given (Art. 134). Its function is to change the E.M.F. of an alternating current circuit; and it is in this connexion that the special advantages of alternating currents begin to appear. In order to raise the E.M.F. of a direct-current circuit from 100 to 1000 volts it is necessary to employ a 100-volt motor to drive a 1000-volt generator; that is, to use rotating machinery. With alternating currents the transformation is performed by a stationary transformer which requires no attention whatever and in addition has a higher efficiency than any machine.

In transmitting power to long distances a change of voltage is sometimes necessary and generally advantageous. Suppose for example that 100 kilowatts have to be transmitted over a distance of 10 kilometres (6 miles). If this is done at 200 volts a current of 500 amperes would have to be sent, and the cost of the cable would be about £7000. If, however, the voltage was raised to 2000 the current would be only 50 amperes, and the

cost of cable £1000. It is evident, therefore, that there is a considerable saving in the cost of copper when high voltages are used, and 50,000 volts is not unusual for long-distance transmission. The high voltages used for transmission, however, would be dangerous for the ordinary user, so that it is necessary to instal a second transformer at the end of the line in order to reduce the voltage again to a convenient amount.

The arrangements for testing a transformer in the laboratory

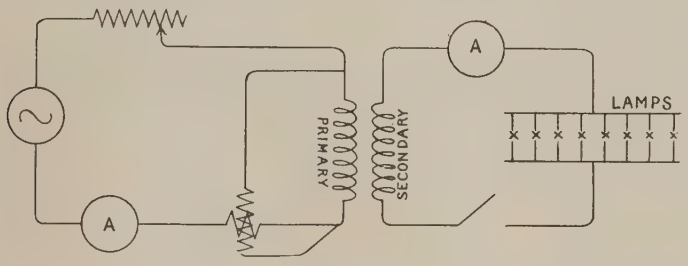


Fig. 221

are shown in Fig. 221. The primary circuit is set up with an ammeter and wattmeter as in Fig. 219, and various currents are taken off the secondary by means of a tray of lamps of suitable resistance (see Art. 63). In addition to the primary and secondary ammeters we require a special key for placing one and the same voltmeter across the primary and secondary terminals. This may be made easily by removing the cross-wires from an ordinary reversing-key.

Readings are taken of all the instruments when 1, 2, 3, ... lamps are placed in parallel in the secondary circuit. It is best to adjust the rheostat each time till the secondary voltage V_2 has an assigned value. The wattmeter gives the primary power W_1 directly. It is not necessary to have a wattmeter in the secondary, because the lamps are practically non-inductive* and the power is given by $W_2 = V_2 I_2$, where I_2 is the secondary current. The *efficiency* of the transformer is $\eta = W_2/W_1$.

* For a single lamp L is of probably the order 200 E.M.U. $= 2 \times 10^{-7}$ henry (cf. Art. 122), and R of the order 300 ohms. Hence when $p = 300$, Lp/R is of the order 2×10^{-7}. The self-inductance of a laboratory circuit of ordinary dimensions is of the order of 2000 E.M.U.

Fig. 222 shows the result of a test on a small laboratory transformer, in which the primary voltage and efficiency are plotted in terms of the secondary power W_2. It will be noticed that V_1 rises as more and more current is taken from the secondary.

Mean radius of ring 7·5 cm., cross-section of core 18·8 sq. cm. Primary 294 turns, resistance 1·76 ohms. Secondary 226 turns, resistance 1·61 ohms. Secondary voltage 53·5.

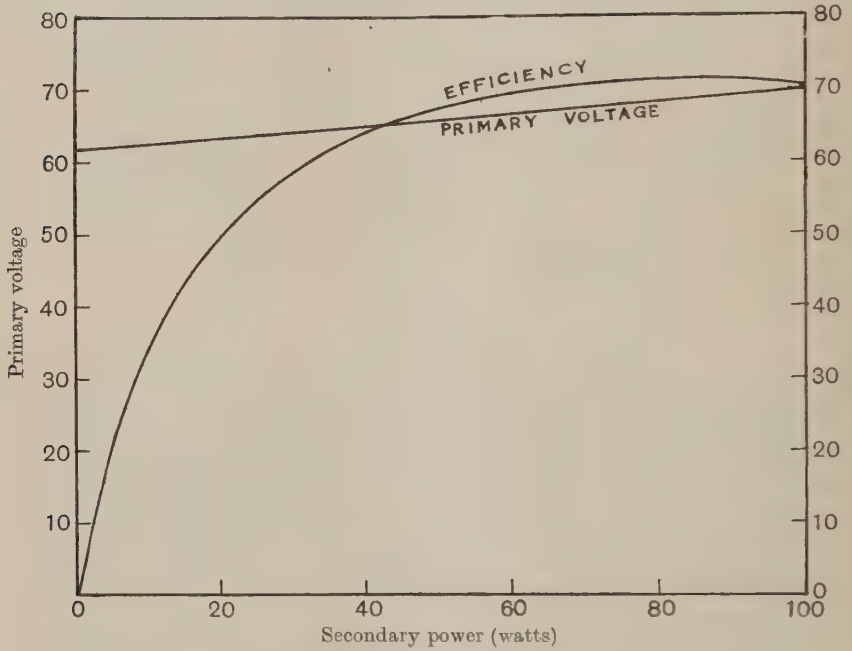


Fig. 222

Hence in order to maintain constant voltage at the consumer's terminals in a lighting system the primary voltage must be increased with increasing load.

The efficiency of commercial transformers is very high, often exceeding 95 per cent. The losses in transformers are much the same as those in other electrical machines, namely hysteresis, Foucault currents, and heating of the copper windings. The modern steel alloys with low hysteresis loss (Art. 108) are extensively used in the construction of transformers. It is also advantageous

to wind the secondary coils closely over or near to the primary coils, so that all the lines of force passing through the primary shall also pass through the secondary.

158. Alternating current motors. Until lately the use of alternating currents for power purposes has been hampered by the difficulty of obtaining a motor at all comparable to the direct-current motor in efficiency and reliability. As it is, there is no alternating current machine having the properties of the direct-current shunt motor, because self-inductance in armature and field-magnets entirely alters the ordinary relationship of shunt circuits. Of recent years, however, alternating current motors have been designed with the properties of the direct-current series motor, which are coming into use for electric railways and traction generally. These motors have a commutator like direct-current machines.

The *alternating current series motor* is similar in principle to the corresponding direct-current machine. At any instant the torque is proportional to the product of the armature current i and the flux N. These are however in phase with each other, so that if both of them are supposed to vary as $\cos pt$ the couple at any time is proportional to $\cos^2 pt$. There will thus be a definite average couple tending to set the armature in rotation. The modifications required for converting a direct into an alternating current series motor are twofold. In the first place, the field-magnet core must be laminated as well as the armature in order to avoid excessive development of Foucault currents. Secondly, steps must be taken to diminish what corresponds to the self-inductance of the whole circuit. In order to see how necessary this is, consider the analogous case of a simple coil without iron attached to a source of alternating E.M.F. Here the power consumed is $W = VI \cos \alpha$, where $\tan \alpha = Lp/R$. If L is excessive α becomes nearly equal to $\frac{1}{2}\pi$, and the power becomes small in comparison with the amplitudes of the voltage and current. In other words, a motor of any power would be required to carry unduly heavy currents and would therefore be very bulky and expensive.

The method of overcoming this defect in the series motor is

very interesting. We clearly cannot short-circuit either field-magnets or armature without preventing the motor from working: but the inductive effect of the armature can be neutralised by making it the primary of a transformer, the secondary winding being a fixed coil short-circuited on itself.

In order to see how this comes about, let N denote the flux of magnetic induction through the secondary, R the resistance and i the current in the secondary. Then $dN/dt + Ri = 0$. When R is small, Ri would remain finite if all the inductance of the primary was in the armature (Art. 134); but since this is not so Ri becomes small with R. Hence the flux N is small, which is the principle of compensation referred to. Since the magnetic field due to the armature currents is perpendicular to the main field, the additional windings on the field-magnet frame are placed at right angles to the ordinary coils.

The alternating current series motor was first used in England on a large scale on the London Bridge—Victoria section of the London, Brighton and South Coast Railway.

Another alternating current commutator motor is the *repulsion motor*, invented by Elihu Thomson. In this machine alternating currents are supplied to the field-magnets, and the armature, which is the same as that of a direct-current motor, is short-circuited across the brushes. The varying flux in the field-magnets first induces currents in the armature and then acts on them mechanically just as in Elihu Thomson's experiments (Art. 136), which were, in fact, devised to illustrate the principle of the repulsion motor. It has the advantage that the armature is entirely insulated from the field.

159. Polyphase currents and the rotating field. Let AB, CD (Fig. 223) represent a vertical view of two coils at right angles to one another, AB being parallel to the axis of x and CD to the axis of y.

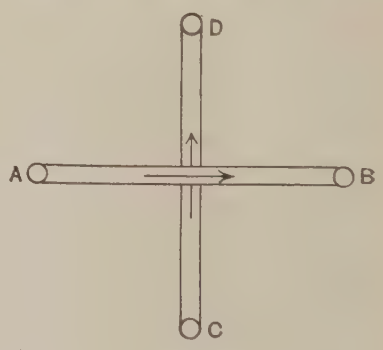


Fig. 223

The current in AB is reckoned as positive when it flows upwards at A and downwards at B, so that the current in the top half of the coil is in the direction of the arrow. Similarly the current in CD is reckoned as positive when it flows upwards at C and downwards at D. Now let alternating currents of equal amplitude be sent through the coils, the phases however differing by the angle $\frac{1}{2}\pi$, the current in AB being $\cos pt$ when that in CD is $\sin pt$. The magnetic force at the centre of the coils is then compounded of a force $A \cos pt$ parallel to the y-axis and $-A \sin pt$ parallel to the x-axis, where A is a constant. The resultant is of magnitude A and makes an angle $\frac{1}{2}\pi + pt$ with the axis of x. Hence the effect of the two coils is to produce at the centre a magnetic field of constant intensity, which rotates with the angular velocity p. If iron is present the same is true of the induced magnetisation, and thus we have the remarkable phenomenon of magnetic poles rotating in a solid mass of iron at rest, caused by two alternating currents in different phases.

The same effect can also be produced by three (or more) alternating currents. Let AB, CD, EF (Fig. 224) be traversed

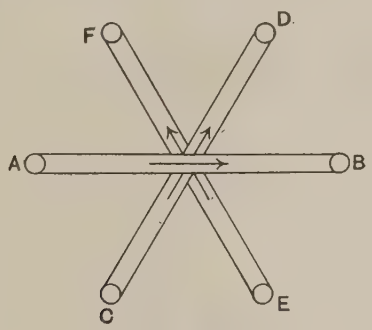


Fig. 224

by the currents $\cos pt$, $\cos\left(pt + \frac{\pi}{3}\right)$, $\cos\left(pt + \frac{2\pi}{3}\right)$ respectively. Then the components of magnetic force at the centre are

$$A\left\{-\frac{\sqrt{3}}{2}\cos\left(pt + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2}\cos\left(pt + \frac{2\pi}{3}\right)\right\}$$

and $$A\left\{\cos pt + \tfrac{1}{2}\cos\left(pt + \frac{\pi}{3}\right) - \tfrac{1}{2}\cos\left(pt + \frac{2\pi}{3}\right)\right\},$$

or $$\frac{3A}{2}\sin pt \quad \text{and} \quad \frac{3A}{2}\cos pt.$$

The resultant field $\frac{3A}{2}$ therefore rotates with angular velocity p as before.

The method of generating polyphase electromotive forces will be understood from Fig. 213. If an extra set of coils is inserted half-way between those shown, and brought out to two additional terminals, the phenomena of induction in them will clearly take place a quarter of a period after those in the first set of coils. A machine so constructed would be a *two-phase generator*. Similarly a three-phase generator can be constructed by arranging three sets of coils between every two consecutive field-magnet coils.

Two-phase currents may also be taken from a rotary converter by suitable arrangement. For this purpose four points on the armature, separated from each other by 90°, are connected to four slip-rings mounted on the axle. If the vectorial angles of OP and OQ in Fig. 214 are ωt and $\omega t + \pi$ at time t, the vectorial angles corresponding to the other phase (say OR, OS) are $\omega t - \frac{1}{2}\pi$ and $\omega t + \frac{3}{2}\pi$. It has been shown (Art. 153) that the difference of potential between P and Q at time t is $V_0 \sin \omega t$, and hence the difference of potential between R and S is

$$V_0 \sin(\omega t + \tfrac{1}{2}\pi) = V_0 \cos \omega t.$$

The two potentials have therefore the proper phase-relation to give two-phase currents. Similarly with six slip-rings a three-phase current can be taken off.

In practice it is possible, by using devices known as the star and mesh groupings, to obtain three-phase currents with three, or at most four, wires instead of six. It is not however proposed to describe these methods here.

160. The induction motor. One form of rotating field motor is that known as the *squirrel-cage* motor, the principle of which is shown in Fig. 225. The currents, in this case two-phase,

are led through two coils AB, CD in a fixed laminated iron frame called the *stator*. The moving part, or *rotor*, is made up of thin

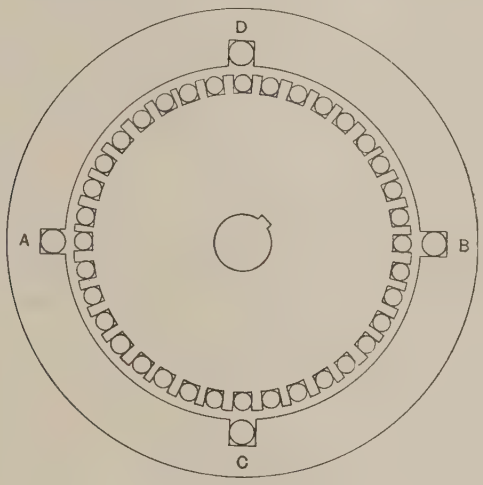


Fig. 225

discs threaded on a spindle like the drum armature of a direct-current machine. The slots in the discs hold a number of stout wires or strips of copper: these however are not wound according to any system, but simply connected together at both ends by a solid copper ring, so that they are all short-circuited.

In order to see how the squirrel-cage motor works, let us begin by calculating the currents in the bars under the assumption that the inductive effect of the bars on one another can be neglected. Consider two opposite bars P, Q (Fig. 226) situated in a uniform magnetic field H which rotates with angular velocity p, the angular velocity of the rotor being ω. Any current that may flow up the bar P will enter the binding ring at the nearer end of the rotor, and by symmetry will flow down the opposite bar Q. Hence we may imagine the bars P, Q joined at top and bottom so as to form an insulated circuit of resistance R, where $\frac{1}{2}R$ is the resistance of a single bar. Let A be the area of this imaginary circuit, and write $N_0 = HA$ for the greatest possible flux of force through a coil. The actual flux through the coil PQ is then $N = N_0 \sin \theta$. In calculating dN/dt we must remember that the

coil is rotating with angular velocity ω, so that $\frac{d}{dt}(\theta + pt) = \omega$. Hence $d\theta/dt = -(p - \omega)$ and the current i in the coil is given by

$$Ri = -\frac{dN}{dt} = -N_0 \cos\theta \cdot \frac{d\theta}{dt} = N_0 (p - \omega) \cos\theta.$$

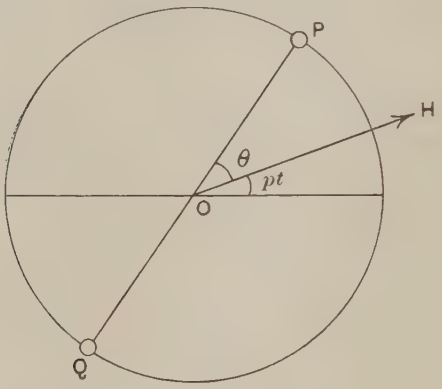


Fig. 226

The induced current vanishes when $p = \omega$, since the coil would then rotate at the same rate as the field and the flux of force would never change. In order that currents may be developed the speed of rotation of the motor must be less than that of the field. However, since R is very small a considerable current will be developed for quite small values of $p - \omega$, so that the difference between the two speeds in practice need not be very great. This difference is called the *slip*, and in large motors does not amount to more than 2 or 3 per cent. at full load. It will be noticed that the current in a bar depends on its position with respect to the field at the time being. In the present case the current is greatest when $\theta = 0$ and vanishes at right angles to the field. The couple on the coil PQ due to the field H is

$$iHA \cos\theta = N_0^2 (p - \omega) \cos^2\theta / R,$$

which is positive for all values of θ. It is obvious therefore that there is always a couple on the rotor when ω is less than p, and that this couple, *caeteris paribus*, is proportional to the slip.

In reality, the mutual inductive effects of the currents in the rotor are very great and have to be taken into account. Now

since the current in a bar determines its inductive effect and the current itself is determined by the total flux through it, it is obviously impossible to determine the currents by a direct method as hitherto.

In spite of this, however, it is possible to find the distribution of current, provided that two simplifying assumptions are made. In the first place the number of rotor bars will be taken to be very great, so that the number between angles α and $\alpha + d\alpha$ is equal to $2n\,d\alpha/\pi$, where $4n$ is the total number of bars on the rotor. Again, the calculation of the mutual inductances obviously gives rise to difficulty and must be replaced by some approximation. We notice that the flux of magnetic force through one coil due to unit current in another is greatest when the coils coincide and zero when their planes are at right angles. Hence we shall not be very far wrong in assuming that the flux is $L \cos \theta$, where L is a constant and θ is the angle between the two coils. We shall also suppose that no iron is present in the field.

We have now to calculate the flux of magnetic force through the coil PQ (Fig. 226) due to the currents in the remaining fictitious coils. If $f(\theta)$ is the current in the bar P, the flux through PQ due to the group of $2n\,d\alpha/\pi$ coils between $\theta = \alpha$ and $\theta = \alpha + d\alpha$ is

$$\frac{2n\,d\alpha}{\pi} f(\alpha)\, L \cos(\theta - \alpha).$$

Hence the required flux is

$$\frac{2nL}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\alpha) \cos(\theta - \alpha)\, d\alpha,$$

and therefore

$$N = N_0 \sin \theta + \frac{2nL}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\alpha) \cos(\theta - \alpha)\, d\alpha.$$

As before, in calculating dN/dt we must remember that

$$d\theta/dt = -(p - \omega),$$

while α is merely a variable of integration. Hence

$$\frac{dN}{dt} = -N_0 (p - \omega) \cos \theta + \frac{2nL\,(p - \omega)}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\alpha) \sin(\theta - \alpha)\, d\alpha,$$

and the law of electromagnetic induction gives

$$Rf(\theta) = N_0(p-\omega)\cos\theta - \frac{2nL(p-\omega)}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\alpha)\sin(\theta-\alpha)\,d\alpha \quad \ldots(18).$$

The equation (18) has to be used to determine the distribution of current along the circumference. Assume a solution

$$f(\theta) = X\cos\theta + Y\sin\theta.$$

Substituting in (18), we have to satisfy the equation

$$R(X\cos\theta + Y\sin\theta) = N_0(p-\omega)\cos\theta - nL(p-\omega)(X\sin\theta - Y\cos\theta)$$

for all values of θ. This can be done provided that

$$XR - YnL(p-\omega) = N_0(p-\omega)$$

and

$$XnL(p-\omega) + YR = 0.$$

Hence we find that

$$f(\theta) = N_0(p-\omega)\frac{R\cos\theta - nL(p-\omega)\sin\theta}{R^2 - n^2L^2(p-\omega)^2} \quad \ldots(19).$$

Thus the distribution of current changes according to a sine-law as we pass round the circumference, as before: but the maximum is displaced and the current is no longer zero at right angles to the field. On multiplying the expression (19) by $N_0\cos\theta \cdot \frac{2n\,d\theta}{\pi}$ and integrating we find that the torque exerted by the motor is given by

$$T = \frac{2n}{\pi}\frac{N_0^2(p-\omega)}{R^2 + n^2L^2(p-\omega)^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{R\cos^2\theta - nL(p-\omega)\cos\theta\sin\theta\}\,d\theta$$

or

$$T = \frac{nRN_0^2(p-\omega)}{R^2 + n^2L^2(p-\omega)^2} \quad \ldots\ldots\ldots\ldots(20).$$

If the motor is working on a constant voltage supply the flux N_0 will not be constant, because the rotor currents react on the stator and induce electromotive forces in it. Nevertheless, the performance of the motor is given in its main features by equation

(20), regarding N_0 as constant. Writing x for $nL(p-\omega)/R$, T is given as a function of the speed ω by the equation

$$T = A\frac{x}{1+x^2},$$

where A is a constant. This expression has a maximum value $\frac{1}{2}A$ when $x = 1$: it vanishes when $x = 0$ and is small when x is large. Hence we should expect the torque exerted by the motor to be greatest for some speed less than p, and to be comparatively small when the motor is starting ($\omega = 0$). This is what actually occurs, the relation between speed and torque being somewhat

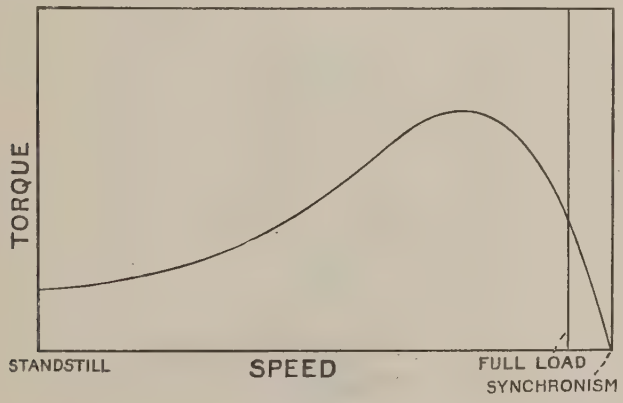


Fig. 227

as shown in Fig. 227. In practice, the starting torque is even smaller than we might expect, because the heavy currents induced in the rotor tend to weaken the field just as the short-circuited secondary of a transformer weakens the flux, so that N_0 is smaller at low than at high speeds.

The combination of low starting torque with high starting current is a serious drawback to the squirrel-cage motor. For this reason motors larger than about five horse-power require some starting device, especially if they are liable to be started against a load. The most satisfactory way is to give up the closed squirrel-cage construction and have regular windings on the rotor, which are brought out to slip-rings situated on the axle. In this way the resistance of the rotor can be made large

to begin with, and gradually cut out as the speed rises. Not only is the current cut down, but also the starting torque may be considerably increased by suitable choice of the starting resistance.

161. The electric arc. The electric arc was discovered in 1806 by Davy, who observed that if two carbon rods, attached to the terminals of a battery of sufficiently high potential, are brought together and then separated, a brilliant light is produced so long as the rods are not too far from one another. Fig. 228

Fig. 228

shows a carbon arc lamp suitable for lantern lighting, in which the screws are used for striking the arc and for adjusting the position of the carbons. The phenomena occurring on lighting are as follows. When the carbons are first brought into contact a fairly large current flows through the circuit. Since, however, the chief resistance in the circuit occurs where the carbons touch, there is a considerable development of heat there, which is sufficient to raise the electrodes to incandescence. On separating the electrodes the intervening space becomes a conductor, partly because air conducts at a sufficiently high temperature and partly on account of the constant emission of electrons from the negative carbon (cf. Art. 226), and the current continues to flow.

The physiological effect of the arc lamp makes it dangerous to fix one's eyes upon it for more than a fraction of a second: but the arc may be conveniently observed by focusing its image with a lens on to a sheet of white paper. The positive carbon, which becomes the hotter of the two, is rounded off and has a white-hot depression in the centre, from which most of the light of the arc comes. The negative carbon is seen to be covered with small bubbles which constantly rise and evaporate and are probably due to impurities contained in the carbon. The air between the carbons emits comparatively little light, consisting of a yellowish patch in the form of a bow touching the negative carbon and a violet portion opposite the hottest part of the positive carbon.

Other materials may be used in the arc as well as carbon. Thus the mineral magnetite may be used for the negative electrode, with a copper rod as the positive, which has the advantage of increasing the brightness of the arc itself. This is attained in another way in the *flame arc*, in which the positive carbon is impregnated with certain salts, usually calcium, which evaporate and impart a brilliant luminosity to the arc stream. The high temperature of the electrodes of the carbon arc (positive about 3700° C., negative about 3000° C.) causes them to be consumed in air at the rate of about 2 cm. per hour, and arc lamps are consequently fitted with automatic devices for keeping the carbons at a constant distance apart. The rate of consumption is reduced to about 1 mm. per hour in the *enclosed arcs*, which are enclosed in a globe which is air-tight, or nearly so. The relative advantages of the various kinds of arc depend very much on circumstances such as the cost of current and carbons, and so forth, and it is difficult to lay down general rules.

The characteristic of direct-current arcs is that, as the current rises, the difference of potential between the electrodes falls. Fig. 229 shows the potential-difference for magnetite arcs of various lengths and for currents up to 8 amperes. With the longer arcs more voltage is required for the same current, and Steinmetz has shown that the voltage can be represented over a large range by the formula

$$V = V_0 + \frac{a\,(l + b)}{\sqrt{i}},$$

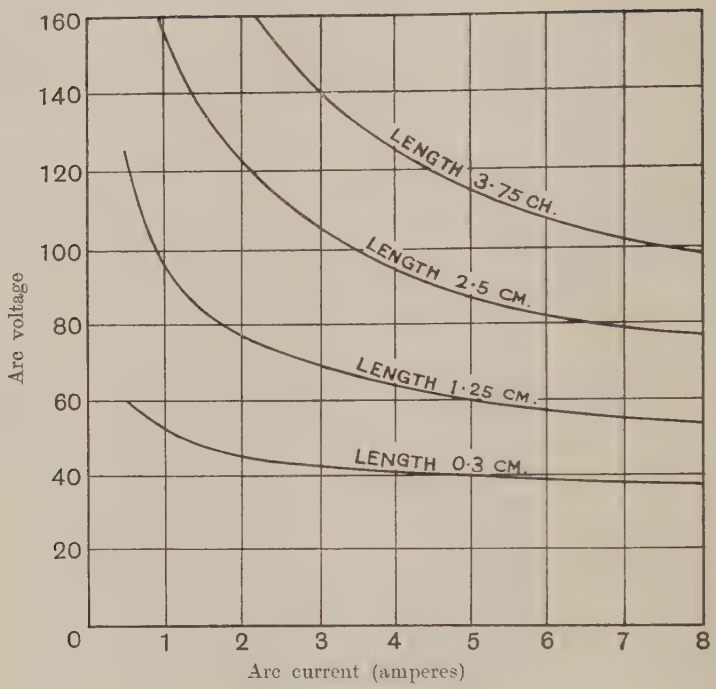


Fig. 229

where l is the length of the arc and V_0, a, b are constants. For the magnetite arc

$$V_0 = 30, \quad a = 48{\cdot}5, \quad b = 0{\cdot}125.$$

The carbon arc has the constants

$$V_0 = 36, \quad a = 51, \quad b = 0{\cdot}8,$$

but the agreement in this case is less close. The quantity V_0 clearly represents the least possible voltage that will burn an arc at all.

Arcs can be produced between metallic terminals, but it may be necessary to employ water cooling in order to prevent the terminals from becoming too hot. The *mercury arc lamp*, which is sometimes used, burns between mercury electrodes in a vacuum. The arc can be struck by tilting the tube until the two streams of mercury run together.

The light from the carbon arc, when examined with the spectroscope, is found to be mainly continuous, being crossed by lines only slightly brighter than the background. Metal arcs, on the other hand, yield brilliant line-spectra. Thus the mercury arc contains about seven bright lines, of which the three strongest are in the blue, green and yellow. The absence of strong lines in the red unfits the mercury arc for certain purposes, as red objects appear black by its light.

The asymmetry of the arc, that is the difference in the appearance at the positive and negative terminals, is an example of the contrast between positive and negative electricity already referred to (Art. 23). This asymmetry has been utilised in the *mercury rectifier*, by which alternating current can be converted into direct current without the use of rotary machinery.

162. Electric lighting. Meters. Electric lighting in rooms is carried out by means of the *electric glow-lamp*, which is familiar to every reader. The form generally used at present contains a filament of pure drawn tungsten wire stretched over a frame consisting of two sets of radial metal strips attached to a glass rod. The ends of the filament are attached to short strips of platinum or nickel-steel wire fused through the glass and making contact with the terminals of the lamp.

A description of the methods of determining the amount of light given out by a lamp, or its candle-power, is beyond the scope of this book: but a *relative* determination may be made in the laboratory to illustrate the conditions regulating the economic emission of light. The comparison is conveniently carried out by means of the paraffin-block photometer, which is the most accurate of the simple instruments. It consists of two similar blocks of paraffin-wax pressed together after being slightly warmed, with a sheet of tinfoil between. Paraffin is translucent, and when illuminated from one side one-half of the compound block appears bright and the other half dark. When illuminated on opposite sides by lamps of candle-power I, J at distances r, s respectively, the two halves will appear equally bright when $I/r^2 = J/s^2$, so that the ratio of I to J can be determined by experiment. For our present purpose it is sufficiently

accurate to take as a standard a new lamp by a good maker, run it off the marked voltage (say 100) and assume that its marked candle-power is correct. The lamp to be tested is also run off the same mains in series with an ammeter and a resistance box, so that various currents can be sent through it. More than the normal current can be used by placing one or more accumulators in series with the lamp.

Suppose that at any particular stage the applied voltage is V and the current i, and that the resistance in circuit, other than the lamp itself, is R. The difference of potential between the terminals of the lamp will be $V - Ri$, so that $Vi - Ri^2$ watts are consumed in the lamp. Dividing this by the candle-power of the lamp, as found with the photometer, we can find the watts required per candle-power for any current. The resistance of the filament will rise as its temperature increases, and is equal to $\frac{V - Ri}{i}$ for current i. Fig. 230 shows the result of such an experiment.

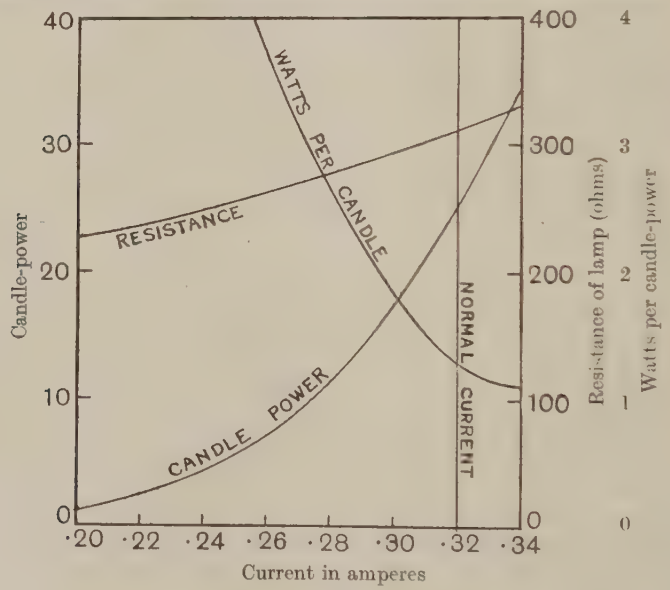


Fig. 230

The efficiency of a lamp is measured by the number of watts consumed per candle-power. This is about 1·25 for the tungsten

lamp. The efficiency falls very quickly when less than the normal current is sent through the lamp, because the filament emits less light in proportion to its total radiation at lower temperatures. Conversely, high efficiency results from the passage of a large current; but if this is carried too far the life of the filament, which is limited by the gradual evaporation of the tungsten, is unduly shortened. The average life under normal conditions is roughly 1500 hours of actual running.

It is evident from what has been said that an efficient glow-lamp must contain a filament of high melting-point. The melting-point of tungsten, which is about 3000° C., is the highest among the metals. Carbon vaporises at about 3700° C., but the evaporation becomes appreciable very much below this point, and lamps with carbon filaments cannot be advantageously used above 1900° C. A recent advance in the manufacture of tungsten lamps has been to place the filament in an atmosphere of nitrogen, which permits the use of even higher temperatures than in a vacuum.

The *Nernst lamp* has now gone out of general use, but is employed in laboratories for lamp and scale work, since practically all the light comes from a single point. It consists essentially of a short stout filament composed of the oxides of thorium and cerium, which becomes incandescent when heated by an electric current and gives off a brilliant white light. As, however, the

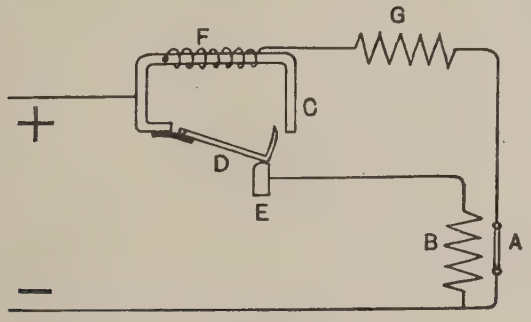


Fig. 231

filament will not conduct at ordinary temperatures special means have to be adopted for heating it, as indicated in Fig. 231, which

shows the arrangement of the complete lamp. A represents the filament and B the heating coil, composed of a fine platinum wire contained in a porcelain tube. C is a small piece of iron shaped somewhat like a horse-shoe, over which is wound a coil of wire F attached to C at one end, but otherwise insulated from it. The iron piece D is attached to C by a spring, and rests normally against another metallic piece E, thus completing the circuit through B. As soon as the filament A becomes hot enough to conduct, a current passes through F which magnetises the iron core and attracts the moveable piece D. The heating circuit is thus automatically broken.

The following table gives the approximate efficiency of various lamps:

Lamp	*Watts per candle-power*
Mercury arc	0·25
Flame arc	0·25
Carbon arc	0·5
Tungsten filament lamp in nitrogen	0·5
Tungsten filament lamp *in vacuo*	1·25
Nernst lamp	1·6
Carbon filament lamp	3·5

Direct-current supply in towns is at a constant voltage, usually about 220, but sometimes 100. Hence the electrical energy used by a consumer can be ascertained by measuring the total quantity of electricity passing through the circuit. Ampere-hour meters on the electrolytic principle are extensively used.

An interesting type of meter is the Kelvin *watt-hour meter*, which registers the power consumed on direct or alternating current circuits in any interval of time. It consists of a small commutator motor without iron, of which the field is produced by the main current i flowing in a coil. The voltage terminals are connected through the armature and also through a compensating coil assisting the field coil. The armature axle carries an aluminium disc which lies between the poles of a permanent magnet. When the impressed voltage is V, the driving couple on the motor is $(\lambda i + \mu V)\, V$, where λ and μ are constants, and

the retarding couple due to Foucault currents is $\nu d\theta/dt$. Hence the equation of motion of the armature is

$$I\frac{d^2\theta}{dt^2} = \lambda iV + \mu V^2 - \nu\frac{d\theta}{dt} - R,$$

where R is the retarding couple due to friction. On integrating over a long time T the left-hand side becomes negligible, and we have

$$\lambda W + \mu\overline{V^2}T - \nu\alpha - RT = 0,$$

where $W = \int iV dt$ is the power consumed and $\alpha = \int d\theta$ the angle turned through by the armature. The compensating coil is adjusted as nearly as possible so that $R = \mu\overline{V^2}$. Hence $\lambda W = \nu\alpha$. Thus W can be measured by means of a clockwork attachment which registers α.

CHAPTER X

ELECTROLYSIS

163. Experimental evidence for Faraday's laws. We have now to consider the laws of electrolytic decomposition, which were dismissed somewhat briefly in Art. 57. In Faraday's original experiments the first step was to investigate carefully the electrolysis of diluted sulphuric acid in order to see whether consistent results could be obtained irrespective of the size of the electrodes, the absolute magnitude of the current, and the strength of the acid. The method was to place several voltameters in series, so that the same amount of electricity must pass through all. Thus in one case three voltameters were placed in series, the first having small electrodes of platinum wire and the second having plates of the metal as electrodes, while in the third the products of electrolysis were collected separately, instead of together as in the first two. The sum of the volumes of hydrogen and oxygen in the three cases were respectively 74·30, 73·25 and 73·65 in arbitrary measure. The small differences that occurred were such as might be explained by the solubility of the gases in the liquid, and Faraday concluded that the greatest accuracy was obtained with sulphuric acid of density about 1·25, collecting the hydrogen only.

As an example of Faraday's second law we may take the electrolysis of lead chloride ($PbCl_2$). Faraday found that the mass of lead deposited was to the mass of hydrogen liberated from a water voltameter in the same circuit in the ratio 100·85 to 1. The ratio of the atomic weights is 207, and the valency of lead is here 2, so that the agreement is fairly satisfactory. Similarly the amount of tin deposited from stannous chloride

($SnCl_2$) was about 58 times the amount of hydrogen liberated, the ratio of the atomic weights being 118.

It is not by any means every substance that lends itself to direct investigation, because in many cases there are disturbing reactions which mask the effect sought for. Thus Faraday failed to obtain the expected amount of lead by electrolysis of fused lead iodide, and attributed the discrepancy to the formation of a periodide.

Lüpke recommends the following solutions for verifying Faraday's second law by a simple laboratory experiment:

(1) Solution of 3 grams of silver nitrate and 5 grams of potassium cyanide in 200 c.c. of water;

(2) Solution of 3 grams of cuprous chloride in hydrochloric acid, diluted with 200 c.c. of water;

(3) 100 grams of a saturated solution of copper sulphate added to 15 c.c. of nitric acid and diluted with 100 c.c. of water;

(4) One gram of stannic chloride dissolved in 100 c.c. of water and added to 100 c.c. of a saturated solution of ammonium oxalate.

These are placed in voltameters with clean platinum cathodes, in series with a water voltameter. The following is the result of an experiment described by Lüpke:

Cation	Valency	$\frac{\text{Atomic weight}}{\text{Valency}}$	Electrolytic deposition in arbitrary units
Hydrogen	1	1·008	1·008
(1) Silver	1	107·88	109·0
(2) Copper	1	63·57	64·1
(3) Copper	2	31·78	32·3
(4) Tin	4	29·75	28·6

The more accurate verifications have referred mainly to the ratio of the depositions of silver and copper. Shaw, working with solutions of silver nitrate and copper sulphate, tried to ascertain whether there was any error due to the solubility of the copper in the sulphate solution, by arranging his results in the order of increasing current-density. The results did not entirely agree

in showing that this was the only cause, but when corrected in the light of this hypothesis they became more consistent. Finally only the results with the higher current-densities were taken, and no correction was applied. The ratio of the depositions of silver and copper was found to be 3·3998, while the value to be expected on the basis of the atomic weights is 3·3946.

It has recently been shown that accurate experiments can be made with a cadmium sulphate solution. Taking silver as a standard, the value 112·31 is obtained for the atomic weight of cadmium, the accepted value from chemical data being 112·40.

164. The theory of electrolytic dissociation. The view of electrolysis now generally adopted rests on a theory of the nature of solutions, which we proceed to expound. It is desirable in the first place to get clear ideas, from the physical point of view, of what is meant by *atom* and *molecule*. According to the ordinary definition, an atom is the smallest amount of an element that can enter into chemical combination. The physical molecule is the smallest amount of a substance capable of permanent independent existence, as in the kinetic theory of gases. The question of molecular weight is partly a physical one, as it is based in simple cases on *Avogadro's principle*, namely that the number of molecules per cubic centimetre of a gas at an assigned temperature and pressure is a definite constant, independent of the nature of the gas. This principle, which cannot be regarded as self-evident, is shown in the kinetic theory of gases to depend on a statistical theorem, that of the *equipartition of energy*, namely that two sets of molecules in presence of one another (as in a mixture of gases) ultimately reach a state in which the average kinetic energy of a molecule of either kind is the same. In the liquid and solid states of aggregation the molecule is defined by chemical properties which cannot be discussed here*.

If p is the pressure and ρ the density of a substance in gaseous form at absolute temperature θ, then the combined laws of Boyle and Gay-Lussac are expressed by

$$p = R\rho\theta/M \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

* See Nernst, *Theoretical Chemistry*, pp. 165–7.

where M is the molecular weight and R the *universal gas-constant* $= 8{\cdot}31 \times 10^7$.

Somewhat similar considerations apply to osmotic pressures, though the subject is by no means free from difficulty. There seems, however, no reason to doubt the conclusion that the osmotic pressure of a substance in dilute solution is the same as would be produced by the same number of independent particles existing in the gaseous form. The term *independent particles* is used intentionally instead of molecules, because we have so far no means of knowing what happens to the molecules of a substance when it is dissolved. If the particles were really molecules of the substance the osmotic pressure would be given by an equation analogous to (1), namely

$$P = Rc\theta/M \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where c is the concentration of the solution in grams per c.c.

If this equation is found to hold good, therefore, we may conclude that the particles in solution are molecules.

For example, Pfeffer found that a 1 per cent. solution of cane sugar in water had the osmotic pressure $6{\cdot}73 \times 10^5$, while the above formula ($c = 1/100$, $M = 342$) gives $6{\cdot}80 \times 10^5$.

As a matter of fact, most of the experimental work has been done on the depression of the freezing point of dilute solutions, which is easier to determine than the osmotic pressure. This depression is connected with the osmotic pressure by the equation $\delta\theta = P\theta/J\lambda\rho$, where J is the mechanical equivalent of heat, θ the absolute temperature of freezing of the solvent, λ its latent heat of fusion and ρ its density*. Using equation (2) we find for water solutions the equation

$$\delta\theta = 1850\,\frac{c}{M}.$$

This equation has been verified in the case of cane sugar by Griffiths, who found that at great dilution $\delta\theta = 1858c/M$.

If each molecule of the dissolved substance gave rise on the average to k independent particles from the point of view of osmotic pressure, instead of one, the last equation would become $\delta\theta = 1850kc/M$. The factor k can therefore be determined by

* See Whetham, *Theory of Solution*, pp. 122–6.

experiment. It is found that, while the equation $\delta\theta = 1850c/M$ holds for non-electrolytes ($k = 1$), *it does not hold for electrolytic solutions*, for which k exceeds unity. Thus with potassium chloride at extreme dilution Griffiths found that $\delta\theta = 3720c/M$, indicating that $k = 2$ very nearly. The following table gives some other results, derived chiefly from experiments by Loomis, Jahn and Bedford:

Substance	$10^5\,c/M$	Factor k	Substance	$10^5\,c/M$	Factor k
H_3PO_4	1·0	1·52	K_2SO_4	1·0	2·62
			$BaCl_2$	1·0	2·70
KOH	1·0	1·86	,,	0·02	3·02
NaOH	2·0	1·87	H_2SO_4	0·1	2·70
KNO_3	1·0	1·87	,,	0·03	2·85
NH_4Cl	1·0	1·93	K_2CO_3	1·0	2·74
NaCl	2·5	1·93	Na_2CO_3	1·0	2·75
KBr	2·5	1·93	$MgCl_2$	1·0	2·78
NH_4NO_3	1·0	1·94			
HCl	1·0	1·95	$K_2Cr_2O_7$	very dilute	>3·02
HNO_3	0·1	1·97	Na_2SiO_3	1·0	3·50
$KMnO_4$	0·06	1·98	$K_3Fe(CN)_6$	0·02	3·98
$MgSO_4$	0·02	2·00	$SnCl_4$	1·0	6·82

With certain exceptions the values of k have a remarkable tendency towards being whole numbers, which is the more pronounced the more dilute the solution. For example, in dilute hydrochloric acid ($k = 1{\cdot}95$) we conclude that practically all the molecules are split up into their constituent atoms. The weaker the solution the less chance there is of a hydrogen atom meeting a chlorine atom and becoming temporarily united with it, so that we should expect k to be larger in this case. The case in which k approaches the value 3 is attributed to the presence of three dissociation products, it being necessary to suppose that complete dissociation is not so easy to attain: and similarly the other figures receive a more or less satisfactory explanation. The chief difficulty occurs with stannic chloride, which has the factor $k = 6{\cdot}82$ while yielding at the most five atoms. Loomis attributes this to certain disturbing effects.

This theory of *electrolytic dissociation*, namely that the molecules of electrolytic solutions are split up to a greater or less degree

even in the absence of electric force, was first clearly enunciated by Arrhenius in 1887. Among known solvents, water is the one which most readily produces dissociation. This has been attributed by Nernst to its high dielectric constant, which would tend to weaken the attraction between the products of dissociation, assuming these to be charged.

165. Ionic theory of electrolysis. A neutral molecule, as has already been said, is regarded as containing a certain normal number of electrons. It is to be supposed that one or more of these electrons are effective in holding the molecule together: if an element, in holding the component atoms together, and if a compound, in linking the constituents. We have seen in the last article that in many cases the molecules of a body in solution are wholly or partly dissociated, and we shall modify the phraseology of Faraday by giving to these parts the name of *ions*. Since all such solutions conduct electricity we conclude that the separate ions are charged; that is, that one of the dissociation products contains one or more electrons in excess while others are in defect. The number of extra electrons in either case will be provisionally called the *valency* of the ion.

On this view the primary function of the electrodes in electrolysis is to maintain a field of force in the liquid. A positively charged ion (or *cation*) is dragged along by the field and ultimately appears at the cathode, provided that no secondary reaction takes place. The negatively charged ions (or *anions*) move backwards along the lines of force and may appear at the anode.

We have now to show that the ionic theory accounts for Faraday's laws of electrolysis. Let e be the charge on the electron, M the atomic or molecular weight of an ion and p its valency. Then we should expect the passage of a charge pe to correspond to the transfer of an amount λM of the ion, where λ is some constant. Hence the mass of the ion liberated by unit charge is $\mu M/p$, where $\mu = \lambda/e$ is also a constant. This, however, is precisely what would be given by Faraday's laws, provided that the number p is identified with the chemical valency. Reviewing the theory as a whole, therefore, we find that it involves the following hypotheses:

(1) A chemically univalent ion carries a certain definite elementary charge e independent of its nature;

(2) A p-valent ion carries a charge pe.

The first hypothesis, as was pointed out by Helmholtz in 1881, is sufficient of itself to lead to the theory of the atomic constitution of electricity. From our present point of view, Faraday's laws of electrolysis form an experimental confirmation of electron theory.

This conclusion is not invalidated when secondary actions occur at the electrodes, for we may suppose that the ion, instead of being liberated, enters into combination and releases a chemically equivalent amount of some other substance. The amount of matter actually liberated has therefore the same relation to the charge as if the liberation had been caused by direct transfer of ions.

As regards the constitution of the ions in liquids, we notice that in the electrolysis of hydrochloric acid hydrogen appears at the cathode and therefore carries a positive charge, the chlorine ion carrying an equal and opposite charge. The separation of the molecules of the solute into cations consisting of hydrogen or metals, and anions consisting of halogens or basic radicles, is typical of those electrolytes for which k approaches 2. It is natural on other grounds to suppose that the line of cleavage is drawn in this way, since the basic radicles (NO_3, SO_4, etc.) show great chemical stability in that they will bodily replace the halogens in compounds. Thus the simplest type of dissociation is exhibited by the equations

$$HCl = \overset{+}{H} + \overset{-}{Cl},$$

$$KNO_3 = \overset{+}{K} + \overset{-}{NO_3},$$

and so on.

In magnesium sulphate, however, the SO_4 radicle is divalent and therefore carries a double (negative) charge. Hence we can also have the type

$$MgSO_4 = \overset{+}{\underset{+}{Mg}} + \overset{-}{\underset{-}{SO_4}}.$$

Three ions must arise from the molecule when k approaches 3. The type is that of sulphuric acid

$$H_2SO_4 = \overset{+}{H} + \overset{+}{H} + \overset{-}{\underset{-}{SO_4}},$$

agreeing with the charges assigned to the H and SO_4 ions in other cases. There is also the complementary type represented by

$$BaCl_2 = \overset{++}{Ba} + \bar{Cl} + \bar{Cl}.$$

The evidence from osmotic pressure is less definite for the higher values of k.

It must not be concluded that the charge on an ion is the same under all conditions. Thus the dissociation of ferrous chloride may be expected to be represented by

$$FeCl_2 = \overset{++}{Fe} + \bar{Cl} + \bar{Cl},$$

while that of ferric chloride is similarly

$$FeCl_3 = \overset{+++}{Fe} + \bar{Cl} + \bar{Cl} + \bar{Cl}.$$

The electrochemical equivalents of the more common ions are given in the following table, expressed in grams per coulomb. The values are not directly observed, but calculated in terms of silver as the standard.

Cations (+charge)				Anions (−charge)			
	Combining weight	Valency	Electrochemical equivalent $10^{-5}\times$		Combining weight	Valency	Electrochemical equivalent $10^{-5}\times$
H	1·008	1	1·045	O ...	16	2	8·291
NH_4...	18·04	1	18·7	OH ...	17·01	1	17·63
Na	23	1	23·83	F ...	19	1	19·69
K	39·1	1	40·52	CN ...	26·01	1	26·96
Fe (ferrous salts) ...	55·85	2	28·94	S ...	32·07	2	16·62
(ferric salts) ...		3	19·29	Cl ...	35·46	1	36·75
Cu (cuprous salts)...	63·57	1	65·88	$C_2H_3O_2$	59·02	1	61·16
(cupric salts) ...		2	32·94	CO_3 ...	60	2	31·09
Zn	65·37	2	33·87	NO_3 ...	62·01	1	64·26
Ag	107·88	1	111·8	SiO_3 ...	76·3	2	39·54
Cd	112·4	2	58·2	Br ...	79·92	1	82·82
Sn (stannous salts)	119	2	61·66	ClO_3 ...	83·46	1	86·5
(stannic salts) ...		4	30·83	SO_4 ...	96·07	2	49·78
Ba	137·37	2	71·2	CrO_4 ...	116	2	60·11
Hg (mercurous salts)	200	1	207	I ...	126·92	1	131·5
(mercuric salts)		2	103·5	BrO_3 ...	127·92	1	132·6
Pb	207·1	2	107·3				

166. Physical constants connected with electrolysis. There is an important relation connecting the elementary charge e with the number of molecules N of a gas at temperature 15° C. and at a pressure of 760 mm. of mercury. Under these conditions it is known that 1 coulomb liberates 0·123 c.c. of hydrogen. Since each molecule corresponds to the passage of a charge $2e$, we have $0{\cdot}246\,Ne = 1$ coulomb $= 3 \times 10^9$ electrostatic units, so that

$$Ne = 1{\cdot}22 \times 10^{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where e is expressed in electrostatic units.

Further, let m_H be the mass of a hydrogen atom. Since each molecule contains two atoms, and the density of hydrogen at 15° C. and at a pressure of 760 mm. of mercury is $8{\cdot}517 \times 10^{-5}$, we have $2Nm_H = 8{\cdot}517 \times 10^{-5}$, or

$$Nm_H = 4{\cdot}26 \times 10^{-5}.$$

Combining this with equation (3), we have

$$\frac{e}{m_H} = 2{\cdot}87 \times 10^{14} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

The value of N can be found from a study of the Brownian movement in liquids, as was shown by Perrin*. Taking his latest value $N = 2{\cdot}90 \times 10^{19}$, we find that $e = 4{\cdot}2 \times 10^{-10}$, but this number differs slightly from that obtained in other ways†. In this book we shall adopt the value

$$e = 4{\cdot}75 \times 10^{-10} \text{ E.S.U.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Combined with equation (4), this gives

$$m_H = 1{\cdot}65 \times 10^{-24} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and hence we can find the mass of any molecule of known molecular weight.

If m is the mass of an ion or molecule of atomic or molecular weight M, it follows from (4) that

$$\frac{e}{m} = \frac{2{\cdot}87 \times 10^{14}}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

This equation holds whatever value of e is adopted.

* See J. Perrin, *Les Atomes*. Paris, F. Alcan.

† Cf. Art. 212.

167. Polarisation and secondary actions. The E.M.F. of polarisation for various applied potentials may be measured by the method shown in Fig. 232. For this purpose we require a source of electromotive force which can be varied by small

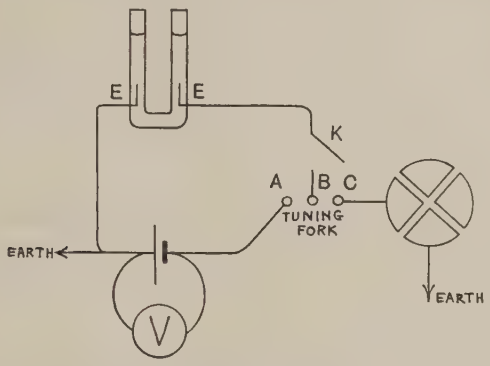


Fig. 232

stages from zero up to about 5 volts. A low-voltage direct-current generator, which can be excited separately by various currents, is very convenient. The electrolyte is contained in a voltameter with electrodes E, E, one of which is earthed. The other is joined through a key K to the insulated prong B of a tuning-fork making contact alternately with the points A, C connected to the generator and an electrometer respectively. When the key is up the electrometer quickly takes up the potential V of the generator, the absolute magnitude of which is given by a voltmeter attached to its terminals. When the key is down the potential V is applied to the electrolytic cell as long as B is joined to A, and when B is joined to C it communicates to the electrometer a potential equal to the E.M.F. of polarisation in the cell. This method is applicable in the case of small electrolytic cells, whose polarisation E.M.F. would disappear rapidly if they were joined directly to a voltmeter.

Fig. 233 shows the relation of polarisation to applied E.M.F for dilute sulphuric acid between electrodes of platinum foil. When V is small the polarisation E.M.F. is nearly equal to V, and there is no appreciable electrolysis. But as V increases the

polarisation E.M.F. tends to a maximum value which in this case is about 2 volts, but depends very much on the state of the electrodes.

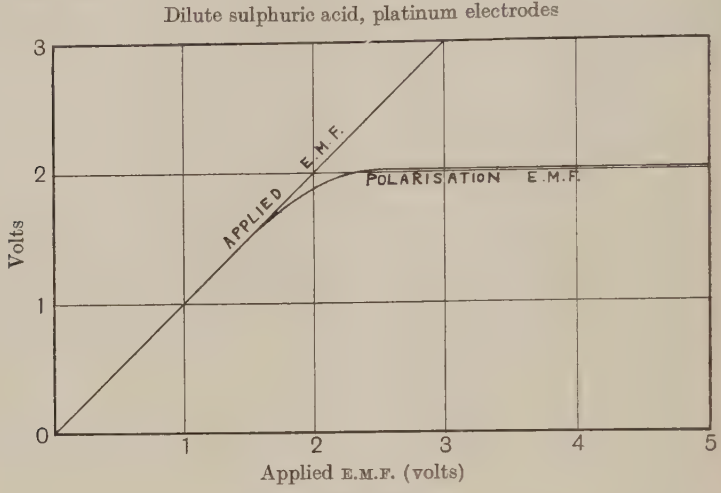


Fig. 233

Except from the statement that polarisation is due to the formation of an electrical " double layer " on or near the electrodes, i.e. a layer of positively charged particles in face of a layer of negatively charged ones, no physical theory of the phenomenon seems to have been offered.

The electrolysis of copper sulphate with copper electrodes, as we have seen, results in a loss of copper by the anode and an equal gain by the cathode. Phenomena of this kind are classed under the title of *secondary reactions*, and must occur when one of the ions in the liquid (here the SO_4 anion) is a radicle incapable of permanent separate existence. In such cases the ion in question combines with the matter of the electrode or solvent and causes the liberation of some secondary product. In the copper voltameter the SO_4 ion attacks the anode with the formation of copper sulphate, with the result that the solution is not weakened by electrolysis. With a platinum anode the radicle attacks the water in preference to the metal, and oxygen is liberated, the reaction being

$$2H_2O + 2SO_4 = 2H_2SO_4 + O_2.$$

The electrolyte therefore becomes more and more acid at the expense of the salt.

The liberation of the dissociation products of water from weak solutions of salts or acids depends on a similar reaction. Thus in the case of sulphuric acid the ions actually present in the liquid are two hydrogen cations to every SO_4 anion. The hydrogen is directly liberated, and the SO_4 ion takes part in the above reaction at the anode, thus conserving the strength of the acid solution throughout.

Other examples of secondary action can be given, which we shall not consider here. It is interesting to notice the difference in chemical behaviour between the charged ion, moving unaltered through the liquid, and the same ion when discharged, which may attack the electrode or the solvent.

168. Accumulators. Accumulators (see Art. 58) are very convenient for laboratory purposes, as they give an extremely constant electromotive force over the interval of time required for an ordinary experiment. The "Exide" accumulator, made by the Chloride Electric Storage Company, is an example of a good modern accumulator. A grid on each electrode holds the active material, which is thereby prevented from falling away when in use. The plates are separated by wooden partitions, rendered porous by a special treatment, which keep the plates safely apart and are found not to add to the internal resistance of the cell. Each cell contains about a dozen positive and a like number of negative plates, contained in celluloid boxes, and two cells in series form a four-volt accumulator (Fig. 234) suitable for laboratory work. As illustrating the proper treatment of accumulators, we may mention that one of these cells, which will deliver 1 ampere continuously for 20 hours, should first be filled with the purest sulphuric acid diluted with distilled

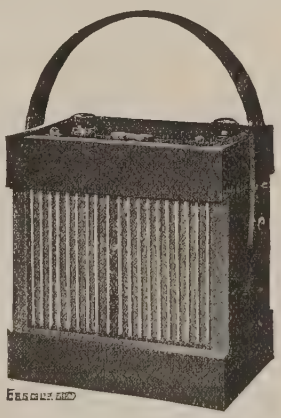


Fig. 234

water till its density is 1·25. It is then charged at the rate of 2 amperes for at least 40 hours, when gas is evolved freely from all the plates. Subsequent charging can be done at the rate of 4 amperes for 10–15 hours, or in any case till each cell gives a potential-difference of 2·55 volts. Loss of liquid by evaporation is made up by the addition of distilled water.

The Edison accumulator is unique in not employing lead and sulphuric acid. The positive plate is of nickel oxide and the negative of iron oxide, subjected to a special process, the electrolyte being a 20 % solution of potassium hydroxide. The E.M.F. of each cell is 1·2 volts, and it is claimed that more power can be obtained, weight for weight, than from the lead accumulator.

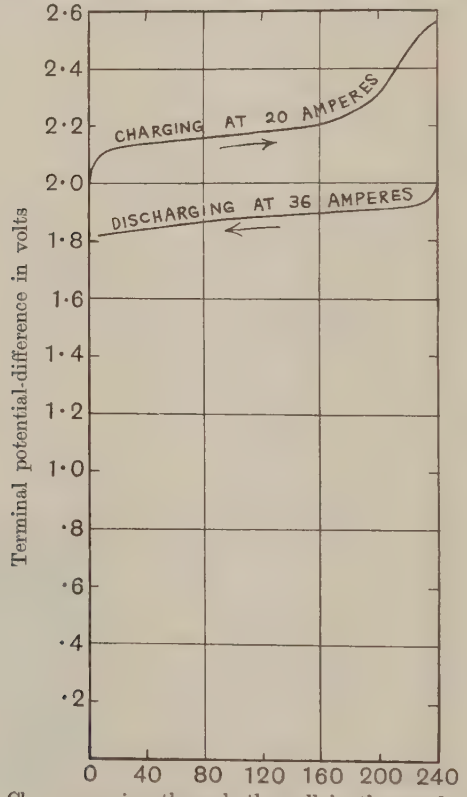


Fig. 235

The changes of the difference of potential between the terminals of a lead accumulator during charging and discharge are shown in Fig. 235, which gives the results of a typical experiment (Cohen and Donaldson). It will be noticed that while practically all the electricity used in charging is recovered on discharge, the latter occurs throughout at a lower voltage, so that by no means all the *energy* is recoverable. The charging energy is represented by $\int V_1 de$ and the electrical work available on discharge by $\int V_2 de$, where V_1, V_2 are the ordinates of the upper and lower curve respectively for the abscissa e. In this way we find that about $5{\cdot}3 \times 10^{12}$ ergs are used in charging the accumulator, of which only $4{\cdot}4 \times 10^{12}$ ergs are recovered, giving an efficiency of 83 per cent. This is a very favourable value, which is not usually approached under the conditions of practical working.

The chemistry of lead cells is complicated and difficult. It is, however, most generally maintained that the principal part of the reaction is as follows. At the end of a normal discharge both electrodes are covered with a layer of lead sulphate, $PbSO_4$. When charging begins the hydrogen proceeding from the electrolysis of sulphuric acid reacts with the lead sulphate at the cathode and forms metallic lead according to the equation

$$H_2 + PbSO_4 = Pb + H_2SO_4.$$

At the anode the sulphate ion SO_4 reacts with the lead sulphate and the water of dilution to form lead peroxide and sulphuric acid, thus:

$$SO_4 + PbSO_4 + 2H_2O = PbO_2 + 2H_2SO_4.$$

Hence the passage of one coulomb round the circuit is accompanied by the disappearance of one equivalent of sulphuric acid; but three equivalents are formed by secondary action, so that the strength of the acid increases during charging. This process continues until the anode, which is to become the positive pole of the accumulator, is covered with a layer of lead peroxide and the cathode with a layer of pure lead; the potential-difference then being about 2·2 volts. Further charging results in the liberation of free hydrogen and oxygen from the electrodes, coupled with a sharp rise in the potential. Potentials above 2·3 volts are somewhat spurious, as they disappear very quickly

on cessation of charging, even when no current is taken from the cell.

During normal discharge the current flows inside the cell from the negative to the positive pole, the latter forming the cathode in the electrolysis of the acid. The reaction at the positive pole is given by

$$H_2 + PbO_2 + H_2SO_4 = PbSO_4 + 2H_2O,$$

and that at the negative pole by

$$Pb + SO_4 = PbSO_4.$$

The visible reactions are therefore exactly reversed, the acid becoming weaker and lead sulphate appearing at both poles. It is generally inadvisable to discharge the cell to a potential-difference lower than 1·8 volts.

169. Measurement of electrolytic resistance. Kohlrausch was the first to show that, when the effects of polarisation are eliminated or avoided, electrolytes obey Ohm's law and have a real resistance. If small, rapidly alternating currents are used, and electrodes of sufficient size, the products of polarisation form but a very thin layer on the surface and are in no case actually liberated. Under these circumstances it is reasonable to suppose that the polarisation E.M.F., whose direction is opposed to that of the impressed E.M.F., is proportional to the charge e that has passed up to the time considered. Assuming provisionally the existence of a resistance R in the electrolyte, the current i is given by

$$Ri = v - Pe,$$

where v is the applied E.M.F. and P a constant. Writing $v = V \cos pt$, and remembering that $i = de/dt$, we find

$$R\frac{di}{dt} + Pi = -Vp \sin pt.$$

Hence

$$i = \frac{\frac{V}{R}\left(\cos pt - \frac{P}{Rp}\sin pt\right)}{1 + \frac{P^2}{R^2p^2}}.$$

It follows that, if P is small in comparison with Rp, the current does not differ much, either in amplitude or phase, from its value

$V \cos pt/R$ in the absence of polarisation. The coefficient P is smaller the greater the surface of the electrode. A large effective surface can be obtained with quite small electrodes by depositing on them platinum in a finely divided form. These considerations enabled Kohlrausch to enunciate the conditions favourable to success with electrolytes; namely high frequency of alternation, moderately large resistance, and platinised electrodes.

The method of carrying out the measurement is as follows. The platinum electrodes of the electrolytic cell are first cleaned in a solution of sulphuric acid and potassium bichromate, washed well and placed in the platinising liquid. This consists of 3 grams of chloroplatinic acid and ·02 gram of lead acetate dissolved in 100 c.c. of water. A small current is passed between the electrodes so that there is a slight but steady evolution of gas, and is reversed at intervals until both electrodes are covered with a fine coating of platinum black. The electrodes are then placed in dilute sulphuric acid and a current passed in both directions in order to remove the last traces of the platinising solution. They are then washed well with distilled water and replaced in the electrolytic cell. This may be of the form shown in Fig. 236 if the electrolyte is a bad conductor: for better conductors one like that in Fig. 237, but without the side tubes, is more suitable. The electrolytic cell must be of insoluble Jena glass and the contents kept at a constant temperature by means of a thermostat, on account of the large variation of resistance with temperature.

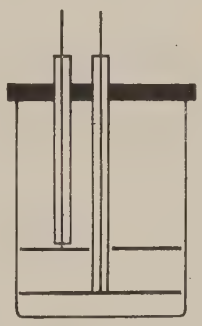


Fig. 236

The measurement of resistance is made on the Wheatstone's bridge. The alternating current is furnished by a high-frequency alternator or a small induction coil, and the galvanometer is replaced by a telephone receiver (cf. Art. 133). The usual relation between the resistances in the arms of the bridge holds when no sound is heard in the telephone. In practice, traces of inductance and capacity in the arms may prevent this condition from being actually realised, and all that is heard is a *minimum* sound in the telephone.

The existence of a definite resistance is proved by the agreement of the results obtained under different experimental conditions. In one of his many experiments performed with this end in view, Kohlrausch measured the ratio of the resistances of various solutions in containing vessels A, B, with the following result:

Solution	Method of measurement	R_A/R_B
NaCl ...	Alternator and dynamometer	22·676
„ ...	Coil and dynamometer	22·693
„ ...	Coil and telephone	22·697
KCl ...	„ „	22·652
HCl ...	„ „	22·683
H_2SO_4 ...	„ „	22·686

Kohlrausch showed further that the error due to inductance and capacity in ordinary resistance boxes was negligible for coils up to 1000 ohms.

The following method of measuring liquid resistance is in

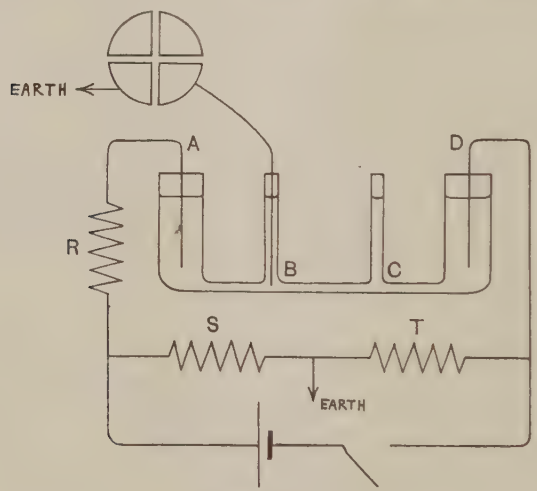


Fig. 237

use in the Electrical Laboratory, Oxford. The electrolytic cell, in addition to the ends carrying the electrodes AD, has two side-tubes B, C fixed on to the main tube. Let it be required to find the resistance in the tube between B and C. The resistance

R is fixed at a suitable amount, and $S + T$ is also kept constant during the experiment. In this way currents i, j are made to flow through the electrolytic cell and shunt circuit respectively. The electrometer, here shown connected to B, can also be connected to the electrode A or the tube C at will. The shunt resistances are altered in each case until the electrometer shows no deflexion. If S_1, S_2, S_3 are the values of S when the electrometer is connected to A, B, C respectively, we have

$$Ri = S_1 j$$

and $$(\text{resistance of } BC)\, i = (S_3 - S_2)\, j.$$

Hence $$\text{resistance of } BC = R\,\frac{S_3 - S_2}{S_1}.$$

This method eliminates altogether any contact difference of potential between the liquid and the rod inserted in the side tubes.

170. Conductivity of electrolytes. When the current flows in straight lines between the electrodes, as in Fig. 236, the value of the conductivity σ of the electrolyte can be deduced from the measurements. If A is the available area of the electrodes, d their distance apart and R the measured resistance, we have

$$R = \frac{l}{\sigma A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Similar considerations apply to the case in which the electrolyte is contained in a tube.

The water of solution has always an appreciable conductivity, which must be taken into account in weak solutions. The conductivity of ordinary distilled water is of the order 2×10^{-6}; for good specimens σ may be as low as 6×10^{-7}. The existence of a limiting conductivity for very weak solutions is clearly shown in Fig. 238, which represents the results of some experiments by Whetham on sulphuric acid at extreme dilution.

Kohlrausch and Heydweiller found that when water was very carefully purified by distillation *in vacuo* its conductivity was about $4{\cdot}3 \times 10^{-8}$. Although every purification of water causes a reduction in its conductivity, yet there is reason to believe that absolutely pure water still conducts appreciably. This is

interesting in indicating the presence of a slight degree of natural dissociation in water.

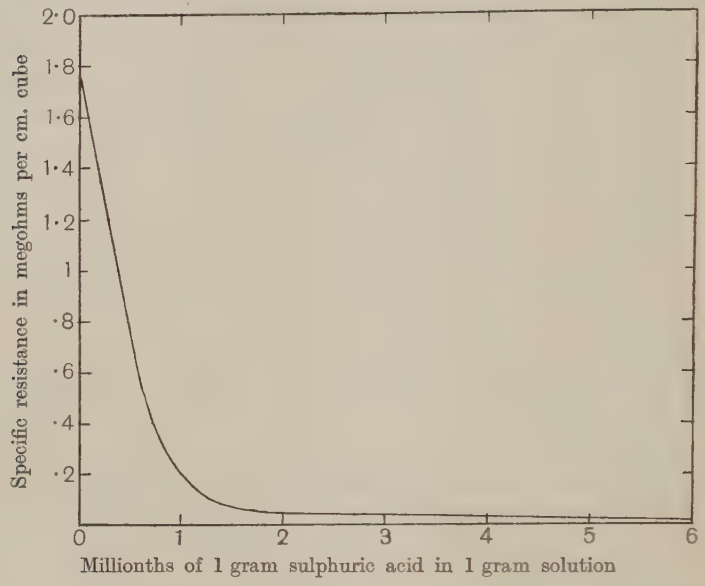


Fig. 238

The following table, due to Kohlrausch, gives the conductivities of some concentrated solutions at 18° C. in absolute measure (reciprocal ohms per centimetre cube). The strengths of the solutions are expressed in percentages of solute in the whole by weight.

	5 %	10 %	20 %	30 %	40 %	50 %	60 %	70 %	80 %
KCl	·069	·136	·268	—	—	—	—	—	—
NH_4Cl	·092	·178	·337	—	—	—	—	—	—
NaCl	·067	·121	·196	—	—	—	—	—	—
K_2SO_4	·046	·086	—	—	—	—	—	—	—
$MgSO_4$	·026	·041	·048	—	—	—	—	—	—
$ZnSO_4$	·019	·032	·047	·044	—	—	—	—	—
$CdSO_4$	·015	·025	·039	·044	—	—	—	—	—
$CuSO_4$	·019	·032	—	—	—	—	—	—	—
$AgNO_3$	·026	·048	·087	·124	·157	·186	·210	—	—
KOH	·172	·315	·499	·542	·450		—	—	—
NaOH	·195	·309	·328	·207	·121	·082	—	—	—
HCl	·395	·630	·762	·662	·515	—	—	—	—
HNO_3	·258	·461	·711	·785	·733	·631	·513	·396	·267
H_2SO_4	·209	·392	·653	·740	·680	·541	·373	·216	·111

Fig. 239 shows the relation of conductivity to temperature for normal solutions* of certain salts. At ordinary temperatures

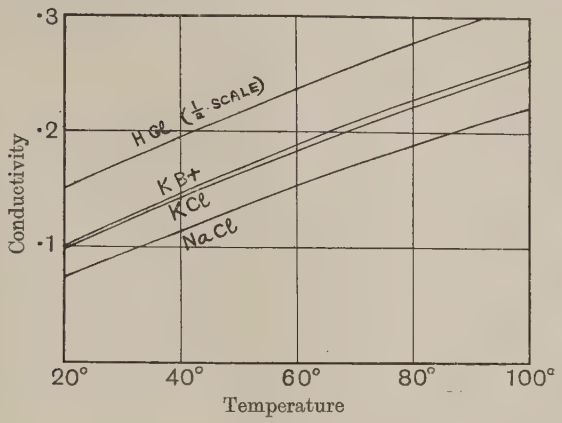


Fig. 239

the rise of conductivity of salt solutions is of the order of 2 per cent. per degree, of acids somewhat less. It is clear, therefore, that care is necessary in measuring the temperature, and in keeping it uniform, if accurate results are to be obtained.

171. Voltaic cells. Faraday's laws give the connexion between the amount of chemical action in a voltaic cell and the charge that has passed; but we require some further explanation of their mode of working. For example, in a Daniell cell made of pure materials, very little zinc dissolves as long as the cell remains on open circuit; but as soon as the circuit is completed the zinc terminal begins to dissolve fairly rapidly, while copper is deposited on the copper terminal. Nernst has given an interesting theory of these phenomena, which we shall now explain.

The theory is based on the fact that all metallic ions are *cations*, and therefore positively charged. It follows that a metal dipped in pure water tends to go into solution positively

* A normal solution is one which contains M grams of the dissolved substance per litre, where M is the molecular weight. It follows from Art. 166 that 1 c.c. of such a solution contains a definite number ($6{\cdot}15 \times 10^{20}$) of molecules, irrespective of the nature of the dissolved substance.

charged, leaving the metal charged negatively. This can be verified easily with the aid of an electrometer. The amount of the metal actually dissolved is very small, for the positive charge in the liquid creates a field of electric force tending to prevent any more ions leaving the metal with a positive charge, and thus a limit to the amount of solution is soon reached. For the sake of illustration, suppose that the metal is in the form of a spherical shell entirely surrounding the water, say of 10 cm. radius. In order to produce at the surface an electric force of 1 volt per centimetre, or 1/300 electrostatic units, the charge e in the liquid is given by $e/100 = 1/300$, or $e = \frac{1}{3}$. This corresponds to about 7×10^8 ions in solution. Large as this number seems, it represents the solution, relatively, of a very small amount of metal, since a layer of metal one molecule thick over the surface contains about 10^{18} molecules.

Another factor comes into play when metals are immersed in solutions of their salts. Here the metallic ions already present tend to separate out on the metal after giving up their positive charge to it. Thus copper becomes positively charged by contact with a concentrated solution of copper sulphate.

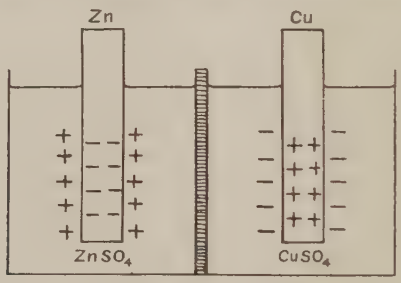


Fig. 240

In the case of the Daniell cell we suppose that zinc tends to go readily into solution. Thus in a short time the electrolyte is charged positively and the zinc terminal negatively. Now the positive charge in the liquid, which resists the solution of the zinc, tends equally to deposit positive ions on the copper terminal, in this way assisting the action of the copper sulphate solution. Thus copper will dissolve less than it would normally, and may have ions deposited on it instead even in a weak solution. Under

these conditions the state of the cell is as shown in Fig. 240. On open circuit the actual amount of solution and deposition on making up the cell is very small, as already explained, the main result being that the potential of the copper is higher than that of the zinc. On completing the circuit extra electrons from the zinc pass over into the copper: the negative charge on the zinc tends to diminish, so that more zinc is enabled to dissolve. Similar but opposite effects occur at the other terminal, and thus the process goes on continually. It follows from this theory that the E.M.F. of the cell will increase if anything is done to promote solution of the zinc or deposition of copper, i.e. if the zinc sulphate solution is diluted or the copper sulphate solution made stronger. This is found to be the case.

Similar considerations to those which apply to metals also hold for hydrogen. It has a tendency to go into solution in the form of positive ions. It can therefore be driven out of the ionic and into the gaseous form at surfaces where there is sufficient electric force, directed away from the liquid. We can thus explain not only the evolution of hydrogen at a cathode in electrolysis, but also the spontaneous decomposition of water by the alkali metals. On this theory potassium, for example, has such a great tendency to go into solution that the resulting electric force is sufficient to liberate hydrogen in this way, as there are always a number of hydrogen ions present in water.

Nernst in his presentation of the theory speaks of an *electrolytic solution pressure* tending to drive a metal into solution in ionic form, which does so until this is compensated by electrostatic action and by the osmotic pressure of the positive ion in the liquid. In a really kinetic theory, however, the conception of pressure should be replaced if possible by something more dynamical. The existence of a definite magnitude characteristic of a metal under all circumstances, such as a solution pressure in the metal should be, does not appear to have yet been placed beyond doubt.

Returning to the Daniell cell, we see that there will in general be three places where the potential undergoes an abrupt change. These are (1) at the surface of separation of the zinc and its solution, (2) at the copper, (3) where the two liquids meet. In

(1) the potential certainly rises in passing into the liquid; in (2) it rises or falls according to circumstances, but in any case to a less degree; while the effect (3) is small and can be calculated (cf. Art. 175 below). If the external circuit is completed there is a fall of potential in the liquids due to their electrolytic resistance: this resistance is called the *internal resistance* of the cell.

Many methods have been devised for measuring the internal resistance of cells. Great accuracy is seldom required in the measurement, and the following method due to Nernst and Haagn will be found convenient, except in the case of accumulators, whose resistance is too low for the purpose. The cell is placed in a Wheatstone's bridge together with two adjustable resistances R_1, R_2, preferably accurate resistance boxes containing fractions of an ohm. The other arms of the bridge are condensers C_1, C_2 of the order of 10 microfarads. Alternating current is supplied by a small induction coil or high-frequency alternator, placed in series with a third large condenser. In this way the circulation of continuous currents in the bridge is prevented, and the bridge is balanced for no current in the telephone T. It is easy to show in this case, as in Art. 133, that

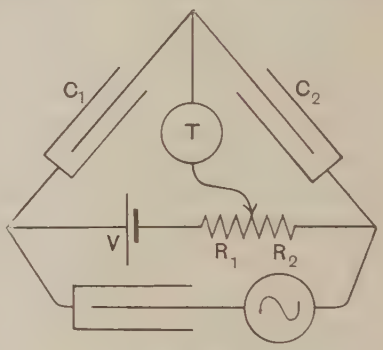


Fig. 241

$$\frac{V + R_1}{R_2} = \frac{C_2}{C_1},$$

and hence V can be found. The resistance of an ordinary Leclanché cell is about 1 ohm, that of a small Weston cell 500 ohms.

The resistance of an accumulator can be found by connecting its terminals to a voltmeter while it is also delivering current to an external circuit. If V_0 is the voltmeter reading on open circuit, V its value when a current i is flowing, the resistance of the cell is given by $V_0 - V = Ri$. A medium-sized 8-volt battery has a resistance of the order of ·05 ohms.

172. Theory of reversible cells. The delivery of a current by a cell is reversible in the thermodynamic sense when the chemical and physical processes in the circuit can be made to go on in the opposite direction by an infinitesimal change in the circumstances under which we are operating. Thus we may imagine a cell whose E.M.F. on open circuit is V sending a current round a circuit containing a direct-current generator of adjustable E.M.F. V'. Then the process may be reversible when V exceeds V' only infinitesimally, because then a slight increase of V' raises it above V and sends a current round the circuit in the opposite direction. In addition, however, the cell must be such that the chemical actions are exactly undone by the reverse transfer, so that the cell returns to the same state whenever the total charge that has passed any point of the circuit, measured algebraically, is zero. The two conditions necessary for a cell to be reversible are therefore (1) that it delivers only an infinitesimal current; (2) that the chemical phenomena at the electrodes are reversible in the ordinary sense. The second condition is satisfied very nearly in cells which do not evolve gas, such as the Daniell, Weston and Clark cells and (practically) the accumulator. The original Volta cell $\mathrm{Zn} \mid \mathrm{H_2SO_4} \mid \mathrm{Cu}$, on the other hand, is essentially irreversible on account of the evolution of gas at the copper terminal. If current is sent through the cell in a direction opposite to the normal, gas is now evolved at the zinc terminal, zinc not being deposited as it would be with the Daniell cell.

Let θ be the absolute temperature of the cell, η its entropy, and let an amount of charge de (coulombs) pass round the circuit in the ordinary direction. The work done *on* the cell in the process is $dW = -Vde$ joules or $-10^7 Vde$ ergs, where V is the E.M.F. in volts, and the heat given to it is $dQ = \theta d\eta$ calories. Hence the gain of internal energy of the system, measured in calories, is

$$d\epsilon = \theta d\eta - \frac{10^7}{J} Vde.$$

Thus
$$d(\epsilon - \theta\eta) = -\eta d\theta - \frac{10^7}{J} Vde,$$

from which it follows that

$$\left(\frac{\partial V}{\partial \theta}\right)_e = \frac{J}{10^7}\left(\frac{\partial \eta}{\partial e}\right)_\theta.$$

This equation contains the thermodynamic theory of the reversible cell. To interpret it, we notice that the left-hand side is merely $dV/d\theta$, where V is the E.M.F. on open circuit. Hence on multiplying by θ we have

$$\theta \frac{dV}{d\theta} = \frac{J}{10^7}\left(\frac{\theta d\eta}{de}\right)_\theta.$$

The right-hand side is the heat actually taken up by the cell during the isothermal passage of unit charge round the circuit, say λ'. If λ is the heat *evolved* by the corresponding chemical change, *when conducted in a calorimeter without the passage of electric charge*, then the difference between the heats λ' and $-\lambda$ is due to the fact that in the former case an amount $10^7 V$ ergs of electrical work is performed, and in the latter case none. Hence

$$\lambda' = \frac{10^7 V}{J} - \lambda.$$

Substituting in the above equation and remembering that $J = 4{\cdot}18 \times 10^7$, we have

$$V = 4{\cdot}18\lambda + \theta \frac{dV}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

This equation is due to Helmholtz.

If the temperature-coefficient $dV/d\theta$ is sufficiently small we have $V = 4{\cdot}18\lambda$; in other words, the electrical work is "equivalent" to the heat of reaction. In the Daniell cell the effective reaction corresponding to the passage of one coulomb is the replacement of one electro-chemical equivalent of copper by zinc in sulphate solution, for which $\lambda = {\cdot}2592$. Thus $4{\cdot}18\lambda = 1{\cdot}09$, a number agreeing very well with the observed voltage. It is, however, clear both theoretically and experimentally that the above relation does not hold in general. Thus Bugarszky has found a combination for which λ and V are of opposite signs, so that the cell at once absorbs heat (from the atmosphere) and delivers current.

An accurate verification of Helmholtz's formula (9) was carried out by Jahn in 1886. The cell to be examined was enclosed in a Bunsen ice calorimeter, and closed through a resistance and also through a voltmeter circuit of high resistance. The heat

actually developed in the cell during the passage of a known current at 0° C. was measured, and from it Jahn deduced the chemical heat λ. It will be seen from the following table that the agreement between theory and observation is very satisfactory:

Cell	V	λ	$dV/d\theta$	$V - 4{\cdot}18\lambda$	$\theta dV/d\theta$
Cu, $CuSO_4 + 100\ H_2O$, $ZnSO_4 + 100\ H_2O$, Zn	1·0962	·26027	+ 0·000034	+ ·0083	+ ·0093
Cu, Cu $(C_2H_3O_2)_2$ aq, Pb $(C_2H_3O_2)_2 + 100\ H_2O$, Pb	0·4764	·08582	+ 0·000385	+ ·1177	+ ·1051
Ag, $AgNO_3$, Cu $(NO_3)_2$, Cu	0·4580	·15598	− 0·000708	− ·1940	− ·1930
Ag, $AgNO_3$, Pb $(NO_3)_2$, Pb	0·9320	·26413	− 0·000632	− ·1720	− ·1725
Ag, AgCl, $ZnCl_2 + 100\ H_2O$, Zn	1·0306	·27088	− 0·000409	− ·1017	− ·1117
Ag, AgCl, $ZnCl_2 + 50\ H_2O$, Zn	1·0171	·25485	− 0·000210	− ·0482	− ·0573
Ag, AgCl, $ZnCl_2 + 25\ H_2O$, Zn	0·9740	·24480	− 0·000202	− ·0493	− ·0551
Ag, AgBr, $ZnBr_2 + 25\ H_2O$, Zn	0·8409	·20736	− 0·000106	− ·0281	− ·0289

173. Measurement of ionic velocity. If the electric force in an electrolyte is doubled, the current is also; from which it follows that the ions move with velocities proportional to the electric force. The velocity of an ion under unit electric force (usually one volt per centimetre) is called its *mobility*, so that if u is the mobility the velocity under an electric force E is given by $v = uE$.

The mobility of ions in certain cases may be determined by the following method. It is applicable to dilute solutions of electrolytes in which only two ions are present.

The first part of the process, due to Hittorf, consists in finding the ratio of the mobilities by measuring the changes which take place in the concentration of the solution. In any practical case there will always be a portion in the centre of the electrolyte where the concentration is unaltered, even after the lapse of an hour or two, with an applied potential of a few volts. In Fig. 242

the thick line represents a plane near the centre of the electrolyte, and the black and white spheres stand respectively for negative and positive ions, so that the electric force is from left to right. At the beginning of the experiment the dissociated molecules

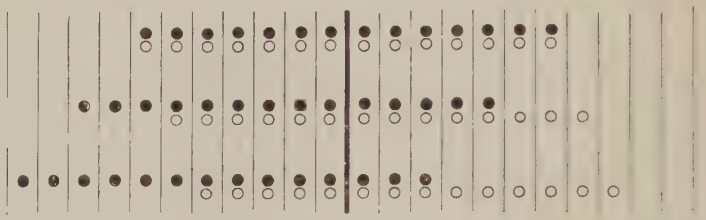


Fig. 242

near the plane are represented in the top diagram, and the other two diagrams show the position of the same ions at two subsequent epochs, the negative ion being supposed for simplicity to move twice as quickly as the positive. It is evident from the figure that the left-hand side of the plane has lost two paired ions while the right-hand side has lost four. Thus the strength of the solution to the left, as measured by the number of pairs of ions still remaining, has decreased half as much as the strength on the other side of the plane. It is easy to see in general that the loss of salt in the anode region (up to where the solution is unaltered) is to that in the cathode region in the ratio u_1 to u_2, where u_1 is the mobility of the positive ion and u_2 that of the negative. These losses can be found by analysing the two portions of the liquid, and it is also easy to find whether the concentration in the central part of the liquid has altered during the experiment.

Looked at in this way, the theory of the experiment is simple enough; but it becomes a little perplexing if the regions near the electrodes are considered instead of the central portion of the electrolyte. Thus in the figure the negative ions are advancing to the anode twice as fast as the rate of advance of the positive ions. This, however, cannot hold right up to the electrodes, because then there would be two equivalents of the negative ion liberated for one of the positive, which would contradict Faraday's laws. The behaviour near the electrodes is therefore somewhat complex, and electrostatic forces come into play which equalise matters in this respect.

The sum of the ionic mobilities can be calculated from the conductivity of the electrolyte, as was shown by Kohlrausch in 1879. Let there be n molecules of the salt per c.c. before dissociation. Then in a dilute solution there will be, say, n_1 positive ions of valency p_1, and n_2 negative ions of valency p_2, where

$$n_1 p_1 = n_2 p_2 = n.$$

If v_1, v_2 are the actual velocities of the ions, the current-density in electrostatic units is

$$\begin{aligned} j &= n_1 p_1 e v_1 + n_2 p_2 e v_2 \\ &= ne\,(v_1 + v_2). \end{aligned}$$

If j is to be measured in amperes per square centimetre, this equation is replaced by

$$j = \frac{ne\,(v_1 + v_2)}{3 \times 10^9}.$$

If u_1 and u_2 are the mobilities and E the electric force in volts per centimetre, $v_1 = u_1 E$ and $v_2 = u_2 E$. Also $j = \sigma E$, where σ is the conductivity, so that

$$\sigma = \frac{ne\,(u_1 + u_2)}{3 \times 10^9}.$$

Let c be the concentration of the solution, m the mass of a molecule of the solute. Then $c = mn$, and therefore

$$\frac{\sigma}{c} = \frac{u_1 + u_2}{3 \times 10^9} \cdot \frac{e}{m}.$$

Hence from equation (7) we have

$$\frac{\sigma}{c} = \frac{u_1 + u_2}{3 \times 10^9} \cdot \frac{2{\cdot}87 \times 10^{14}}{M},$$

where M is the molecular weight of the solute. Thus

$$u_1 + u_2 = 1{\cdot}04 \times 10^{-5} \frac{M\sigma}{c} \quad \ldots\ldots\ldots\ldots (10).$$

Since u_1/u_2 is given by Hittorf's method, and $u_1 + u_2$ by equation (10), both u_1 and u_2 can be found.

The theory is only strictly applicable to the case in which all the molecules of the electrolyte are dissociated, which is by no means easy to realise (cf. Art. 164). It follows that the velocities

in solutions of ordinary strengths cannot be determined with great accuracy. In dilute solutions the velocity of, say, a potassium ion may be expected to be independent of the nature of the other ion present, since the two ions are supposed to migrate independently. Thus the mobility of the potassium ion in 0·01 normal solutions of KCl, KI and KNO_3 is found to be $6{\cdot}26 \times 10^{-4}$, $6{\cdot}34 \times 10^{-4}$ and $6{\cdot}08 \times 10^{-4}$ respectively. The following table gives the mobilities of various ions, as estimated by Kohlrausch for infinite dilution:

Cations	$10^4 u$	Anions	$10^4 u$
H	31·5	OH	17·4
K	6·46	CrO_4	7·2
NH_4	6·4	SO_4	6·8
Pb	6·1	Br	6·7
Ba	5·5	I	6·65
Ag	5·43	Cl	6·55
Cu (in cuprous salts)	4·6	NO_3	6·17
Zn	4·6	ClO_3	5·50
Cd	4·6	F	4·66
Mg	4·5	$C_2H_3O_2$	3·5
Na	4·35		
Li	3·34		

In addition to the above indirect method, a direct method of measuring ionic velocities has been used by Lodge, Whetham and others. On examination, however, this method, apparently so simple, is seen to involve far more delicate considerations than the preceding, and will not be described here.

174. Theory of the motion of ions in electrolytes. Since the osmotic pressure of the dissolved particles in a liquid is the same as the pressure they would exert in the gaseous form, it is natural to regard both as due to the same cause, namely the motion of agitation of the particles. We shall adopt this view and regard the dissolved particles as moving in the solvent in the same way that the molecules of one gas move among those of another gas with which it is mixed. The motion of an ion in a liquid probably approximates more to a continuously curved trajectory than to a succession of zigzags; but nevertheless there is one thing in common, namely a certain average interval of

time T in which the velocity of an ion in any particular direction can be wiped out, so that all directions of motion afterwards are equally probable. In the case of a gas this is comparable with the time of describing a mean free path. Disregarding the differences between the two cases, therefore, we can speak of a *free path* of an ion in a liquid and of its *collisions* with molecules of the solvent.

For the sake of simplicity we shall further assume that all the free paths are of length l and all the velocities of agitation V. Let e be the charge and m the mass of an ion moving in an electric field of E electrostatic units. Then in the interval between collisions the acceleration of the ion is eE/m, so that the ion has drifted along the lines of force a distance $eET^2/2m$ further than it would have done in the absence of the force. The number of collisions per second is $1/T$, and therefore the amount of drift per second, or the velocity of the ion under the electric force, is $eET/2m$. Since $T = l/V$, we have $v = eEl/2mV$. Hence if u is the *velocity under unit electrostatic force*,

$$u = \frac{el}{2mV} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The mean free path l in the liquid can be estimated from this equation. The velocity of agitation V is given by the principle of equipartition of energy. Thus if Ω is the velocity of agitation of the molecules of hydrogen at pressure p and density ρ and at the same temperature as the liquid, we have $\frac{1}{2}mV^2 = \frac{1}{2} \cdot 2m_H \cdot \Omega^2$ and $\Omega^2 = 3p/\rho$. Hence

$$V^2 = \frac{6p}{\rho} \frac{m_H}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

For the hydrogen ion at 15° C. we thus find $V = 2{\cdot}67 \times 10^5$. The velocity under a force of 1 volt per cm. ($E = 1/300$) is

$$v = 3{\cdot}15 \times 10^{-3}.$$

Hence we have $l = 1{\cdot}8 \times 10^{-9}$ cm. This is of the order of one-tenth the diameter of a hydrogen molecule, assuming the ion to consist of a single charged molecule. Hence we conclude either that the collision with a molecule is more sudden than in gases, or else that the ion is accompanied on its course by a number of

molecules of the solvent which cluster round it. The latter is the most plausible hypothesis.

When the ionic concentration is not the same at all points of a liquid diffusion comes into play, and there is on the whole a drift of ions even in the absence of electric force. The ordinary theory of diffusion is based on an analogy with the theory of the flow of heat, and supposes that if n is the number of ions per c.c. at the point (x, y, z), then the number of ions crossing an imaginary surface-element dS per second is

$$-K\left(\lambda\frac{\partial n}{\partial x}+\mu\frac{\partial n}{\partial y}+\nu\frac{\partial n}{\partial z}\right)dS,$$

where (λ, μ, ν) are the direction-cosines of the normal to dS. The quantity K is called the *coefficient of diffusion*.

In the general case in which an electric field is acting as well as diffusion, the number of ions crossing dS per second becomes

$$\left[-K\left(\lambda\frac{\partial n}{\partial x}+\mu\frac{\partial n}{\partial y}+\nu\frac{\partial n}{\partial z}\right)+n(\lambda v_x+\mu v_y+\nu v_z)\right]dS,$$

where v is the velocity of drift under the electric field. The result of integrating this expression over a closed surface S must however be equivalent to $-\int\frac{\partial n}{\partial t}\,d\tau$ extended through the interior of S, since it represents the rate at which ions are leaving the surface. Transforming the surface-integral by Gauss' theorem we thus find

$$\frac{\partial}{\partial x}\left(-K\frac{\partial n}{\partial x}+nv_x\right)+\frac{\partial}{\partial y}\left(-K\frac{\partial n}{\partial y}+nv_y\right)+\frac{\partial}{\partial z}\left(-K\frac{\partial n}{\partial z}+nv_z\right)=-\frac{\partial n}{\partial t},$$

or

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv_x)+\frac{\partial}{\partial y}(nv_y)+\frac{\partial}{\partial z}(nv_z)-K\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}+\frac{\partial^2 n}{\partial z^2}\right)\quad..(13).$$

This is the equation of continuity of a fluid in which diffusion is going on.

The physical meaning of the constant K is well illustrated by the following theorem, which follows from the last equation:

If R^2 is the mean value of the squares of the distances of a large number of ions in unlimited liquid from any fixed point in space, and if no electric force is acting, then $d(R^2)/dt = 6K$.

Let N be the total number of ions, n the number per c.c. at the point (x, y, z) at time t. Then N is equal to $\int n\,d\tau$, extended over the whole of space, and is also a constant. Taking the fixed point as origin, we have by definition

$$NR^2 = \int r^2 n\,d\tau.$$

Hence
$$N\frac{d(R^2)}{dt} = \int r^2 \frac{\partial n}{\partial t}\,d\tau.$$

If there is no electric force, $v = 0$ and

$$\frac{\partial n}{\partial t} = K\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right).$$

Thus
$$N\frac{d(R^2)}{dt} = K\int r^2\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right)d\tau.$$

Now it follows from Green's reciprocal theorem applied to the whole of space that

$$\int (U\Delta V - V\Delta U)\,d\tau = 0.$$

Putting $U = r^2 = x^2 + y^2 + z^2$ and $V = n$ in the preceding equation, the right-hand side becomes $6K\int n\,d\tau = 6KN$. Hence the theorem is proved.

It follows that ions tend to diffuse so that the mean value of the square of their distances from any point increases uniformly with the time; and hence they must ultimately diffuse to infinite distances. This theorem enables us to calculate K in terms of molecular quantities. For consider the ions which have collisions in the neighbourhood of a point P. After a time l/V they are found uniformly spread over a sphere of radius l with P as centre.

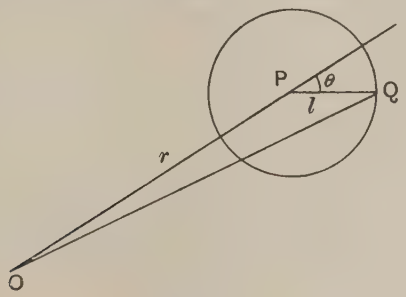


Fig. 243

If Q is one of these ions, the square of its distance from O has increased from r^2 to $r^2 + 2lr\cos\theta + l^2$. Hence on the average

R^2 increases by an amount l^2 in time l/V, or by an amount lV per second. Hence $d(R^2)/dt = lV$, and

$$K = \tfrac{1}{6}lV \quad \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

From (11) and (14) we obtain by division

$$\frac{u}{K} = \frac{3e}{mV^2}.$$

Let N be the number of molecules per c.c. in a gas at the temperature of the liquid and pressure Π. Then mV^2 is twice the mean kinetic energy of a molecule of the gas, so that $\Pi = \frac{1}{3}NmV^2$. Hence

$$\frac{u}{K} = \frac{Ne}{\Pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Although the present theory does not aim at rigour, the important relation (15) is exact.

175. Concentration cells. When two solutions of the same salt, but of different concentrations, are placed in contact, they will gradually diffuse into one another and attain the same concentration. This however is a matter of days, and there is another effect which takes place much sooner, namely the smoothing down of the surface of contact with the formation of a transition layer. It is this effect which we shall examine here, as it is accompanied by important electrical effects.

Let the plane $x = 0$ originally separate two dilute solutions of the same salt, of concentrations c_1, c_2 respectively. Then after a short time the region on either side of the plane will form a transition layer in which the concentration changes rapidly but continuously from c_1 to c_2.

In general the ions have different rates of diffusion, so that at first more charge will pass over the plane in one direction than

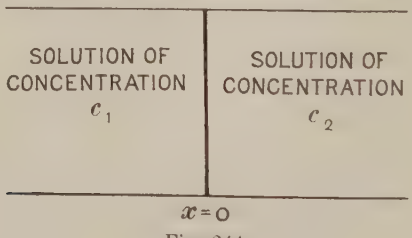


Fig. 244

the other. This however results in an accumulation of positive charge in one of the solutions which tends to prevent further separation of the ions, and ultimately a steady state is attained in which as much charge passes in one direction as in the other. When this has occurred let n be the number of dissociated molecules (ionic pairs) per c.c. at distance x from the original plane of separation, E the electric force caused by the separation of the ions. As has been explained already, the actual amount of separation required to produce electric forces of ordinary magnitudes is very small, so that we may put

$$n_1 p_1 = n_2 p_2 = n \ldots\ldots\ldots\ldots\ldots\ldots (16),$$

where n_1 is the number of positive ions per c.c. at any point, p_1 the valency of the positive ion, and the suffix 2 similarly distinguishes the negative ion. The number of positive ions crossing unit area of a plane parallel to $x = 0$ per second is $-K_1 \frac{dn_1}{dx} + n_1 v_1$, and each carries a charge $p_1 e$, so that the transfer of charge from left to right per second is $p_1 e \left(-K_1 \frac{dn_1}{dx} + n_1 v_1\right)$. Hence in the steady state we have

$$p_1 \left(-K_1 \frac{dn_1}{dx} + n_1 v_1\right) = p_2 \left(-K_2 \frac{dn_2}{dx} + n_2 v_2\right).$$

But $v_1 = u_1 E$ and $v_2 = -u_2 E$; and equation (15) gives in the present case

$$K_1 = \frac{\Pi}{Ne} \frac{u_1}{p_1}, \quad K_2 = \frac{\Pi}{Ne} \frac{u_2}{p_2}.$$

Making these substitutions we find

$$-\frac{\Pi}{Ne} u_1 \frac{dn_1}{dx} + n_1 p_1 u_1 E = -\frac{\Pi}{Ne} u_2 \frac{dn_2}{dx} - n_2 p_2 u_2 E;$$

or, using (16),

$$n (u_1 + u_2) E = \frac{\Pi}{Ne} \left(\frac{u_1}{p_1} - \frac{u_2}{p_2}\right) \frac{dn}{dx}.$$

This last equation enables us to calculate the natural difference of potential that arises between the solutions. For writing it in the form

$$E dx = \frac{\frac{u_1}{p_1} - \frac{u_2}{p_2}}{u_1 + u_2} \frac{\Pi}{Ne} \frac{dn}{n}$$

and integrating across the whole transition layer, we have

$$V = \frac{\frac{u_1}{p_1} - \frac{u_2}{p_2}}{u_1 + u_2} \frac{\Pi}{Ne} \log_e \frac{\nu_1}{\nu_2},$$

where $n = \nu_1$ in the undisturbed parts of the left-hand solution and $n = \nu_2$ on the right. Since $\nu_1/\nu_2 = c_1/c_2$ this gives

$$V = \frac{\frac{u_1}{p_1} - \frac{u_2}{p_2}}{u_1 + u_2} \frac{\Pi}{Ne} \log_e \frac{c_1}{c_2} \quad \ldots\ldots\ldots\ldots\ldots (17).$$

Here V is reckoned as positive when the left-hand solution is at the higher potential, and if Ne is taken from equation (3), V comes out in electrostatic units.

The theory of the potential-differences arising at the boundary of any two electrolytes in contact is somewhat beyond the scope of this book; but there is one important general property which must be mentioned, namely that it is only the *ratio* of the concentrations, and not their absolute values, which enters into the expression for V. For at a point where the electric force is E the condition of zero total flow of electricity is

$$\Sigma e_r \left(- K_r \frac{dn_r}{dx} + n_r u_r E \right) = 0,$$

where e_r is the charge on an ion of the rth kind and summation is taken over all types of ion present. Hence E would be unaltered if all the n's were increased in the same ratio; that is, if the concentrations of the two solutions were so increased.

Cells in which differences of potential are produced by differences in the concentrations of the solutions are known as *concentration cells*. If we carry out the calculation of the surface potential-differences at 15° C., which is sufficient in view of the uncertainty existing as to the mobilities, and consider only univalent ions, equation (17) becomes

$$V = 0{\cdot}0574 \frac{u_1 - u_2}{u_1 + u_2} \log_{10} \frac{c_1}{c_2} \text{ volts} \quad \ldots\ldots\ldots (18).$$

In order to verify this theory, Nernst set up a number of cells in which the effect of the electrodes is eliminated and the only

potentials it is necessary to consider are those due to changes of concentration. One example is the cell

KCl, 0·1 normal	KCl, 0·01 normal	HCl, 0·01 normal	HCl, 0·1 normal	KCl, 0·1 normal
(1)	(2)	(3)	(4)	(5)

with any similar electrodes. Here the potential arising at the interface (2, 3) is equal and opposite to that arising at (4, 5), since the ratios of the concentrations are the same in both cases, and the effect of the electrodes also cancels out. Hence the only effective electromotive forces are those between (1, 2) and between (3, 4). The most general cell of the above type may be written symbolically

$$| c_1u_1u_2 | c_2u_1u_2 | c_2u_3u_4 | c_1u_3u_4 | c_1u_1u_2 |,$$

where u_1, u_2 are the ionic mobilities in the first electrolyte, u_3, u_4 those in the second, and c_1, c_2 the two concentrations occurring. The total E.M.F. of the cell is found from (18) to be

$$V = 0{\cdot}0574 \left(\frac{u_1 - u_2}{u_1 + u_2} - \frac{u_3 - u_4}{u_3 + u_4} \right) \log_{10} \frac{c_1}{c_2}.$$

The result of Nernst's experiments is shown in the following table:

Electrolytes	E.M.F.	
	observed	calculated
KCl, HCl	− 0·0357	− 0·0380
KCl, HNO_3	− 0·0378	− 0·0390
NH_4Cl, NaCl ...	0·0098	0·0109
KCl, NaCl	0·0111	0·0112

In addition there are many concentration cells in which the effect of the electrodes is not negligible. The theory of these cells has been developed by Nernst.

CHAPTER XI

ELECTRIC OSCILLATIONS

176. Theory of condenser discharges. Let C be a condenser whose plates are connected by a circuit of self-inductance

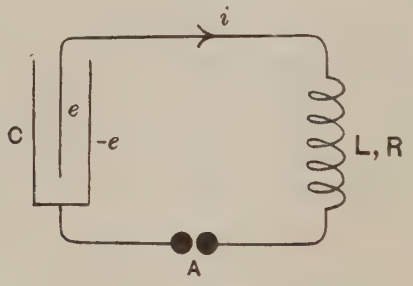


Fig. 245

L and resistance R. The function of the gap A in the circuit will be considered later, and for the present the gap may be supposed not to exist. If e is the charge on the inner plate at time t, the current is $i = -de/dt$ and the difference of potential between the plates $\frac{e}{C}$. Hence the equation $L\frac{di}{dt} + Ri = \frac{e}{C}$ gives

$$LC\frac{d^2e}{dt^2} + RC\frac{de}{dt} + e = 0 \quad \ldots\ldots\ldots\ldots(1).$$

The current i and the potential-difference V between the plates of the condenser both satisfy an equation of the same form. The equation (1) therefore determines the general type of electrical phenomena which can occur in a condenser circuit left to itself; and as we shall see that the quantities involved are usually

oscillatory in nature, the equation may be said to represent the *electric oscillations* of the system.

When $R = 0$, equation (1) becomes $d^2e/dt^2 + n^2e = 0$, where

$$n^2 = 1/LC \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence in this case e is a simple-harmonic function of the time, of period

$$T = 2\pi/n = 2\pi\,(LC)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

It follows that, if a condenser is charged up to a certain potential and then discharged through a circuit of negligible resistance but appreciable self-inductance, the discharge is oscillatory, electricity surging backwards and forwards and charging the plates alternately positively and negatively. It had been observed by Savary in 1827 that when a needle was placed in a coil and magnetised by the discharge of a Leyden jar, the direction of magnetisation might vary according to circumstances. The present theory was first given by Lord Kelvin in 1853, formula (3) being known by his name.

Returning to the general equation (1), and writing

$$\mu = \frac{R}{2L} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

we have

$$\frac{d^2e}{dt^2} + 2\mu\frac{de}{dt} + n^2e = 0,$$

where $n^2 = 1/LC$ as before. The solution of this equation in real form differs according as $n \gtrless \mu$. If $n > \mu$ it is

$$e = \{A\cos(n^2 - \mu^2)^{\frac{1}{2}}t + B\sin(n^2 - \mu^2)^{\frac{1}{2}}t\}\,e^{-\mu t} \quad \ldots (5),$$

while when $n < \mu$ it becomes

$$e = \{Ae^{(\mu^2 - n^2)^{\frac{1}{2}}t} + Be^{-(\mu^2 - n^2)^{\frac{1}{2}}t}\}\,e^{-\mu t} \quad \ldots\ldots\ldots (6).$$

We have thus to distinguish two types of discharge; the *oscillatory*, given by equation (5) and occurring when $R^2 < 4L/C$, and the *non-oscillatory* which occurs when R^2 exceeds this value. Oscillatory discharges, which are the most usual as well as the most important type, are thus represented in general by a vibration of *damped simple-harmonic type*. The periodic time is most conveniently defined as twice the interval between two consecutive

epochs at which the charge or current vanishes, and is then given by $T = 2\pi (n^2 - \mu^2)^{-\frac{1}{2}}$. In general, therefore, the period exceeds that given by Kelvin's formula.

The quantity $$\delta = \frac{2\pi\mu}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

is called the *damping coefficient*, or *decrement*. It is such that the exponential factor sinks to $e^{-\delta}$ after one period given by Kelvin's formula. The discharge is only oscillatory when $\delta < 2\pi$: but even the value $\delta = 1$ represents a very highly damped discharge. For weakly damped oscillations in which δ is so small that its square is negligible, Kelvin's formula gives the periodic time with sufficient accuracy.

177. Discussion of a particular case. Let the condenser be first charged to potential V_0 and the circuit then closed. Thus at time $t = 0$ we have $e = CV_0$ and $i = -de/dt = 0$. If we confine our attention to the case of small damping, equation (5) becomes $e = (A \cos nt + B \sin nt)\, e^{-\mu t}$. The above initial conditions give $A = CV_0$ and $B = \mu CV_0/n$, so that the solution is

$$e = CV_0 \left(\cos nt + \frac{\mu}{n} \sin nt\right) e^{-\mu t} \quad \ldots\ldots\ldots (8).$$

By differentiation we have

$$i = nCV_0\, e^{-\mu t} \sin nt \quad \ldots\ldots\ldots\ldots\ldots (9),$$

neglecting μ^2 once more in comparison with n^2. We require also a rough estimate of the maximum current flowing in the circuit at any time. Since the damping is small, the variation of the exponential factor from time $t = 0$ up to the time of the first maximum of i is negligible, and we have with sufficient accuracy

$$i_{\max} = nCV_0 \quad \ldots\ldots\ldots\ldots\ldots (10).$$

The decrement is found from (7) to be

$$\delta = \pi R \left(\frac{C}{L}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (11).$$

In these formulae all quantities are understood to be expressed in the electromagnetic system of units, absolute or derived. Thus if a condenser of 1 microfarad capacity is charged up to 100 volts

and then discharged round a circuit of inductance 1/100 henry and resistance 1 ohm, we have $C = 10^{-6}$, $L = 10^{-2}$, $R = 1$ and $V_0 = 10^2$. Hence we find $n = 10^4$, $T = 1/1600$ second, $i_{max} = 1$ ampere, $\delta = \cdot 03$. The condition of small damping is clearly satisfied, as the exponential factor sinks to $1/e$ only after 30 complete oscillations. This is the type of comparatively slow oscillation with large inductance and capacity: with ordinary laboratory resources it is difficult to produce oscillations slower than about 100 to the second without recourse to iron in the inductance, which of course alters the phenomena somewhat. The resistance necessary to make the discharge non-oscillatory would be 200 ohms, from which it follows that the discharge of a condenser in an experiment such as that of Art. 84 may very well be oscillatory. This, of course, does not impair the validity of the ordinary formulae, which go on the assumption that the charges $\pm CV_0$ originally on the plates are neutralised in a comparatively short time.

As an example of more rapid oscillations we may take the case of a Leyden jar of capacity 1000 E.S.U. charged to 5000 volts and then discharged round a coil of a few turns of wire, say of inductance 10,000 E.M.U. and resistance 1/10th ohm. Here the periodic time of the oscillations is $6 \cdot 6 \times 10^{-7}$ second and the maximum current 5·3 amperes, while $\delta = \cdot 003$. The exponential factor sinks to $1/e$ in 1/5000 second, during which time the system has carried out 300 complete oscillations. The resistance necessary to cause an aperiodic discharge would be 190 ohms.

178. Production and demonstration of electric oscillations. In order to realise experimentally the conditions of the last article, namely to charge up a condenser to a certain voltage and then discharge it, use is made of the properties of the electric spark. Let A in Fig. 245 represent two zinc or brass balls brought to within a short distance (say 1 mm.) of one another, and connected to the terminals of the secondary of an induction coil. The "break" of the primary coil gives rise to a rapidly increasing difference of potential between the terminals of the secondary, which at first simply charges up the plates of the condenser. As soon as this potential-difference rises to a certain amount, which

in this case is about 3000 volts, the spark-gap gives way and the condenser begins to discharge. Now a general characteristic of sparks is that when a current has once passed it can continue flowing even when the potential falls below that necessary to start a spark in the first place. As soon as the spark passes, therefore, the gap A may be imagined to be bridged across by a conducting wire, which is broken again when the amplitude of the oscillations falls below a certain amount. The course of the oscillations produced by an induction coil and spark-gap is

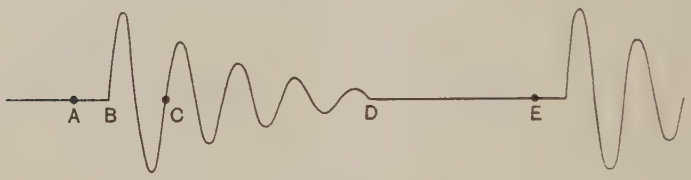


Fig. 246

illustrated in Fig. 246, which represents the current as a function of the time. If the "break" of the coil occurs at A, the condenser is charging up between A and B without passage of current. At B the oscillations begin, and BC represents the periodic time. The oscillations cease at D, after which nothing happens until the next "break" of the coil at E, when everything goes on as before. In the Leyden jar discharge considered at the end of the last article BC is of the order of one-millionth of a second. In one-thousandth of a second the amplitude of the oscillations is reduced to the fraction $1/e^5 = 1/150$ of its original value by the action of resistance alone, without taking into account the additional damping due to the spark-gap. Since the time between two successive "breaks" of an ordinary coil is of the order $\frac{1}{50}$th second, it follows that the idle interval of time DE may much exceed the active interval BD.

The oscillatory nature of condenser discharges in circuits of sufficiently low resistance was first shown by Feddersen (1861), who examined the image of the spark-gap in a rotating mirror. His experiments have been repeated and modified by numerous observers. The following table gives the results of some recent experiments by Diesselhorst:

T observed	*T calculated from Kelvin's formula*
$1{\cdot}628 \times 10^{-6}$	$1{\cdot}646 \times 10^{-6}$
$2{\cdot}293 \times 10^{-6}$	$2{\cdot}313 \times 10^{-6}$
$4{\cdot}967 \times 10^{-6}$	$4{\cdot}967 \times 10^{-6}$
$5{\cdot}130 \times 10^{-6}$	$5{\cdot}141 \times 10^{-6}$
$7{\cdot}734 \times 10^{-6}$	$7{\cdot}731 \times 10^{-6}$

The agreement between theory and observation is thus very close.

In laboratory experiments with a rotating mirror it is advisable to proceed as follows. The capacity C, of the order of 10,000 electrostatic units, consists of a number of large Leyden jars in parallel. The inductance is a coil of many turns of thick wire, L being of the order of 500,000 electromagnetic units at least. The rotating mirror is fixed on the axle of a small electric motor, and the image of S projected after reflexion on a ground-glass screen at a suitable distance away. A train of separate sparks

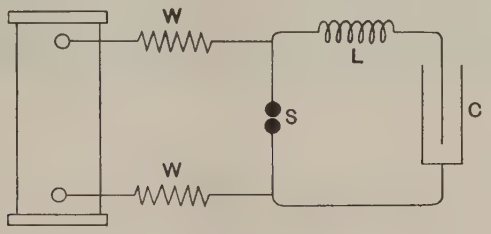


Fig. 247

is seen for each interruption of the hammer-break of the coil. That the coil has, under certain circumstances, a disturbing effect may be shown by short-circuiting the water-resistances W, W, when the sparks of the induction coil themselves occur in sets, which can be observed with a mirror rotating much less rapidly than before. The secondary of an induction coil, in fact, has a large self-inductance and something analogous to capacity, and therefore forms an oscillation circuit by itself.

Fig. 248 is a reproduction of a curve* representing the current obtained by charging up a condenser and then discharging it through a circuit without a spark-gap. The frequency here was 250 per second and the decrement δ about 0·12. The record was

* I am indebted to Prof. J. Zenneck for this figure and for Fig. 262.

not made with an ordinary oscillograph, but with a Braun tube in a manner to be explained later (Art. 206). Zenneck has also examined circuits containing spark-gaps, with remarkable results.

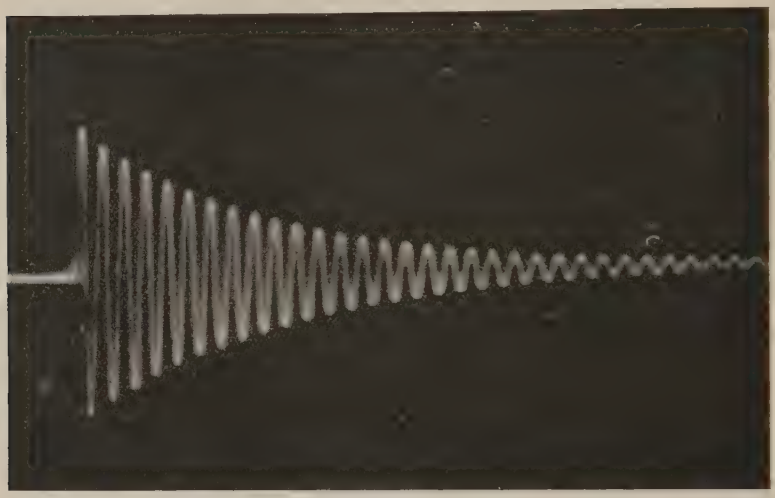


Fig. 248

Thus if the electrodes are silver or copper the amplitude of current sinks almost linearly, and not exponentially, to zero. This fact shows that the effect of the spark-gap in oscillation circuits cannot be entirely neglected.

In spite of the simplicity of its manipulation, the induction coil is not well suited for work with condenser discharges. In the first place it usually gives far more voltage than is required

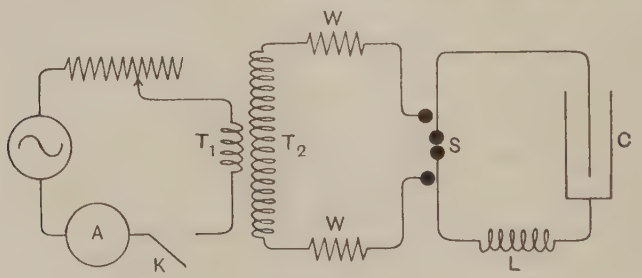


Fig. 249

to start the spark. The platinum contact-pieces of the hammer-break are burnt away more or less rapidly, giving rise to irregularity in the action of the coil, and requiring constant adjustment. For observations extending over any length of time the induction coil should be replaced by a transformer giving from 10,000 to 30,000 volts, in conjunction with a rotary converter. Fig. 249 shows a complete arrangement of this kind for producing electric oscillations. Water-resistances W, W are sometimes useful to prevent a large alternating current from passing through the spark-gap, and if the main spark-gap S is to be immersed in oil subsidiary spark-gaps in the leads are also required, otherwise the discharge cannot be made to begin sufficiently abruptly.

Fig. 250 shows a type of *variable inductance* for altering the

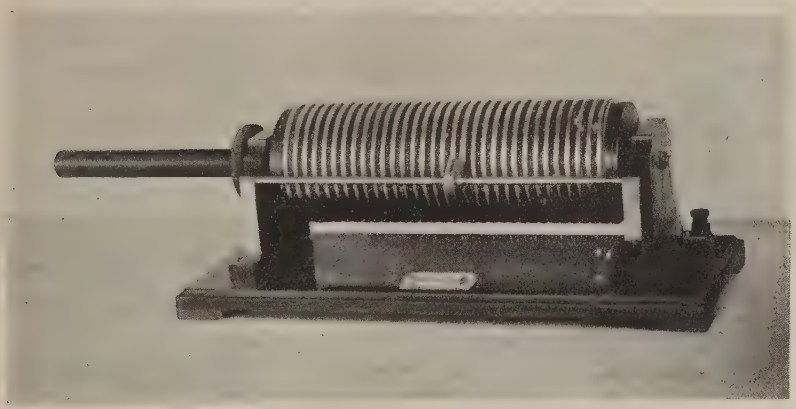


Fig. 250

frequency of the oscillation within certain limits. Here various lengths of a solenoid of thick wire, wound on ebonite, are taken off by a sliding contact in the form of a wheel rolling on the wire.

179. Instruments for detecting and measuring high-frequency currents. If a single train of oscillations in a circuit C_1 induces oscillations in a second circuit C_2, then it can be shown that the total charge $\int i_2 dt$ passing round the second circuit is zero. This is evidently true if C_2 contains a condenser:

if not, let i_1 be the current in C_1 at time t. Then with the usual notation

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2 i_2 = 0.$$

Integrating with respect to t over the whole time occupied by the oscillations, we have $\int i_2 dt = 0$. It follows that the oscillations cannot be detected by inserting a direct-current galvanometer, however sensitive, in C_2. Moreover, a telephone receiver in the circuit will give no sound; for the frequency of electric oscillations is too high to excite the vibrations of the membrane, and the telephone will respond only to the integral effect of the whole train, that is most probably to $\int i_2 dt$.

For measuring purposes the most generally useful instruments are the *thermal galvanometers* described in Art. 104. As they depend on the heating effect of the current, they give deflexions proportional to $N\int i_2^2 dt$, where N is the number of trains of oscillations per second. Duddell's thermo-ammeter is particularly convenient for laboratory work in which the greatest sensitiveness is not required.

The necessity for sensitive detectors* of high-frequency currents has arisen in connexion with the transmission of electric waves to a distance (Art. 199).

The simplest of the sensitive detectors now used is the *crystal detector*. It has been known for some time that certain crystals, such as psilomelane, exhibit unilateral conductivity when held between two metallic plates; i.e. for a given applied

* The first of these seems to have been Minchin's *impulsion cell* (1890). It consists of a small glass tube filled with an alcohol and containing two metallic plates connected by platinum wires to a quadrant electrometer. One of the plates is clean and the other sensitised by a peculiar process. A cell so prepared is sensitive to light, giving an E.M.F. of about $\frac{3}{4}$ volt for diffused daylight: but if it is tapped lightly it becomes insensitive, a second tap restoring the sensitive state, and so on. Minchin found that the insensitive state could be converted to the sensitive when the spark of a Hertzian oscillator passed at a distance of 30 feet or more from the cell. A very sensitive detector, the *coherer*, was subsequently invented by Branly. It consists of a mass of metallic filings contained in a glass or ebonite tube between metallic electrodes, which becomes conducting in the presence of electric waves. Branly, however, did not recognise as clearly as Minchin the fact that its action was due to the inductive effect of Hertzian oscillations. By the end of 1891 Minchin had succeeded in detecting the effect of a Hertzian oscillator at a distance of 150 feet.

voltage more current passes in one direction than the other. Fig. 251 shows the relation between voltage and current for a specimen of carborundum. Such a crystal will probably convert

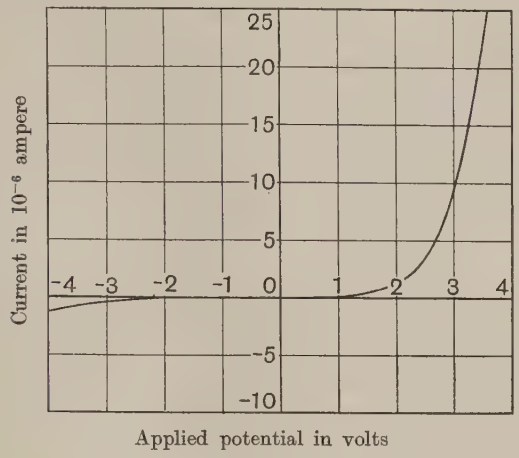


Fig. 251

high-frequency alternating currents more or less into direct currents, as it will let through more current one way than the other and give for $\int i dt$ a finite value different from zero. A single train of electric oscillations, inducing on a loop of wire containing a crystal and telephone in series, will therefore have an integral effect, and a sharp click will be heard in the telephone. In practice, the receiving circuit will usually contain a condenser, and the telephone and crystal are in series across it. The sensitiveness can generally be increased by applying a steady electromotive force to the crystal by means of a battery and potentiometer as shown in Fig. 252, the best voltage being generally

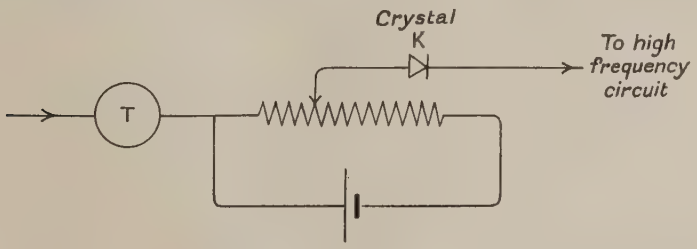


Fig. 252

of the order of half a volt. But this potentiometer attachment would not be used when simplicity of operation is the chief desideratum.

The crystal detectors in use are of three types:

(1) Crystal pressed between two flat metallic plates (carborundum).

(2) One of the metallic contacts in the form of a point, the pressure being adjustable by means of a screw (molybdenite, galena).

(3) Two crystals making contact with each other at a point, the pressure being likewise adjustable (zincite-copper pyrites, zincite-bornite).

These types do not differ in principle from one another, as contact takes place in all cases over a very small area. Some difference of opinion exists as to what crystals are the best to use. Thus carborundum has the advantage of being extremely hard, so that a slight change in the pressure to which it is subjected does not affect its sensitiveness. On the other hand ordinary specimens of it are not as sensitive as the zincite-copper pyrites combination.

The precise cause of the peculiar rectifying action of crystal detectors is not known; but some experiments of Pierce may be mentioned in this connexion. With carborundum he found that the difference between the positive and negative currents, for the same voltage, became much less when the faces of the crystal were platinised so as to increase the area of effective contact with the electrodes. The action therefore appears to have its seat at or near the points where the crystal touches the electrodes.

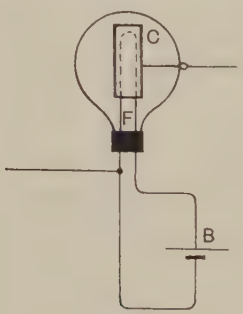


Fig. 253

Fleming's *oscillation valve* (Fig. 253) consists of a glow-lamp with a tungsten filament F, surrounded by a carbon cylinder C, the filament being heated with an auxiliary battery B. Its action depends on a peculiar property of incandescent metals in high vacua, namely that they will allow negative electricity to leak from them freely, while positive electricity can hardly escape at all. This is shown by the diagram, Fig. 254.

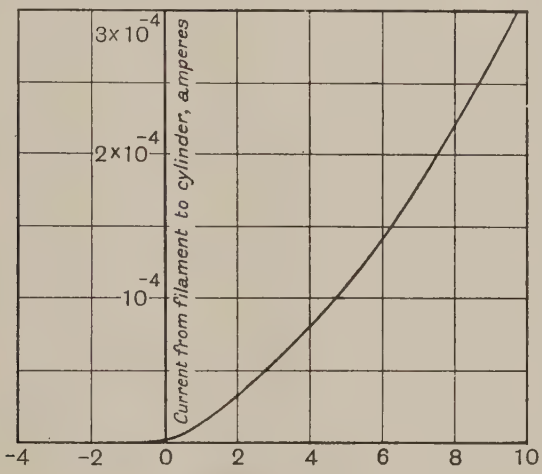


Fig. 254

Very considerable improvement results from the insertion of a grating between the filament and cylinder, the three being raised to different potentials. This device is due to de Forest. Such a contrivance is no longer a valve, any more than a crystal is; but it is easy to see that *any apparatus which does not obey Ohm's law may act as a rectifying detector*. Let us assume that the relation of current to potential is the same for high frequencies as for low frequencies or steady currents, and given by $V = f(i)$; and suppose that the apparatus in question is placed in a circuit of self-inductance L and resistance R, so that a small current i flows in it at time t. If this current is induced by a current j in a neighbouring oscillation circuit, we have

$$L\frac{di}{dt} + M\frac{dj}{dt} + Ri + V = 0.$$

Expanding the function $f(i)$ as far as i^2 by Maclaurin's theorem, we have

$$L\frac{di}{dt} + M\frac{dj}{dt} + \{R + f'(0)\}\, i + \tfrac{1}{2}f''(0)\, i^2 = 0,$$

since $V = 0$ when $i = 0$. On integration over the interval occupied by a single train of oscillations, this gives

$$\{R + f'(0)\} \int i dt = -\tfrac{1}{2}f''(0) \int i^2 dt,$$

so that $\int i dt$ certainly differs from zero if $f''(0)$ does; that is, if the (i, V) or characteristic curve is bent in the neighbourhood of the origin.

The above is a kind of general theory of crystal and valve detectors. It must however be remembered that no adequate proof has yet been adduced as to whether crystals behave in the same way at high and low frequencies.

180. Properties of high-frequency currents. The current in an oscillation circuit may reach a considerable magnitude. Thus if $C = 5000$ E.S.U., $V = 30{,}000$ volts and $n = 10^6$, $i_{\max} = 170$ amperes. As regards its inductive effect, however, this current is equivalent to an alternating current of no less than 10^6 amperes at a frequency of 25 per second. It follows that the induction effects are extraordinarily great. A glow-lamp in series with a loop of wire of one or two turns will glow brightly when brought near an oscillation circuit.

The most remarkable property of the currents is that they are not uniformly distributed over the cross-section of the wires in which they flow, but concentrated near the outer surface, and this more and more the higher their frequency. The exact theory of this effect is given in Art. 203: for the present we may regard it as an experimental fact. It can be demonstrated with the apparatus shown in Fig. 255. A represents a hollow metal

Fig. 255

cylinder about 5 cm. in diameter, B a coaxial metal rod, the two being placed in metallic contact by plates at the ends. Both

rod and cylinder are cut at one point, and a small discharge tube inserted across the inner gap is viewed through a small hole in the cylinder. Send a low-frequency current through the apparatus from an induction coil, and adjust the outer gap so that the current will just pass through the inner tube in preference to going through it. If now the apparatus is placed in an oscillation circuit, the converse will be found to occur. This illustrates the preference of the current for the outer portions of a wire.

The effect of this unequal distribution is to increase the apparent resistance of the wire. Imagine the cross-section of the wire to be cut into a large number of equal areas, and let i be the current through one of them. Then for a given total current Σi has an assigned value. But the heating effect of the current is proportional to Σi^2, and by a known algebraical theorem* this is least when all the i's are equal. Thus the heating effect of a given current is least when it is distributed uniformly over the cross-section of the wire.

The heating effect of high-frequency currents has been measured by several observers. The apparatus used by Fleming for this purpose is shown in Fig. 256. Two wires W_1, W_2, as nearly

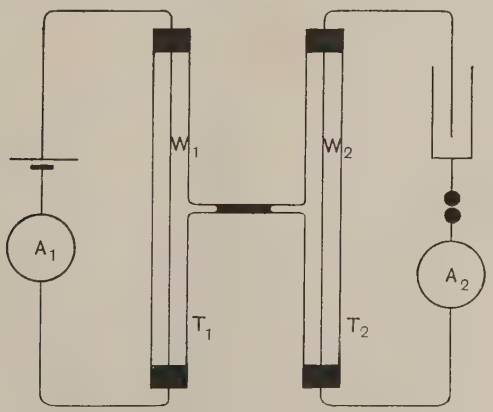


Fig. 256

as possible identical, are passed down two tubes T_1, T_2 connected by a cross-tube containing a thread of oil, so that the whole

* Cf. Chrystal, *Algebra*, vol. II. p. 49.

forms a differential air-thermometer. The left-hand wire is traversed by direct current and the right-hand wire by high-frequency currents from a condenser discharge, the effective values of the currents being measured by thermal ammeters A_1, A_2 in the circuits. The direct current is adjusted by means of a rheostat until the oil thread remains in its ordinary position when both currents are flowing. Let R_1 be the normal resistance of the wire W_1 for steady currents, kR_1 its resistance for the high-frequency currents. Then the heat developed in the left-hand tube is represented by $R_1 I_1^2$, and in the right-hand tube by $k R_2 I_2^2$. Thus

$$R_1 I_1^2 = \lambda k R_2 I_2^2,$$

where λ is a numerical constant which is equal to unity in a truly symmetrical apparatus, but in practice may have a slightly different value on account of differences in the sizes of the tubes and the thermal emissivities of the wires. λ may be eliminated by performing a second experiment in which high-frequency current flows through the left-hand wire and direct current through the right-hand wire. Thus

$$k R_1 I_3^2 = \lambda R_2 I_4^2,$$

giving

$$k = \frac{I_1 I_4}{I_2 I_3}.$$

The fractional increase of resistance can thus be found experimentally.

The change in the distribution of current in a circuit also affects its self-inductance; but whereas the resistance increases indefinitely as the current flows more and more on the surface, the inductance tends to a new limiting value, called the *high-frequency inductance*. The theory of Art. 121 must be modified in this case, since the internal filaments of the wire no longer contribute to the magnetic flux. Instead of the result $L = M + \frac{1}{2}l$ we now find the simple result $L = M$. Thus *the high-frequency inductance of a circuit is equal to the mutual inductance of the line of centres of the cross-sections and a parallel curve drawn in the surface of the wire.* As an example of this rule we have the following formulae for high-frequency inductance:

(1) *Long solenoid of n turns, length l, area A.*

$$L = 4\pi n^2 A/l,$$

the distribution of current being unimportant.

(2) *Rectangular circuit a by b, diameter of wire 2r.*

$$L = 9{\cdot}210 \left\{ c \log_{10} \frac{2ab}{r} - a \log_{10} (a+d) - b \log_{10} (b+d) \right\} - 8c + 8d,$$

where c is the half perimeter, d the diagonal of the rectangle.

(3) *Circular coil of one turn, radius a.*

$$L = a \left(28{\cdot}93 \log_{10} \frac{8a}{r} - 25{\cdot}13 \right).$$

The preceding remarks show that we must be rather careful in speaking of the resistance and self-inductance of a circuit for high-frequency currents. Serious practical difficulties do not arise, since the resistance of oscillation circuits can be calculated with sufficient accuracy for the particular frequency in question. The difference between the high and low-frequency inductance of a circuit amounts to exactly half the total length of wire used, and is usually insignificant.

181. Tesla experiments. The idea of *transforming* electric oscillations is due to Tesla. His apparatus, called a Tesla trans-

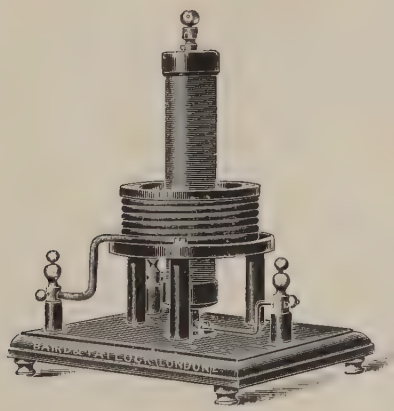


Fig. 257

former, is shown in Fig. 257. The primary consists of a small number of turns of thick wire inserted in the oscillation circuit;

the secondary coil is a solenoid of many turns carefully insulated from one another*. In this way an extremely high voltage (of the order 10^6) is reached, and a variety of striking experiments may be made.

If the ends of the secondary are connected to two unequal metallic rings (diameters 15 and 10 cm., about 10 cm. apart) a beautiful conical discharge takes place between them, the whole space being traversed by filmy blue streamers. Leads of bare and even insulated wire are likewise surrounded by a luminous halo. The secondary terminals may be joined to a pair of wires 10 metres or so long stretched parallel to one another about 10 cm. apart, when a filmy discharge will take place along their entire length. Long crackling sparks are obtained from a pointed terminal attached to the top of the secondary. Glow-lamps and discharge tubes light up at a considerable distance from the transformer. If a discharge tube is held in one hand while the secondary is touched with the other it will light up, and with a powerful apparatus the currents may be made to pass through the bodies of several persons. In performing these experiments it must be remembered that it would be dangerous to touch the *primary* of the Tesla transformer (or any other oscillation circuit).

182. Electrical resonance. When an electromotive force $V \cos pt$ is placed in a circuit containing resistance, inductance and capacity in series, the current is given (Art. 130) by

$$i = I \cos (pt - \alpha),$$

where
$$I = \frac{V}{\left\{\left(Lp - \frac{1}{Cp}\right)^2 + R^2\right\}^{\frac{1}{2}}}, \quad \tan \alpha = \frac{Lp - \frac{1}{Cp}}{R} \quad \ldots(12).$$

Regarding p as the variable, I assumes its maximum value V/R when $LCp^2 = 1$; that is, when the periodic time of the applied E.M.F. is the same as that of the natural electric oscillations in the circuit. If R is small this maximum current may be much greater than that corresponding to slightly larger or smaller

* For really powerful effects the secondary should be immersed in oil.

values of p. This effect is called *resonance* by analogy with the corresponding effect in acoustics. The phase-difference between current and electromotive force vanishes when resonance occurs, and the circuit then behaves exactly like a simple coil of resistance R and inductance zero.

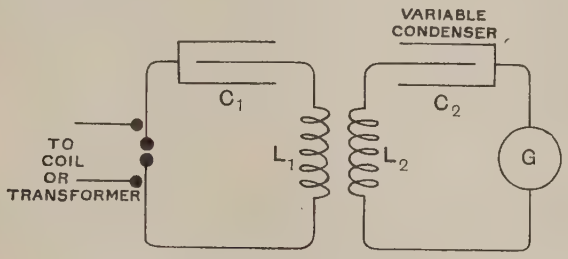


Fig. 258

Fig. 258 shows an arrangement for examining the resonance of two electric oscillation circuits. L_1, L_2 are two small inductances formed by stretching insulated wire over wooden frames. Three turns of wire of 1 mm. diameter, placed about 2 cm. apart on a rectangular frame 60 cm. by 30 cm., will give a suitable inductance (of the order 10,000 absolute electromagnetic units). C_1 is a Leyden jar, C_2 a variable condenser of somewhat larger capacity, G a thermo-ammeter or other sensitive thermal galvanometer. Care should be taken to start with the frames a considerable distance apart, otherwise the current in the secondary may be too large. On turning the handle of the variable condenser a large deflexion of the galvanometer will occur at a well-marked position.

A curve connecting the capacity of the secondary with the current may be called a *resonance curve*. This name is however usually reserved for a particular curve obtained as follows. Imagine the periodic time $T = 2\pi (LC)^{\frac{1}{2}}$ of the natural oscillations of the secondary circuit calculated for all values of its capacity. Then at a particular periodic time T_0 the current in the secondary will have its maximum effective value I_0. If I is the effective value of the current for any other period T, the resonance curve is one with T/T_0 as abscissa and I/I_0 as ordinate. An

example is given in curve I of Fig. 259 ($L_1 = L_2 = 9800$ E.M.U., $C_1 = 1250$ E.S.U., resistance of secondary 1·5 ohms).

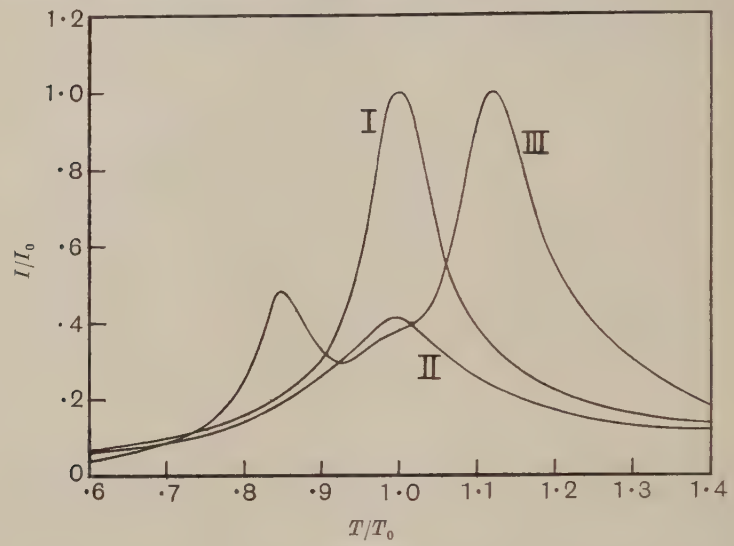


Fig. 259

The exact theory of resonance curves, and the way they depend on the decrements of the circuits, will be considered in Art. 203. The general form of the curves is fairly obvious from what has been said above. Resonance occurs when the natural frequency of the secondary coincides with that of the primary, that is when $L_1C_1 = L_2C_2$. Neglecting the damping of the primary, the resonance curve will have a more and more pronounced peak the less the resistance of the secondary. It will be shown that the sharpness of the peak depends essentially on the *sum* of the decrements of the primary and secondary. Curve II of Fig. 259 shows the effect of inserting a non-inductive resistance of 9 ohms in the secondary, the scale of current being the same as that in curve I, from which it is clear that the sharpness of resonance is considerably reduced.

183. Frequency-meters for electric oscillations. If the self-inductance of the secondary circuit in Fig. 258 is known, and also the capacity for various positions of the variable condenser, we have clearly an instrument for determining the frequency

of the oscillations in the primary. The condenser is adjusted, with the circuits still a considerable distance apart, till the thermal galvanometer shows the maximum deflexion, and then $T = 2\pi (L_2 C_2)^{\frac{1}{2}}$ is the periodic time of the primary oscillations. An instrument of this type was first put on the market by Dönitz. The most accurate method of measurement is to draw the whole resonance curve and determine the position of maximum current graphically, otherwise it is rather difficult to find this position on account of the slight unsteadiness of the oscillations which always occurs in practice.

A very convenient instrument is the portable frequency-meter of the Marconi Company, the internal arrangement of which is shown in Fig. 260. Here the detector is a carborundum crystal K of comparatively high resistance in series with a telephone T, the whole placed as a shunt across the variable condenser C. The inductance L is built into the lid of the containing box. In use, the meter is first held in the hand and the handle of the condenser turned until there is a maximum sound in the telephone. The observer now moves away until he can only just hear the sound when the handle is in a particular position. In this way it is possible to determine the position of resonance with great nicety.

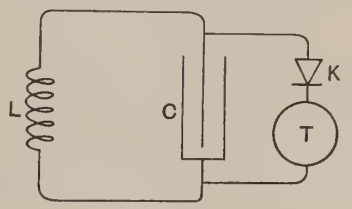


Fig. 260

An ingenious direct-reading frequency-meter has been invented by Hirsch. One plate of the variable condenser is kept in continuous rotation by a motor. Attached to the edge of this plate is a small vacuum tube containing helium, which is also connected through a sliding contact to the fixed plates. When the condenser passes through the position of resonance the difference of potential between the plates is a maximum, and the discharge tube can be made to light up in this position, and this only. A graduated scale under the discharge tube then gives the frequency by inspection.

For absolute determinations of frequency it is necessary to standardise the instrument in some way. The method of calibration adopted at the Reichsanstalt, Berlin, is to compare the

readings of a given meter with one or more standard meters, the inductance and capacity of which are determined once for all by calculation. A careful series of measurements was carried out by Diesselhorst in 1908 in which the accuracy of these standards was checked by observations with a rotating mirror, so that the absolute readings are known to be correct to a few parts in a thousand.

It will appear shortly that electromagnetic effects are propagated through space with a finite velocity c, equal to the ratio of the electromagnetic to the electrostatic unit of charge. Thus electric *oscillations* are accompanied by electric *waves*, the wave-length λ being connected with the period of oscillation by the relation $\lambda = cT$. For this reason frequency-meters are usually described as wave-meters, because they also measure the wave-length of the corresponding electric waves.

In Kelvin's formula $T = 2\pi (LC)^{\frac{1}{2}}$ both L and C are supposed to be measured in the same system of units, for example the electromagnetic. But now if C is measured in electrostatic units, L remaining in absolute electromagnetic units, the formula becomes $T = 2\pi (LC/c^2)^{\frac{1}{2}}$. With this understanding we therefore have the simple result

$$\lambda = 2\pi (LC)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

giving λ directly in centimetres. When not specially stated we shall always understand L and C to be measured as here described.

On account of the ease with which observations can be made the wave-meter becomes the most convenient instrument for measuring small inductances and capacities. If a standard inductance is at hand, or one whose magnitude can be calculated from its dimensions, then the capacity of any Leyden jar is found by making up an oscillation circuit of the two (taking care not to introduce more inductance than necessary in the leads) and measuring the corresponding wave-length. The method is applicable to cases in which the leakage is too great to permit of electrostatic measurements. Again, with a known condenser self-inductances can be measured; and this is the most accurate method of measuring small inductances. Knowing the self-inductances of two coils, their mutual inductance may be found by a method already explained (Art. 132).

184. Coupled oscillation circuits. When the two circuits in Fig. 258 are brought so close together that their mutual inductance is appreciable, the oscillations in the circuits disturb each other. In order to examine this more particularly in the case in which the resistances in the circuits are negligible, suppose that $\pm e_1$ are the charges on the plates of C_1 at time t, $i_1 = -de_1/dt$ the primary current, and e_2, i_2 the corresponding quantities for the second circuit. Then we have

$$\left.\begin{aligned} L_1C_1\frac{d^2e_1}{dt^2} + MC_1\frac{d^2e_2}{dt^2} + e_1 = 0 \\ MC_2\frac{d^2e_1}{dt^2} + L_2C_2\frac{d^2e_2}{dt^2} + e_2 = 0 \end{aligned}\right\}.$$

Write $n_1^2 = 1/L_1C_1$, $n_2^2 = 1/L_2C_2$, and try a solution $e_1 = A_1 \cos pt$, $e_2 = A_2 \cos pt$. Substituting in the above equations, we have

$$(n_1^2 - p^2)\,A_1 = \frac{Mp^2}{L_1}A_2 \quad \text{and} \quad \frac{Mp^2}{L_2}A_1 = (n_2^2 - p^2)\,A_2.$$

Eliminating the ratio $A_1 : A_2$, the admissible values of p are given by

$$(n_1^2 - p^2)\,(n_2^2 - p^2) = k^2p^4 \quad \ldots\ldots\ldots(14),$$

where

$$k = \frac{M}{\sqrt{L_1L_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

The quantity k is called the *coefficient of coupling*: the circuits are said to be weakly coupled when k is small, strongly coupled when k approaches unity. Equation (14) shows that there are two possible oscillation frequencies in a pair of coupled circuits, which in general are not the same as the natural frequencies of the two circuits by themselves.

In the special case of resonance ($n_1 = n_2 = n$), (14) gives $n^2 - p^2 = \pm kp^2$. Hence if T_0 is the common periodic time of the circuits separately, the combined periods are given by

$$T = T_0\sqrt{1 \pm k} \quad \ldots\ldots\ldots\ldots\ldots(16).$$

The actual currents in the circuits are obtained by superposing solutions of the above fundamental type. The general solution of the differential equations is now

$$e_1 = A\cos pt + B\sin pt + C\cos qt + D\sin qt,$$

$$e_2 = \frac{M}{L_2}\left\{\frac{p^2}{n^2-p^2}(A\cos pt + B\sin pt) + \frac{q^2}{n^2-q^2}(C\cos qt + D\sin qt)\right\},$$

where $$p^2 = \frac{n^2}{1+k}, \quad q^2 = \frac{n^2}{1-k}.$$

When $t = 0$ we have $e_1 = C_1V_0$, $i_1 = 0$, $e_2 = 0$, $i_2 = 0$. These conditions determine the arbitrary constants A, B, C, D, and thus we find

$$e_1 = \tfrac{1}{2}C_1V_0(\cos pt + \cos qt), \quad e_2 = \tfrac{1}{2}\left(\frac{L_1}{L_2}\right)^{\frac{1}{2}}C_1V_0(\cos pt - \cos qt).$$

Hence $$i_1 = \tfrac{1}{2}C_1V_0(p\sin pt + q\sin qt),$$

$$i_2 = \tfrac{1}{2}\left(\frac{L_1}{L_2}\right)^{\frac{1}{2}}C_1V_0(p\sin pt - q\sin qt).$$

The frequencies p and q, if not too close, may be detected separately with a wave-meter, and appear in the form of two humps when a resonance curve is taken of the two circuits together by means of a third circuit loosely coupled with them. An example is given in curve III of Fig. 259, from the results of an experiment in which $L_1 = 9800$, $L_2 = 18{,}400$, $M = 3600$, $C_1 = 1250$, $C_2 = 665$, $k = 0{\cdot}268$. The positions of the maxima, as calculated from (16), are given by $T/T_0 = 0{\cdot}856$ and $1{\cdot}13$, which agree well with those observed.

As in acoustics, the existence of two frequencies gives rise to

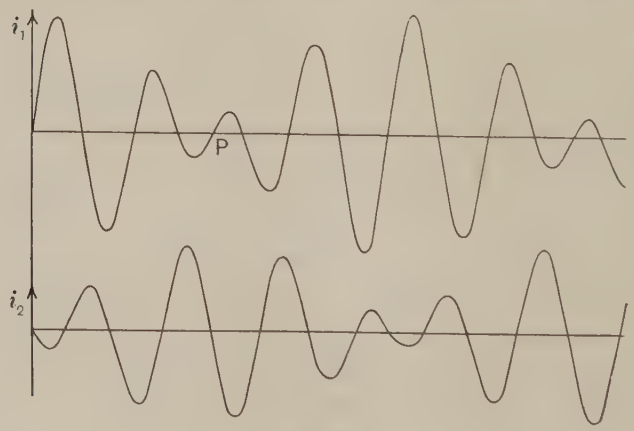


Fig. 261

beats, or periodic variations of the amplitude of the oscillations. The curves in Fig. 261 correspond exactly to curve III of Fig. 259,

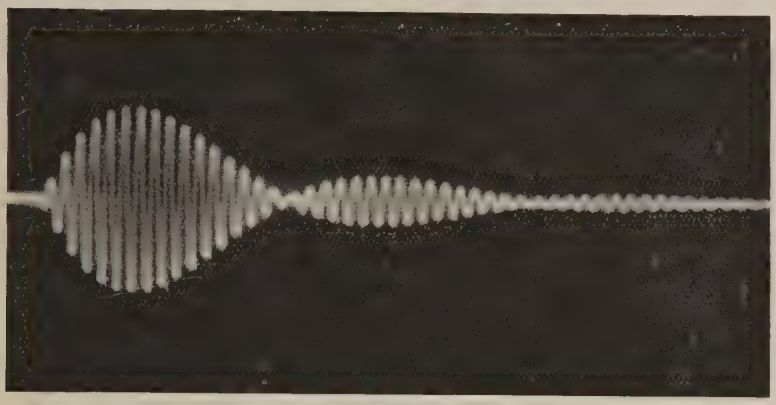


Fig. 262

except that in calculating them the damping has been neglected. Fig. 262 is an actual oscillograph record of the secondary current in a pair of coupled circuits.

An important practical consequence of the above is that a wave-meter cannot be brought too near to an oscillation circuit without reacting on it and altering its frequency. A sensitive detector is therefore essential for accuracy.

185. Damping in oscillation circuits. It is clear that electric oscillations will be damped out by any process involving a dissipation of energy. The formula (11), which takes into account the heating of the wires of the circuit alone (and that only if R is the resistance calculated for the corresponding frequency), therefore gives an inferior limit to the damping that takes place in practice. Other sources of energy-loss can be much reduced by suitable design of the oscillation circuit, the spark-gap excepted.

The effect of the spark-gap on the course of the oscillations has already been mentioned. Magnesium is the best material for the electrodes if small damping is required, but zinc and cadmium are also to be recommended. The spark-gap, again, must not be too short, say at least 2 mm. in length. If these

precautions are taken δ may be got as low as 0·05, but not much lower.

The greater damping effect of short sparks was observed by Perot in 1892, but it has only recently been realised how great this damping may become. The following instructive method has been used by M. Wien. A coupled circuit is set up with the spark-gap to be examined in the primary, and a resonance curve

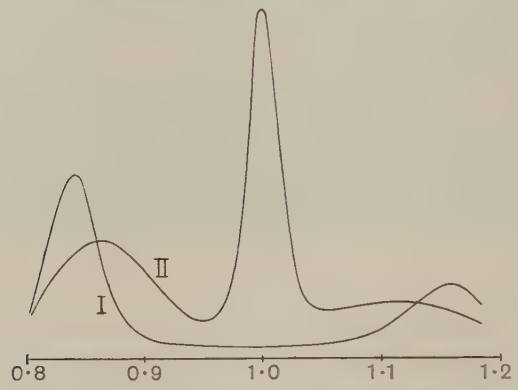


Fig. 263

taken off by means of a third circuit loosely coupled with the first two. Under ordinary circumstances a curve is obtained like that of Fig. 263, I, which corresponds to spark-length $\frac{1}{2}$ mm. If now the spark-length is diminished the humps of the coupled circuit subside and a third hump rises which corresponds to the natural frequency of oscillation of the separate circuits, supposed equal (curve II, spark-length 0·15 mm.). The beginning of this process is already apparent in curve III of Fig. 259.

The obvious interpretation of these results is that with short spark-gaps the primary oscillation may cease altogether, leaving the secondary circuit to oscillate alone with its own natural frequency. Since the secondary contains no spark-gap its damping is small, as may be seen also from the sharpness of the central peak in curve II.

Curiously enough, therefore, the problem of producing weak damping in one circuit depends on the production of very strong

damping in another. This is known as the method of the *quenched spark*. An effective type of quenched spark is the plate discharger, consisting of two flat plates separated from one another by a thin ring of mica. Since, however, the sparking potential, and therefore the amount of energy available, is small with a single short spark-gap, several plates are placed in series. The plates may be strung on an insulating rod and separated from one another by thin discs of mica, as shown in Fig. 264. The nature of the intervening gas has an effect on the amount of quenching, hydrogen being better than air and carbon dioxide worse.

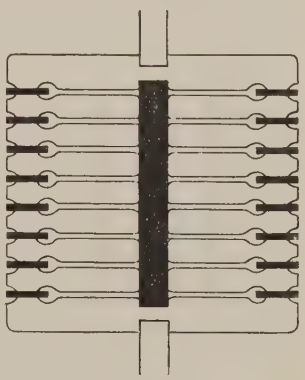


Fig. 264

The amount of coupling between the two circuits is also of importance, as will be understood by a reference to Fig. 261. Every spark-gap will of course become non-conducting if the current is allowed to become very small over a sufficient length of time. At a point such as P the amplitude of current is small over one or two oscillations; but whereas an ordinary spark-gap would get over this awkward period, a properly designed quenched spark will become non-conducting and the secondary will henceforth oscillate alone. The chance of this happening is clearly greatest when the beats are long; that is, when the frequencies of the coupled oscillations are nearly equal. It follows that if the coefficient of coupling between the two circuits exceeds a certain limit the spark-gap will cease to act effectively.

186. The singing arc. The problem of exciting *continuous* oscillations in condenser circuits, as distinct from trains of damped oscillations, was solved by Duddell in 1900. If a condenser C, of the order of 5 microfarads, and a self-inductance L, of the order of $\frac{1}{50}$ henry, are placed as a shunt across a carbon arc (Fig. 265), slow continuous oscillations are produced, which are accompanied by a musical note from the arc. The carbons should be thin and uncored, and the resistance R adjusted for a

current of 2 to 3 amperes at 100 volts, though a higher voltage is required for really good effects. In actual working the carbons are first brought together till the arc is struck, and then separated

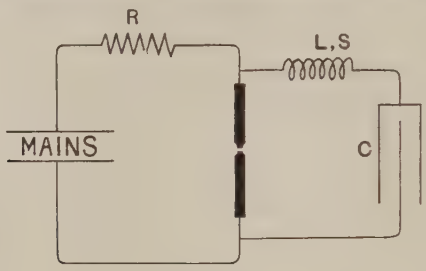


Fig. 265

slowly until singing begins, which occurs just before the separation at which the arc goes out.

The oscillations are detected in the usual way with a secondary circuit containing self-inductance, capacity and a thermal detector; and resonance curves can be obtained showing that the primary frequency is given approximately by Kelvin's formula. The difference of potential between the plates of the condenser may rise to four or five times the potential of the mains, as may easily be seen by placing an electroscope or electrostatic voltmeter across its terminals.

Many striking experiments may be made with the singing arc. A piece of iron placed near the coil L raises the self-inductance and lowers the note emitted by the arc. The note may be altered continuously by making up the self-inductance of two coils in series with one another and moving one of them about so as to change the mutual inductance. It is raised when a secondary circuit in the vicinity is short-circuited, on account of the diminution of the effective self-inductance of the primary (cf. Art. 134). The oscillations may be detected across a room by means of a receiving circuit with a telephone shunted across the condenser. When the capacity of this condenser is changed the intensity of the sound in the telephone changes also, but not the pitch: this illustrates the important principle of forced vibrations, namely that it is the frequency of the forcing agency which is established, and not that of the system acted on.

A complete theory of the singing arc is out of the question, but the following considerations show the possibility of continuous oscillations in the shunt circuit of a direct-current arc. The essential feature of the arc is that as the current in it increases the difference of potential between the carbons becomes less. Assume for simplicity that the current i through the arc is a definite function $f(V)$ of the difference of potential V between the carbons. Then if j is the current in the condenser circuit and V_0 the voltage of the mains, we have

$$L\frac{d^2j}{dt^2}+S\frac{dj}{dt}+\frac{j}{C}=\frac{dV}{dt},$$

$$V_0=V+R(i+j).$$

The last equation gives

$$j=\frac{V_0-V-Rf(V)}{R}=\psi(V), \text{ say.}$$

Again $$2L\frac{dj}{dt}\frac{d^2j}{dt^2}+2S\left(\frac{dj}{dt}\right)^2+\frac{2}{C}j\frac{dj}{dt}=2\frac{dj}{dt}\frac{dV}{dt},$$

which gives on integration

$$L\left(\frac{dj}{dt}\right)^2+2S\int\left(\frac{dj}{dt}\right)^2dt+\frac{j^2}{C}=\text{const.}+2\int\psi'(V)\left(\frac{dV}{dt}\right)^2dt.$$

If j is to execute periodic changes $(dj/dt)^2$ and j^2 resume their original values periodically. The left-hand side thus becomes effectively $2S\int(dj/dt)^2\,dt$, which increases continually as time goes on. The integral on the right must therefore increase with t. Hence periodic changes cannot take place if $\psi'(V)$ is negative, that is if $Rf'(V)+1>0$, so that a necessary condition for the production of oscillations is that somewhere or other

$$\frac{di}{dV}+\frac{1}{R}<0.$$

This condition cannot be satisfied by an ordinary resistance, or even by a glow-lamp, for which di/dV is always positive. In the arc between solid carbons, however, di/dV is negative.

The above theory is nothing more than an illustration, since the behaviour of the arc for frequencies as low as 50 to the second is very different from its behaviour for steady currents. Interesting

results have been obtained by Blondel by means of the oscillograph. One of his curves is shown in Fig. 266, representing the most

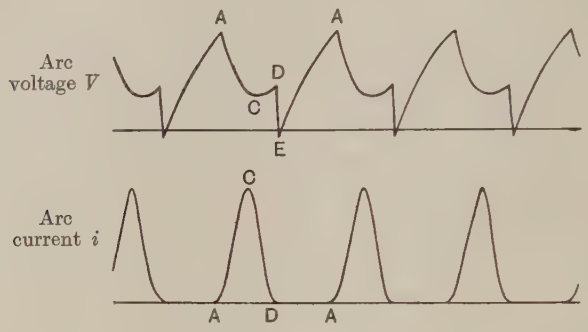


Fig. 266

common case. By taking corresponding values of i and V and plotting them against one another we obtain the "characteristic" curve (Fig. 267). The portion ACD in the curves corresponds to the interval during which the arc is letting current through, DEA to the interval of extinction. At C the arc is fairly hot; but as the current diminishes it cools slightly, so that the potential at D is slightly higher than at C. At D, when the arc ceases to conduct, the state of the rest of the circuit is such that the potential falls to a negative value (at E) and then rises rapidly. Though the arc is cooling all this time it is still hot enough to re-light at A for a certain potential higher than that at D. The cycle

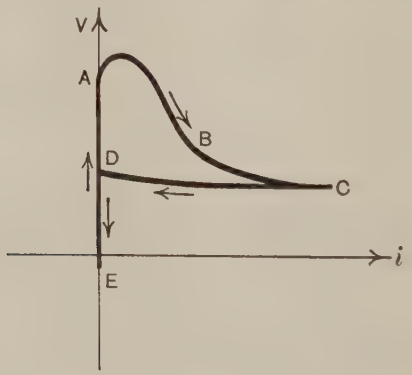


Fig. 267

is then completed along a curve ABC lying uniformly above the curve CD.

187. Hertz's experiments. In all our previous work we have postulated a direct influence of one object A on another B across the intervening space (action at a distance). No objection can be taken to this point of view as long as the phenomena are correctly explained. However, there is no denying that it would be more satisfying if the action was traced all the way from A to B by postulating suitable properties in the separating medium. In 1888 Hertz published a remarkable series of experiments confirming the theory, propounded by Maxwell in 1863, that electric action is propagated in the form of waves having all the properties of light-waves.

Hertz's earlier experiments were performed with the apparatus shown in Fig. 268. Two square brass plates A, A', each of side

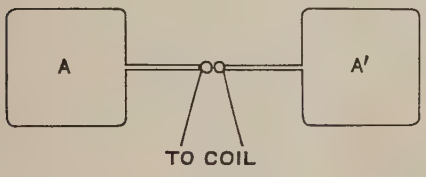


Fig. 268

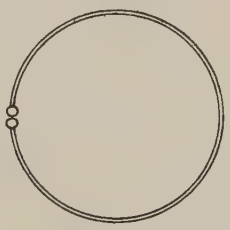


Fig. 269

40 cm., were joined by a copper wire 60 cm. long. The wire was cut in the middle and a spark-gap inserted, the terminals of which were connected to a large induction coil. Although the apparatus does not form anything like a closed circuit, yet the plates A, A' have capacity and the connecting wire something like inductance, so that the discharge is oscillatory and of very high frequency. For a receiver (Fig. 269) Hertz used a wire in the form of a circle of radius 35 cm., likewise interrupted by a spark-gap. When the oscillations fall on the receiver sparks are seen to pass across the gap, whose length can be adjusted by means of a micrometer screw.

The first observations were on reflexion, or rather on interference between the direct beam of rays and the beam reflected

from a conducting surface. Working in a large lecture room, Hertz set up his oscillator in a horizontal position at a distance of 13 metres from a massive wall to which a sheet of zinc 4 metres high and 2 metres wide had been attached. Periodic variations were found in the intensity of sparking when the receiver was placed horizontally at increasing distances in front of the zinc sheet. This was attributed to the formation of standing waves, the wave-length of which was estimated at 9·6 metres.

Subsequently Hertz worked with much shorter waves (65 centimetres), using the apparatus shown in Fig. 270. The oscillator was now a brass cylinder 3 cm. in diameter and 26 cm.

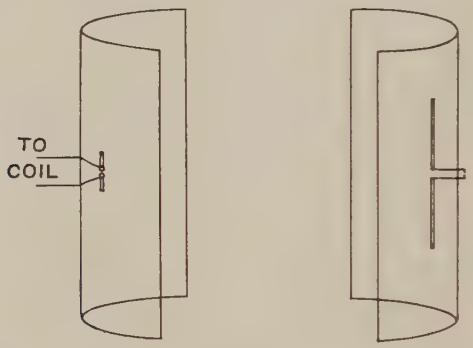


Fig. 270

long, with spark-balls in the centre. It was placed vertically in the focal line of a parabolic mirror of height 2 metres and focal length 12·5 cm., the wires from the spark-gap to the induction coil being led through the mirror. The receiver was formed of two straight pieces of wire, each 50 cm. long and 5 mm. in diameter, placed with their ends 5 cm. apart in the focal line of a second parabolic mirror. From the nearer ends two wires were led through the mirror to a small adjustable spark-gap at the back, in which the passage of sparks was observed.

In judging these experiments it should be borne in mind that the linear receiver is affected most powerfully when it is parallel to the lines of electric force, while the previous circular receiver responds best when its plane is perpendicular to the lines of magnetic force. The greatest distance at which the short waves

could be detected was about 20 metres, but experiments were usually made at a distance of 6 to 10 metres.

(1) *Rectilinear propagation.* A sheet of zinc 2 metres high and 1 metre wide stopped all sparking when placed in the path between the mirrors, but did not when placed on one side. The body of the observer also had the same effect. Insulators such as wood did not stop the rays.

(2) *Reflexion.* When the mirrors are placed at such an angle that no direct action takes place, vigorous sparks could be obtained by placing a sheet of zinc so that the normal to its plane bisected the angle between the axes of the mirrors. The effect was not very sharply marked, sparks being obtainable when the mirror was turned through 15° in either direction. But since the dimensions of the apparatus were of the same order of magnitude as the wave-length diffraction effects must come into play.

(3) *Refraction.* Hertz used a large prism of pitch. The height was 1·5 metres, the cross-section an isosceles triangle of base 60 centimetres and vertical angle 30°. Oscillator, prism and receiver were arranged like the collimator, prism and telescope of a spectrometer, the distance from oscillator to prism and from prism to receiver being 2·6 metres. With an angle of incidence of 25°, sparks began to appear in the receiver at the deviation 11°, became strongest at 22° and vanished again at 34°. The angle of minimum deviation was estimated at about 22°, corresponding to the refractive index 1·69.

(4) *Polarisation.* A large grating was made of copper wires fixed parallel to one another at a distance of 3 centimetres apart. If the parabolic mirrors were vertical and the wires horizontal, practically no effect was produced by interposing the grating between the mirrors: but if the wires were vertical all traces of sparking ceased. This effect is analogous to that of a plate of tourmaline on plane-polarised light, the plate absorbing rays polarised in one direction much more than those polarised at right angles. It is however interesting to consider it from a purely electrical point of view, in order to gain insight into the way in which this comes about. It will appear shortly that the electric force in the wave as it falls on the grating is parallel to the direction of the primary oscillator. Such a wave

induces no currents in the wires when they are perpendicular to the electric force: but if they are parallel to it currents and free charges appear, and their electric and magnetic effects are superposed on those of the original wave. The effects of the separate wires reinforce each other in certain directions (reflexion) and destroy each other in others (absorption). Hertz found, in fact, that the grating would *reflect* perfectly when the wires were vertical, but not when they were horizontal.

Hertz found that no sparks are produced if the oscillator and receiver are at right angles to one another, but they reappear when the grating is interposed in a direction which is not parallel to one of them. Let the lines of the grating make an angle θ with the oscillator. The charges induced in the wires are proportional to the component $E\cos\theta$ of electric force parallel to them, and their electric field makes an angle $\frac{1}{2}\pi - \theta$ with the receiver. Hence the electrical effect in the receiver is roughly proportional to $E\sin\theta\cos\theta$, which has a maximum value when $\theta = \frac{1}{4}\pi$.

188. Experiments of Bjerknes, and of Sarasin and de la Rive. We must now interrupt the main argument in order to describe two investigations which throw light on special points. Bjerknes examined the resonance and damping under various conditions. A set of circular receivers was constructed of various metals, but of geometrically identical form, and resonance curves taken off by altering the dimensions of the oscillator. (The quantity observed was potential, the spark-gap in the receiver being replaced by a small electrometer of special design.) The sharpness of resonance was greatest with the copper receiver, and then followed brass, German silver, platinum, nickel, and iron in this order. The order of the non-magnetic metals is that of their conductivities, but iron and nickel fall out of the series and give proportionally greater damping, which shows that the magnetic properties come into play even for the frequencies of Hertzian oscillations.

Choosing an oscillator of the same natural frequency as his receivers ($7{\cdot}1 \times 10^7$ per second), Bjerknes next tried the effect of depositing thin layers of various metals on the surface of the

receivers. The results are shown in Fig. 271, in which the abscissa represents the thickness of the added layer. Thus an iron wire covered with a layer of copper one-hundredth of a millimetre thick behaves like a solid copper wire*; from a certain thickness onwards the nature of the underlying metal is in all

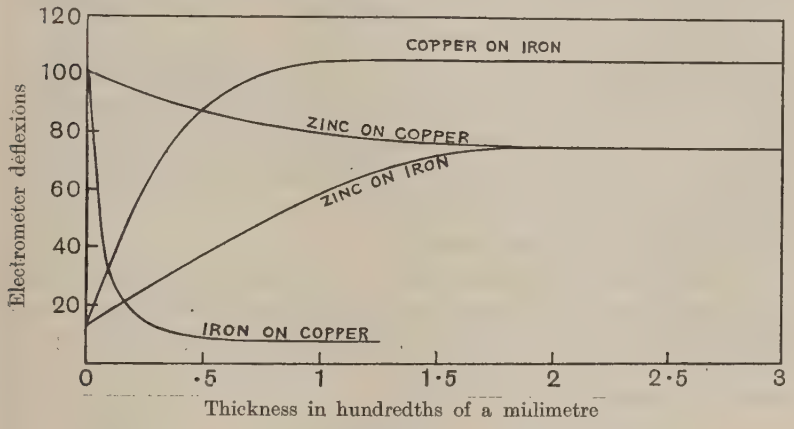


Fig. 271

cases immaterial, the same effect being obtained with zinc on iron and with zinc on copper. Nothing could show more clearly the fact that high-frequency currents flow only on the outsides of wires.

Bjerknes found that the decrement of his oscillator was 0·26, a much higher value than that occurring with ordinary oscillation circuits. Some effect of this kind is to be expected if, as we believe, the Hertzian oscillator is continually emitting energy in the form of waves, so that its oscillations are more quickly damped out. This process is known as *damping by radiation.*

The circular receiver, on the other hand, approximates more to the ordinary re-entrant oscillation circuit. It does not radiate and its decrement is considerably smaller than that of the oscillator. This is brought out most clearly by the experiments of Sarasin and de la Rive. These physicists, in repeating Hertz's experiments on the interference between the direct rays and those

* The slight difference is probably due to the high conductivity of electrolytically deposited copper; while Bjerknes' iron deposit seems to have been of low conductivity.

reflected from a metal sheet, found that *the wave-length obtained depended only on the dimensions of the receiver, and not at all on the oscillator.* When a heavily-damped wave falls on a receiver it sets it into electrical vibration which continues long after the wave has passed over; and on the return of the wave similar free vibrations are set up. The two sets of vibrations will interfere destructively if the time τ required for the wave to move from the receiver to the mirror and back is an odd multiple of $\frac{1}{2}T$, where T is the natural period of vibration of the receiver. Thus the distance l of the first node from the mirror is given by $l = \frac{1}{4}cT$, c being the velocity of propagation of the waves. In this equation there is no quantity characteristic of the oscillator, which explains the observed facts.

Sarasin and de la Rive also examined the formation of stationary electric waves on wires, and found the same wave-length in air as on the wire, as shown in the following table:

Diameter of receiver (cm.)	100	75	50	35 (thick wire)	35 (thin wire)	25	20 (thick wire)	20 (thin wire)	10
λ in air (cm.)	812	564	444	304	320	240	172	204	76
λ on wire (cm.) ..	768	592	392	292	—	224	—	180	—

It thus appears that the wave-length to which a circular receiver responds is nearly eight times its diameter. Experiments on waves along wires had previously been made by Hertz, but the results were less decisive on account of the presence of certain disturbing causes.

189. Laboratory experiments on electric waves. The great size of Hertz's apparatus, and the consequent disturbing effect of the floor and walls of the room, render it unsuitable for regular laboratory use. Of the various arrangements that have been suggested for repeating Hertz's experiments on a smaller scale, that of Lebedew is by far the most reliable. The following simple apparatus, suitable for qualitative experiments, is on this

plan. The oscillator (Fig. 272) is about 7 cm. long, the actual spark-balls consisting of two small platinum knobs formed by fusing the end of a platinum wire in an electric arc. The oscillator is partly immersed in turpentine, in which the main sparks pass,

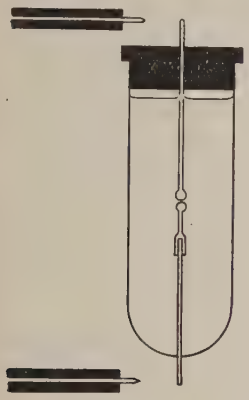


Fig. 272

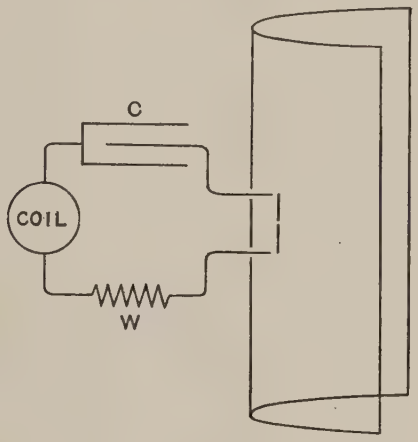


Fig. 273

and the length of the spark-gap is adjusted for maximum effect by means of a screw, the best length being only a fraction of a millimetre. In order to avoid as far as possible the disturbing effects of the coil, a water-resistance W and a Leyden jar C are placed in the secondary circuit (Fig. 273), and the coil is separated from the oscillator by two subsidiary gaps about 1 cm. in length. The parabolic mirror, 40 cm. high and of focal length 4 cm., is of tin plate, perforated with holes to take the leads to the subsidiary spark-gaps, sheathed in ebonite tubes.

The receiver, which is placed in the focal line of a second parabolic mirror, is a copper wire about 15 cm. long interrupted at the centre by a small thermal detector of the type invented by Klemenčič (Art. 104). It is important that the detector should be evacuated and used with a very sensitive galvanometer.

For convenience of manipulation the receiver is mounted on a stand moveable like the arm of a spectrometer, and a prism table is also placed over the central pivot. The whole apparatus will be found very certain in action when first set up; but after

a time the small spark-balls become roughened and require to be remoulded in the electric arc*. The wave-length emitted is of the order of 15 centimetres, and the size of the subsidiary apparatus can be much reduced. Thus refraction can be demonstrated with a prism only 12 centimetres high, though a larger prism is desirable.

In this way ebonite, slate and other insulators can be shown to be more or less transparent, thin glass (even when coloured) almost completely so. Complete absorption is obtained with thin metal sheets, or with a grating of thin copper wires (No. 24 S.W.G.) stretched 1 cm. apart on a wooden frame 40 cm. square, provided that the wires are parallel to the lines of electric force. The same grating when turned through a right angle scarcely absorbs the rays at all. A most interesting result is obtained with a large block of wood whose edges are parallel and perpendicular to the grain. In this case the absorption is greater when the grain is parallel to the electric force than when perpendicular. The block thus acts like a grating, probably on account of the greater electric conductivity along the grain. The experiments on reflexion are also very striking.

Lebedew's original experiments required considerable skill to carry out, as he used a very small oscillator giving a wave-length of only 6 millimetres. The wave-length was determined by the interference of the waves reflected from two parallel metallic sheets in different planes: but the experiment is difficult and not very accurate. Lebedew was able to demonstrate the double refraction of rhombic sulphur for electric waves. The greatest and least refractive indices were found to be 2·25 and 2·00, while Boltzmann found the square roots of the corresponding dielectric constants to be 2·18 and 1·95 (cf. Art. 193).

190. Equations of the free ether. The experiments with the Hertzian oscillator cannot be explained on the ordinary lines by action at a distance due to the charges or currents present

* The reason for this is that the electric discharge from a fine point, or from a roughened surface, takes place at a low potential, and with the small capacities involved it is impossible to get the potential of the oscillator high enough to produce the normal discharge. This is the main source of difficulty in work at very high frequencies.

in the field : the far-reaching analogy with light-waves points rather to wave motion in an imponderable medium (the *ether*), taking place with a finite velocity probably identical with that of light.

Any new theory propounded to cover these effects should keep as close as possible to the existing theories of electrostatics and electromagnetism, because those theories, within their limits, are confirmed most exactly by experiment. We notice that Hertzian and allied effects are best shown with rapid oscillations in rods and other bodies which differ considerably from the re-entrant electric circuit used in ordinary experiments. The fundamental electromagnetic laws however refer to closed circuits, and are otherwise devoid of meaning. What is wanted, therefore, is an adequate theory of *open* circuits. The line of least resistance would seem to be *to make all circuits closed, by counting as a current something which is not a flow of electricity*, and applying the original laws unaltered. This is effectively Maxwell's line of thought, and his views have proved so successful that there is no need to consider alternative explanations.

In order to see what is required, let us imagine space to be filled with moving electricity, ρ being the volume-density (in electrostatic units) and v the velocity at the point (x, y, z) at time t. According to Art. 72 the current-density j would be given by

$$j_x = \frac{1}{c}\rho v_x, \quad j_y = \frac{1}{c}\rho v_y, \quad j_z = \frac{1}{c}\rho v_z.$$

From the general equations

$$\left.\begin{aligned} 4\pi j_x &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ 4\pi j_y &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ 4\pi j_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

we have

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots (18).$$

Using the previous values for j_x, j_y, j_z, we should have

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0.$$

This equation is however incorrect, as it conflicts with the general equation

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0 \quad \ldots\ldots(19)$$

(Art. 5).

Towards resolving this difficulty let us make the hypothesis that Gauss' theorem on the normal component of electric force is universally true, and not only in electrostatics. Expressed analytically, this gives

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad \ldots\ldots\ldots\ldots(20).$$

Equation (19) now shows that

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = -\frac{\partial}{\partial x}\left(\frac{1}{4\pi}\frac{\partial E_x}{\partial t}\right) - \frac{\partial}{\partial y}\left(\frac{1}{4\pi}\frac{\partial E_y}{\partial t}\right) - \frac{\partial}{\partial z}\left(\frac{1}{4\pi}\frac{\partial E_z}{\partial t}\right).$$

Hence we may keep the equation (18) intact by writing

$$j_x = \frac{1}{c}\left(\rho v_x + \frac{1}{4\pi}\frac{\partial E_x}{\partial t}\right), \quad j_y = \frac{1}{c}\left(\rho v_y + \frac{1}{4\pi}\frac{\partial E_y}{\partial t}\right),$$

$$j_z = \frac{1}{c}\left(\rho v_z + \frac{1}{4\pi}\frac{\partial E_z}{\partial t}\right) \quad \ldots\ldots(21).$$

The vector whose components are $\frac{1}{4\pi c}\left(\frac{\partial E_x}{\partial t}, \frac{\partial E_y}{\partial t}, \frac{\partial E_z}{\partial t}\right)$ will be called the *ether-current-density*, so that *a changing electric force counts as a current.*

The fictitious electricity has served its turn and may now be dismissed. For the free ether equations (17) give

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial E_x}{\partial t} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{1}{c}\frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{1}{c}\frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(22).$$

To obtain three other equations the law of electromagnetic induction is invoked as a hypothesis in the following form:

The line-integral of electric force (electromagnetic units) round any circuit C is equal to the rate of decrease of the flux of magnetic force through C.

Since the electric force in electromagnetic units is (cE_x, cE_y, cE_z), this gives

$$c\int_C E_x dx + E_y dy + E_z dz = -\int_S \left(l\frac{\partial H_x}{\partial t} + m\frac{\partial H_y}{\partial t} + n\frac{\partial H_z}{\partial t}\right) dS.$$

Transforming the left-hand side by Stokes' theorem and equating results, we have

$$\left.\begin{aligned} -\frac{1}{c}\frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ -\frac{1}{c}\frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ -\frac{1}{c}\frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (23).$$

It is usual to add the equations

$$\left.\begin{aligned} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0 \\ \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (24),$$

the first of which is obtained by putting $\rho = 0$ in equation (20). The equations (22), (23), (24) are the equations of the free ether. They may be written vectorially in the form

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial E}{\partial t} &= \text{curl } H \\ -\frac{1}{c}\frac{\partial H}{\partial t} &= \text{curl } E \\ \text{div } E &= 0 \\ \text{div } H &= 0 \end{aligned}\right\}.$$

The new theory throws some light on the problem left unsolved at the end of Ch. VIII, namely as to how induction takes place in a fixed circuit when another circuit is moved. We should now regard a changing magnetic force as accompanied by an electric force, because the constitution of the intervening medium requires it to be so. There is thus a certain (real or apparent) antithesis

between induction in a circuit by its own motion and by the motion of neighbouring circuits. In the former case the action of the ether on a moving charge is involved, in the latter the specific properties of the medium as the carrier of electromagnetic action.

191. The electromagnetic theory of light. From equations (22) and (23) we have

$$\frac{1}{c}\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial}{\partial y}\left(\frac{\partial H_z}{\partial t}\right) - \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right)$$
$$= c\left(-\frac{\partial^2 E_y}{\partial x \partial y} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_z}{\partial x \partial z}\right).$$

Hence

$$\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial}{\partial x}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right).$$

Remembering (24) we see that E_x satisfies the differential equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2},$$

which is the equation of wave-motion (Art. 6). Similarly all the components of electric and magnetic force satisfy an equation of the same form. It follows that electromagnetic actions are propagated through the ether with a velocity c equal to the ratio of the electromagnetic to the electrostatic unit of charge*.

According to the most reliable determinations the velocity of light *in vacuo* is as follows:

Michelson (1885)	..	$2{\cdot}9986 \times 10^{10}$,
Newcomb (1885)	..	$2{\cdot}9986 \times 10^{10}$,
Perrotin (1900)	..	$2{\cdot}9990 \times 10^{10}$.

The agreement of these numbers with the values of c given in Art. 140 is very remarkable. The natural consequence is to regard the electric and luminiferous media as the same, and to explain light as a form of electric wave (of very short wave-length). As Hertz remarked in connexion with one of his experiments, it is "a fascinating idea that the processes in air which we have

* The ratio c of the two units has the dimensions of a velocity. The theory of electrical dimensions is otherwise of little interest.

been investigating represent to us on a millionfold larger scale the same processes which go on in the neighbourhood of a Fresnel mirror or between the glass plates used for exhibiting Newton's rings."

The experimental evidence as to the velocity of propagation of electric waves in air is by no means as satisfactory as it might be. If we accept Sarasin and de la Rive's conclusion that this velocity is the same as that of waves on wires, we may utilise the measurements of Blondlot, and of Trowbridge and Duane (Art. 196), which show that the velocity is very nearly that of light. A direct method would consist in measuring the wave-length by an interference method with a resonator of known natural frequency: but the experiments made by MacLean in this direction are not very conclusive.

We now proceed to lend definiteness to the idea of an electric wave by means of a simple example. Consider a plane wave travelling along the axis of z, and write

$$E_x = P\cos p\left(t-\frac{z}{c}\right),\quad E_y = Q\cos p\left(t-\frac{z}{c}\right),\quad E_z = R\cos p\left(t-\frac{z}{c}\right),$$

$$H_x = S\cos p\left(t-\frac{z}{c}\right),\quad H_y = T\cos p\left(t-\frac{z}{c}\right),\quad H_z = U\cos p\left(t-\frac{z}{c}\right).$$

Substituting in the equations of the ether we find

$$S=-Q,\quad T=P,\quad R=U=0.$$

We may take $Q=0$ without loss of generality, and thus we have the particular solutions

$$\left.\begin{aligned} E_x &= P\cos p\left(t-\frac{z}{c}\right),\ E_y=0,\ E_z=0\\ H_x &= 0,\ H_y=P\cos p\left(t-\frac{z}{c}\right),\ H_z=0\end{aligned}\right\}.$$

There is no longitudinal component of either E or H, so that *the waves are transverse*. Moreover, E and H are in the same phase, but at right angles to one another. Fig. 274, which represents

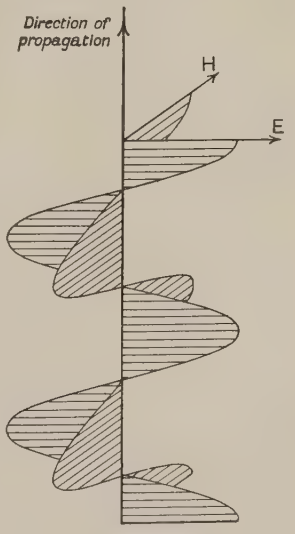


Fig. 274

the electric and magnetic forces at a given time, at various points of the axis of z, shows the relation of the three vectors to one another.

192. Theory of the ideal Hertzian oscillator. Hertz has given an elegant mathematical theory of the waves in the ether in his experiments, in which the oscillator is reduced to its simplest theoretical form. The plate-oscillator and the rod-oscillator have this in common, that electricity surges backwards and forwards across the gap so as to have an electric moment which is sometimes positive, sometimes negative. Let us consider the ideal case of a small electric doublet at the origin, with its axis along the axis of z and its moment a prescribed function $f(t)$ of the time. Towards solving the equations of the ether in the space surrounding the doublet let us make a trial assumption

$$\left.\begin{aligned} E_x &= \frac{\partial^2\theta}{\partial x\,\partial z}, \quad E_y = \frac{\partial^2\theta}{\partial y\,\partial z}, \quad E_z = \frac{\partial^2\theta}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} \\ H_x &= \frac{1}{c}\frac{\partial^2\theta}{\partial y\,\partial t}, \quad H_y = -\frac{1}{c}\frac{\partial^2\theta}{\partial x\,\partial t}, \quad H_z = 0 \end{aligned}\right\} \ldots(25).$$

On substitution we find that all the equations are satisfied if θ satisfies the single condition

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}.$$

This may be further specialised by writing

$$\theta = \frac{1}{r} f\left(t - \frac{r}{c}\right) \ldots\ldots\ldots\ldots\ldots\ldots(26).$$

Substituting this expression in equations (25) we find after some reductions

$$\left.\begin{aligned} E_x &= \frac{3xz}{r^5} f + \frac{3xz}{cr^4} f' + \frac{xz}{c^2 r^3} f'' \\ E_y &= \frac{3yz}{r^5} f + \frac{3yz}{cr^4} f' + \frac{yz}{c^2 r^3} f'' \\ E_z &= \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) f + \left(\frac{3z^2}{cr^4} - \frac{1}{cr^2}\right) f' - \frac{x^2+y^2}{c^2 r^3} f'' \\ H_x &= -\frac{y}{cr^3} f' - \frac{y}{c^2 r^2} f'' \\ H_y &= \frac{x}{cr^3} f' + \frac{x}{c^2 r^2} f'' \\ H_z &= 0, \end{aligned}\right\} \ldots(27)$$

where f, f', f'' are written instead of $f\left(t-\frac{r}{c}\right), f'\left(t-\frac{r}{c}\right), f''\left(t-\frac{r}{c}\right)$ respectively.

So far all that has been found is a particular solution of the equations of the ether. The inverse powers of $1/r$ show that the source of disturbance is situated at the origin. Close up to the source the first term in the expression for the electric force predominates, and we have approximately

$$E_x = \frac{3xz}{r^5} f(t), \quad E_y = \frac{3yz}{r^5} f(t), \quad E_z = \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) f(t).$$

This however is the static force due to an electric doublet of moment $f(t)$, with its axis pointing along the axis of z (cf. Art. 15). Now the finite velocity of propagation does not matter close up to the oscillator: hence we conclude that there actually is an electric doublet of moment $f(t)$ at the origin. The field at any place and time is then given by equations (27).

The magnetic force at any point is at right angles to the plane containing the point and the axis. The electric force is always in that plane. At points on the equatorial plane $z = 0$ we have further $E_x = E_y = 0$, so that the electric force is at right angles to the equatorial plane. These results are perfectly general and hold good whatever the form of $f(t)$.

In order to find as nearly as possible the forces due to an actual Hertzian oscillator, we write $f(t) = A$ for $t < 0$, and $= Ae^{-\frac{\delta t}{T}} \cos \frac{2\pi}{T} t$ for $t > 0$. The form of the solution is then different according to whether r is greater than or less than ct. Outside the sphere $r = ct$ we have simply the static field of the doublet of moment A. We have thus to look on a Hertzian wave as expanding continually outwards and pushing back the original electrostatic field as it goes. Fig. 275 shows the electric force at a point in the equatorial plane as a function of the distance from the oscillator after an interval of five complete periods, δ being taken as 0·2. The electric force outside the wave is, on this scale, too small to be shown. It will be noticed that the phase of the electric force is not the same at all points, but changes sign at regular intervals. This is in most striking disagreement

with the theory of action at a distance, according to which the phase, at a given time, should be the same everywhere. It is interesting to find what the formulae (27) would become on this

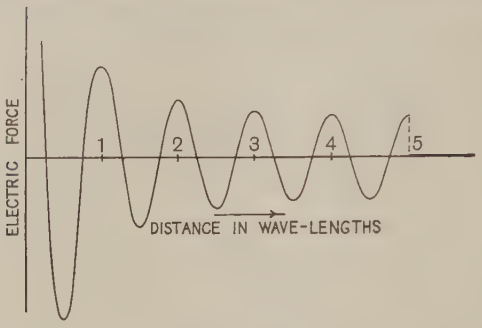


Fig. 275

latter hypothesis. The doublet may be regarded as consisting of a moveable charge e at the point $(0, 0, \zeta)$ and a fixed charge $-e$ at the origin, where $e\zeta = f(t)$; and the magnetic force is then calculated from Ampère's formula, putting ev for ids as in Art. 95. This would give

$$\left.\begin{aligned} &E_x = \frac{3xz}{r^5} f(t), \quad E_y = \frac{3yz}{r^5} f(t), \quad E_z = \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) f(t), \\ &H_x = -\frac{y}{cr^3} f'(t), \quad H_y = \frac{x}{cr^3} f'(t), \quad H_z = 0 \end{aligned}\right\} \quad (28).$$

Hence the theory of action at a distance would lead to the first term in equations (27) in each case, with $t - r/c$ replaced by t.

The difference between the formulae is especially marked at great distances from the oscillator, when (27) reduces to

$$\left.\begin{aligned} &E_x = \frac{xz}{c^2 r^3} f''\left(t - \frac{r}{c}\right), \quad E_y = \frac{yz}{c^2 r^3} f''\left(t - \frac{r}{c}\right), \\ &\qquad\qquad E_z = -\frac{x^2 + y^2}{c^2 r^3} f''\left(t - \frac{r}{c}\right), \\ &H_x = -\frac{y}{c^2 r^2} f''\left(t - \frac{r}{c}\right), \quad H_y = \frac{x}{c^2 r^2} f''\left(t - \frac{r}{c}\right), \quad H_z = 0 \end{aligned}\right\} \quad (29).$$

While the terms in (28) are at most of the order $1/r^2$, those of (29) are of the order $1/r$, and therefore fall off less rapidly with increasing distance.

That the forces in a Hertzian wave are much greater than the corresponding static forces, even as near as one wave-length from the oscillator, is shown in Fig. 276, which shows the electric

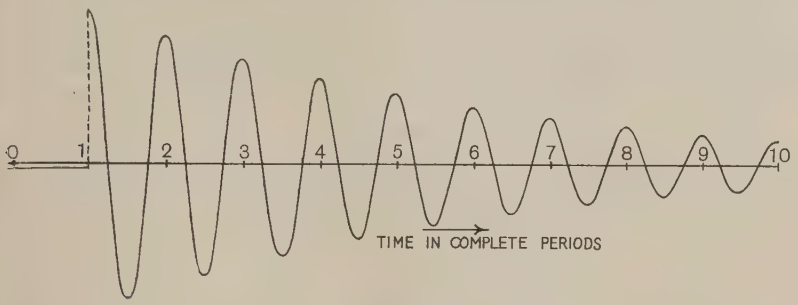


Fig. 276

force in the equatorial plane at that distance as a function of the time. The slow falling off of intensity with distance is one of the reasons why such great distances have been covered in wireless, or electric wave, telegraphy.

193. Propagation of electric waves in dielectrics. In dielectrics there is, in addition to the ether current, a current due to changing polarisation. Consider a simple model dielectric with n molecules per cubic centimetre, each molecule containing a moveable electron e (cf. Art. 47) and a fixed balancing charge $-e$. If (x, y, z) is the normal position of an electron and $(x+\xi, y+\eta, z+\zeta)$ its position at time t, then the components of polarisation are given by $P_x = ne\xi$, $P_y = ne\eta$, $P_z = ne\zeta$. The motion of the electrons constitutes a current, and the current-density (Art. 72) is $\frac{ne}{c}\frac{\partial \xi}{\partial t} = \frac{1}{c}\frac{\partial P_x}{\partial t}$. Adding the ether-current we therefore have

$$j_x = \frac{1}{4\pi c}\frac{\partial}{\partial t}(E_x + 4\pi P_x) = \frac{1}{4\pi c}\frac{\partial D_x}{\partial t},$$

where D is the electric induction (Art. 50). We have thus three equations of the type

$$\frac{1}{c}\frac{\partial D_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}.$$

The second three equations are left unaltered (supposing the dielectric to be non-magnetic), while the equation

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$$

is introduced instead of the corresponding equation in E. The fundamental equations for a dielectric are therefore

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial D}{\partial t} &= \text{curl } H \\ -\frac{1}{c}\frac{\partial H}{\partial t} &= \text{curl } E \\ \text{div } D &= 0 \\ \text{div } H &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (30).$$

We have now three vectors E, H, D occurring instead of the original two, and the equations require supplementing by a relation between D and E. The form of this depends in general on the nature of the molecule and other factors which cannot be universally specified. However, when the oscillations are not too rapid we may expect the molecule to respond to the electric force in the same way that it would to a steady force of the same magnitude, and write $D = KE$, where K is the dielectric constant.

The above equations may now be treated in the same way as the equations of the free ether. The six components of electric and magnetic force satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{K}{c^2}\frac{\partial^2\theta}{\partial t^2}.$$

It follows that electric waves of sufficient length are propagated in a dielectric with the velocity $c/\sqrt{K}$. This explains the refraction of electric waves by insulators. The refractive index, being the ratio of the wave-velocity *in vacuo* to that in the dielectric, is equal to $\sqrt{K}$. Hence we have the following remarkable consequence of Maxwell's theory:

The refractive index of a dielectric for sufficiently long electric waves is equal to the square root of its dielectric constant.

A strict test of this theory with very long electric waves is clearly out of the question, since it would be necessary to use

apparatus of enormous dimensions in order to get rid of diffraction effects. The only test, therefore, is to work with shorter waves of various lengths, and see whether the values of the refractive index thus obtained are the same. If this is so, these values may reasonably be compared with $\sqrt{K}$; but if not, we shall have evidence of *electrical dispersion* analogous to that occurring in optics. The following table gives the experimental results for a few substances:

Water. $\sqrt{K} = 9$.

Wave-length (cm.)	μ	Observer
65	8·88	Ellinger
8·8	8·89	Eckert
5·7	8·79	Eckert
5·0	8·80	Cole
3·7	8·10	Eckert
1·75	7·82	Eckert
0·8	8·97	Lampa
0·6	9·40	Lampa
0·4	9·50	Lampa
*1·26 × 10^{-4}	1·32	Rubens
†5·89 × 10^{-5}	1·33	—

Ethyl alcohol. $\sqrt{K} = 5{\cdot}1$.

Wave-length (cm.)	μ	Observer
65	4·89	Ellinger
5·7	3·4	Eckert
0·8	2·57	Lampa
0·6	2·29	Lampa
0·4	2·24	Lampa
†5·89 × 10^{-5}	1·36	—

Glycerine. $\sqrt{K} = 7{\cdot}5(?)$.

Wave-length (cm.)	μ	Observer
5·7	3·8	Eckert
4·5	4·1	Merczyng
0·8	1·84	Lampa
0·6	1·76	Lampa
0·4	1·62	Lampa
†5·89 × 10^{-5}	1·47	—

Benzene. $\sqrt{K} = 1{\cdot}5$.

Wave-length (cm.)	μ	Observer
9·04	1·50	Kossonogoff
8·5	1·84	v. Lang
6·43	1·52	Kossonogoff
4·30	1·52	Kossonogoff
2·95	1·56	Kossonogoff
1·92	1·56	Kossonogoff
0·8	1·77	Lampa
0·6	1·76	Lampa
0·4	1·74	Lampa
*1·85 × 10^{-4}	1·48	Rubens
†5·89 × 10^{-5}	1·50	—

Paraffin wax. $\sqrt{K} = 1{\cdot}56$.

Wave-length (cm.)	μ	Observer
10	1·42	Wiedeburg
0·8	1·52	Lampa
0·6	1·41	Lampa
0·4	1·39	Lampa

* Heat waves.

† Sodium light.

The results for benzene and paraffin wax lend strong support to Maxwell's theory, considering the difficulty of this kind of measurement. For these and some other substances the dispersion is small*. The behaviour of water is very remarkable. For the longer wave-lengths μ is nearly constant and equal to $\sqrt{K}$, but falls rapidly below the wave-length of the shortest electric waves hitherto produced experimentally. There is evidence to show that the refractive index is quite low for the longest heat-waves, so that the region of transition is to be sought in that portion of the spectrum which is still unknown.

The question of what causes the variation of μ with frequency is answered by the optical dispersion theories†. These theories show that the equation $D = KE$ may be applied to periodic changes as well as to steady states, but K in general depends on the frequency. This generalisation of the idea of dielectric constant will be useful when we come to consider the case of electric waves on wires.

194. Electric waves along a pair of parallel wires. A *perfect conductor* is one for which the lines of electric force leave the surface at right angles, even at the frequencies of electric waves. Subject to subsequent verification we shall assume that all ordinary metals fulfil this condition, at any rate approximately.

We shall now consider the propagation of electric waves along two wires of perfectly conducting material. The wires are taken as parallel to the axis of z, but not necessarily thin or round, the cross-sections by a plane parallel to $z = 0$ being bounded by any two curves C_1, C_2. The theory is based on the following elegant particular solution of the equations of the ether:

$$\left.\begin{aligned} E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = 0 \\ H_x = \frac{\partial U}{\partial x}, \quad H_y = \frac{\partial U}{\partial y}, \quad H_z = 0 \end{aligned}\right\} \quad \dots\dots(31),$$

where $$U + iV = f(x + iy,\ z - ct)$$

f being an arbitrary function of the two arguments $x + iy$, $z - ct$.

* The refractive index of a transparent substance for visible light often agrees well with the square root of its dielectric constant.

† See Drude, *Theory of Optics*, English translation, pp. 382 et seq.

Confining attention to a particular time and a particular cross-section (z, t constant), the condition of perfect conductivity is that V is constant over C_1 and C_2. Thus V is proportional to the potential in the corresponding electrostatical problem (Art. 41). The lines of electric force* coincide with those in the electrostatic problem, the lines of magnetic force with the equipotential curves.

We shall assume that the charge on the wires resides entirely on the surface, and is given by Coulomb's formula $E = 4\pi\sigma$. Suppose now that the wires have a length l great in comparison with their distance apart, and that C is the capacity in the electrostatic problem, which we suppose to have been already solved. In the actual wave-problem it follows from what has been said above that the integral $\int E ds$ is the same for all lines of electric force passing from C_1 to C_2 in the particular cross-section considered. Denoting it by V, V may be called the difference of potential between the wires *at that cross-section*, and is a function of $z - ct$ only. Let e be the charge per unit length of C_1 at the cross-section z, i the current in electromagnetic units at the same place, and $-e$, $-i$ the same quantities for the wire C_2. Then $e = CV/l$. The charge entering the part of C_1 between the planes z and $z + dz$ per second is $-\dfrac{\partial i}{\partial z}dz$. This must be accounted for by the rate of increase of the charge within the section, namely $\dfrac{1}{c}\dfrac{\partial e}{\partial t}dz$. Hence $\dfrac{\partial i}{\partial z} = -\dfrac{1}{c}\dfrac{\partial e}{\partial t}$. Since $\dfrac{\partial}{\partial t} = -c\dfrac{\partial}{\partial z}$ in this case, we have simply $i = e$.

The distribution of current and "potential" corresponding to the wave-length λ in ether is thus given by

$$V = A\cos\frac{2\pi}{\lambda}(z - ct), \quad e = \frac{CA}{l}\cos\frac{2\pi}{\lambda}(z - ct),$$

$$i = \frac{CA}{l}\cos\frac{2\pi}{\lambda}(z - ct) \ldots (32).$$

It will be noticed that the effect of the wires appears only in the boundary conditions. In physical language, the wires serve only to guide the waves, which are really propagated through the ether. The velocity of propagation is c.

* Notice that the electric force is not derivable from V as a potential. Its components are $(-\partial V/\partial x, -\partial V/\partial y, 0)$ and not $(-\partial V/\partial x, -\partial V/\partial y, -\partial V/\partial z)$.

For a wave travelling in the negative direction of the axis of z we have

$$V = A \cos \frac{2\pi}{\lambda}(z + ct), \quad e = \frac{CA}{l} \cos \frac{2\pi}{\lambda}(z + ct),$$

$$i = -\frac{CA}{l} \cos \frac{2\pi}{\lambda}(z + ct) \ldots (33).$$

Adding (32) and (33), with a slight change of notation, we have the solution corresponding to a stationary wave, namely

$$V = A \cos \frac{2\pi z}{\lambda} \cos \frac{2\pi ct}{\lambda}, \quad e = \frac{CA}{l} \cos \frac{2\pi z}{\lambda} \cos \frac{2\pi ct}{\lambda},$$

$$i = \frac{CA}{l} \sin \frac{2\pi z}{\lambda} \sin \frac{2\pi ct}{\lambda} \ldots (34).$$

Three cases have to be distinguished:

(1) *Free ends to both wires.* Here we must have $i = 0$ when $z = 0$ and when $z = l$. The first condition is satisfied already in equation (34), and the second if $2\pi l/\lambda = n\pi$, where n is an integer. Hence the possible wave-lengths of the free stationary waves are given by $\lambda = 2l$, $2l/2$, $2l/3$, etc.

The distribution of the amplitude of current and potential (or

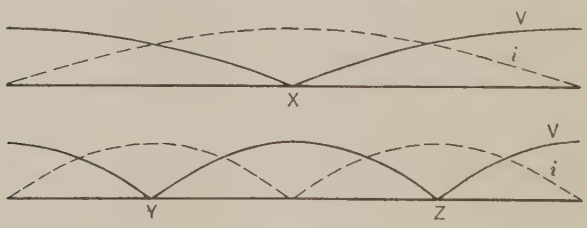


Fig. 277

charge) along the wires is shown in Fig. 277 for the fundamental and first harmonic.

(2) *Free ends at $z = 0$, wires bridged across at $z = l$.* Here

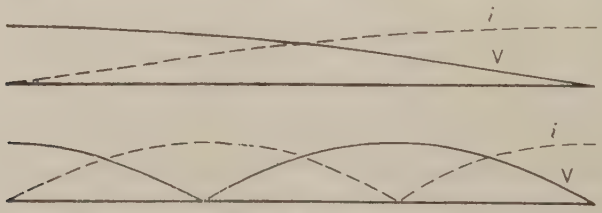


Fig. 278

$i = 0$ for $z = 0$ and $V = 0$ for $z = l$, giving $2\pi l/\lambda = (n + \frac{1}{2})\pi$. Thus the admissible waves are given by $\lambda = 4l$, $4l/3$, $4l/5$, etc. The first two of these are represented in Fig. 278.

(3) *Both ends bridged over.*

Instead of (34) it is convenient to take another fundamental solution, namely

$$V = A \sin \frac{2\pi z}{\lambda} \cos \frac{2\pi ct}{\lambda}, \quad e = \frac{CA}{l} \sin \frac{2\pi z}{\lambda} \cos \frac{2\pi ct}{\lambda},$$

$$i = -\frac{CA}{l} \cos \frac{2\pi z}{\lambda} \sin \frac{2\pi ct}{\lambda} \quad \ldots(35).$$

The terminal conditions are then satisfied if $2\pi l/\lambda = n\pi$, so that $\lambda = 2l$, $2l/2$, $2l/3$, etc. The first two waves are shown in Fig. 279.

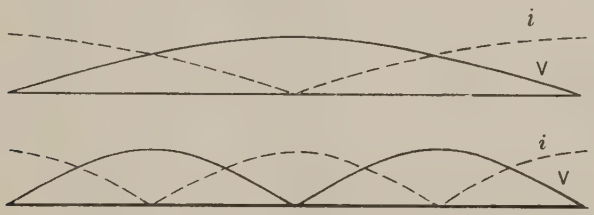


Fig. 279

195. Experiments with stationary electric waves on wires. The production of stationary waves on wires originates with Hertz, whose apparatus has since been modified and considerably improved. Fig. 280 represents a simple arrangement suitable for qualitative experiments. AA' are the plates of a

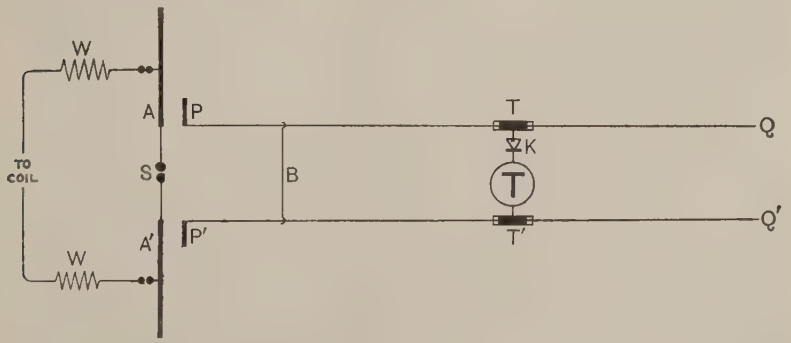


Fig. 280

Hertzian oscillator of the type shown in Fig. 268, the plates being, say, 12 cm. square. Opposite AA' are two smaller plates PP', terminating the parallel wires PQ, $P'Q'$, 6 metres long and 4 cm. apart. In detecting the waves no direct metallic contact with the wires is permissible, on account of the disturbance that would be produced thereby. Two small glass tubes TT, covered with tinfoil, are slipped over the wires, and are connected together through a crystal and telephone. These tubes act as small condensers, and the charges on them at any time are proportional to $\pm e$, so that the intensity of sound in the telephone at any position corresponds to the amplitude of e or V.

On placing a metallic bridge B over the wires between the telephone and oscillator the sound in general disappears, or becomes very faint. But when B is in a position in which the amplitude of V vanishes (as at X, Fig. 277), the corresponding oscillation goes on unchanged and a sound is heard in the telephone. The position of the nodes of potential can thus be found by moving the bridge along the wire with an insulating handle. To isolate the first harmonic in Fig. 277 place the bridge about one-fourth the distance along the wire and adjust for maximum sound (point Y). Put the detector at the centre of the wire and place a second bridge over the wires on the other side. A pronounced maximum sound occurs when this bridge occupies the position Z.

Quantitative results may be obtained by replacing the telephone by a galvanometer, or by using a sensitive thermal detector as was done by Rubens. His results showed that the distribution of charge along the wire was simple when a bridge was laid across, but very complicated when there was no bridge, as we should expect from the mixing of the various harmonics which then occurs.

It is not difficult to see *a priori* that we are here dealing with the *natural* oscillations of the wire-system. The primary Hertzian oscillator AA' is very heavily damped (we may if we wish use a quenched spark-gap at S); while the oscillations in the wire-system are very persistent. The latter will therefore continue alone long after the Hertzian oscillator has ceased to act. It is interesting to notice that the circumstances are entirely altered

if the plates PP' are the same size as AA', and close to them. For in this case $SAPBP'A'S$ will form a closed oscillation circuit which does not give rise to electric radiation, and the primary damping may then be quite small.

The oscillations on wires bridged over at one end (Fig. 278) may be examined by joining QQ' together permanently by a wire. It is sometimes more convenient to excite these oscillations by another method, due to Blondlot (Fig. 281). Here the wires are

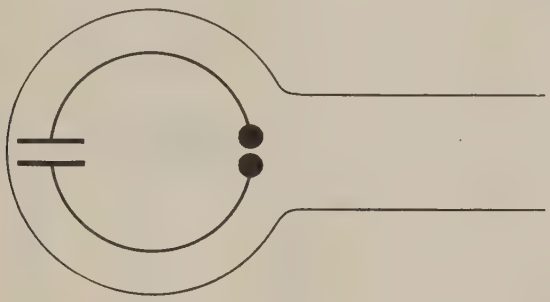


Fig. 281

bent round so as to form a loop enclosing a small condenser circuit (diameter of loop say 8 cm., plates $4 \times 2\frac{1}{2}$ cm., 2 cm. apart) which acts on the loop by electromagnetic induction. The Blondlot oscillator would most naturally give *forced* waves of its own period, but if a short brass spark-gap is used, the primary is usually sufficiently damped to show the natural oscillations of the wire-system, which in this case is closed at the nearer end. By bridging it over at the further end as well we may determine the nodes shown in Fig. 279.

The stationary waves on wires can be demonstrated in an elegant way with the apparatus shown in Fig. 282. The wires

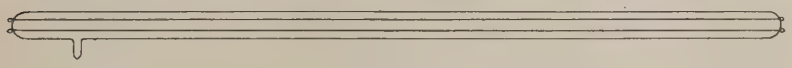


Fig. 282

are enclosed in a long tube containing air at a low pressure, which glows at places where the "potential" is greatest and remains dark at the nodes, so that the tube is filled with alternate bright and dark patches at intervals apart equal to a half wave-length.

196. Measurement of the velocity of waves on wires. The velocity of propagation of electric waves along wires was measured by Blondlot by a direct method. Two Leyden jars (Fig. 283) have their inner coatings continuous, but each of the outer coatings is cut into two parts AB, $A'B'$, separated from one another. The inner coatings are connected to the secondary of an induction coil, in parallel with the spark terminals C, C'. A is joined to A' by a wire interrupted by a small spark-gap DD', and also by a piece of moistened cord E, the object of which will appear shortly. BD and $B'D'$ are joined by wires BFD, $B'F'D'$, each 1000 metres long, doubled back on themselves, BB' being also connected together by a moistened cord E'.

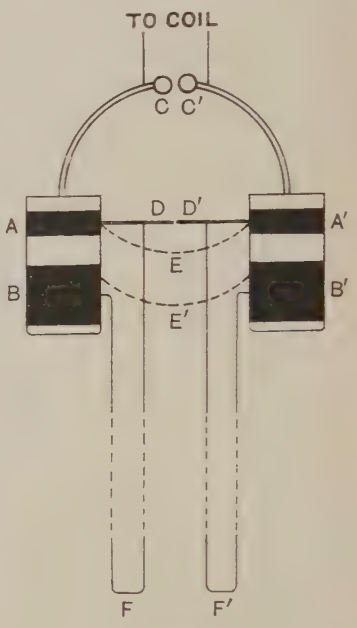


Fig. 283

The working of the apparatus is as follows. While the coil is charging up the inside coatings, and before the spark-gap CC' gives way, A and A' tend to become charged with electricity of opposite signs, and so do B and B'. These charges are however equalised as they appear by the moist cords, which conduct sufficiently well for the purpose. As soon as the potential is high enough to spark across CC' an oscillatory discharge takes place which gives rise to oscillatory potential-differences between A and A'. Since the moist cords cannot neutralise these sufficiently rapidly a spark passes between the terminals DD'. Now while this is occurring the corresponding disturbance due to BB' is travelling along the wires BFD and $B'F'D'$ with the velocity of propagation v, and after a time l/v a second discharge occurs through DD', l being the length of either of the long wires. Blondlot examined the image of the spark in a rotating mirror,

and by measuring the time that elapsed between the two discharges deduced the value of v.

A set of five observations gave as the mean $v = 2{\cdot}964 \times 10^{10}$ cm. per sec., and three similar observations on wires 1800 metres long gave 2·975, 2·985 and 2·980, mean $2{\cdot}980 \times 10^{10}$ cm. per sec.

The weakening of the wave in traversing such a great length of wire was distinctly noticeable, and Blondlot made the coatings BB' larger than AA' in order that the sparks might be of nearly equal intensity.

Another accurate determination of the velocity of waves on wires has been made by Trowbridge and Duane. Their method

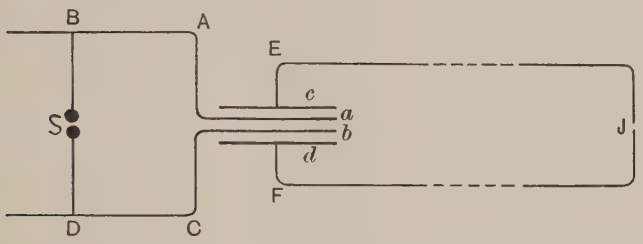


Fig. 284

of excitation is practically that of Fig. 280, adapted to give forced instead of free vibrations on the wires. The primary condenser ab consists of two metal plates 30 cm. square, separated by a sheet of plate glass 2 cm. thick. Sheets of ebonite 1·8 cm. thick are laid on the other side of ab, and metal plates cd, each 26 cm. square, attached to them. These plates form one extremity of the parallel-wire system EJF, 58·6 metres in length, which is interrupted at J by a spark-gap with pointed terminals of cadmium. The primary circuit is completed by a sliding piece BD containing a spark-gap S, the distance of the parallel wires AB, CD from one another being 40 cm. By moving the sliding piece about resonance can be obtained with one or more of the natural periods of oscillation of the wire-system; which is advantageous in giving the greatest possible intensity of light in the spark-gap J. The most powerful effects were obtained when $AB = CD = 85$ cm., in which case resonance seems to have taken place with the first harmonic. Trowbridge and Duane observed the position of the nodes and

also measured the frequency of the oscillatory spark J by means of a very rapidly rotating mirror, with the following results:

Wave-length λ	Periodic time T	Velocity of propagation $v = \lambda/T$
5670	$1{\cdot}871 \times 10^{-7}$	$3{\cdot}030 \times 10^{10}$
5670	1·876	3·022
5690	1·940	2·923
5690	1·897	3·000
5690	1·874	3·036
5690	1·899	2·996
5660	1·878	3·014
	Mean	$3{\cdot}003 \times 10^{10}$

The velocity of propagation of waves on wires is therefore sensibly the same as that of light, a conclusion that has been utilised already. Again, it is clear from these experiments that the waves are capable of traversing considerable distances before being extinguished, so that there is good reason to believe that copper and other metals fulfil approximately the conditions of perfect conductivity as utilised in Art. 194.

197. Measurement of dielectric constants for high frequencies. The theory of waves on a pair of parallel wires may easily be extended to the case in which the wires are surrounded by a dielectric. The fundamental solution (31) is then replaced by

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = 0 \\ H_x &= \sqrt{K}\,\frac{\partial U}{\partial x}, \quad H_y = \sqrt{K}\,\frac{\partial U}{\partial y}, \quad H_z = 0 \end{aligned}\right\} \quad \ldots\ldots(36),$$

where

$$U + iV = f\left(x + iy, z - \frac{ct}{\sqrt{K}}\right)$$

The velocity of propagation along the wires, as in the free dielectric, is $c/\sqrt{K}$; from which it follows that if λ is the wave-length on wires immersed in air, and λ' that on wires immersed in the dielectric, the frequency n being the same in both cases, then $\lambda^2/\lambda'^2 = K$, where K is the dielectric constant for the frequency n.

In order to apply this result for a liquid dielectric Drude used the apparatus shown in Fig. 285. B is a small Blondlot oscillator

in which the condenser is done away with and the ends of the wire cut off flat and left adjacent to one another. When a bridge B_1

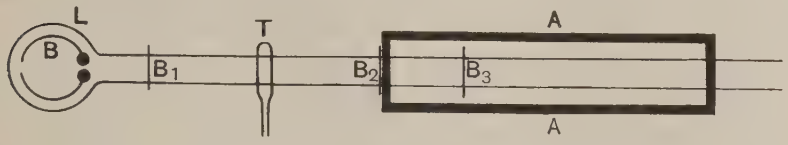


Fig. 285

is laid over the wire the Blondlot oscillator and the loop form together an oscillatory circuit having practically only a single period, which induces forced oscillations in the system to the right of B_1. This may be shown by means of a second bridge B_2 (disregarding the trough for the present), with a small electrodeless vacuum tube T held over the wires as a detector. As B_2 is moved along the wires, B_1 remaining fixed, the tube lights up brightly at regular intervals of, say, $\frac{1}{2}\lambda$. The wave-length λ in air is thus determined.

Having found the position of the first node B_2 a trough AA with perforated ends is slid over the wires until one end coincides with B_2, and the liquid poured into it. A third bridge B_3 may now be laid over the wires inside the trough and moved about as before. It is found that periodic maxima still occur, but the distance between the successive nodes has now a smaller value $\frac{1}{2}\lambda'$. Then λ' is the wave-length in the liquid for a wave-length λ in air, so that K is known.

The object of making the entrance to the trough coincide with a node is to get rid of the strong reflexion which would otherwise occur with liquids of high dielectric constants. Drude further found it advantageous to work the oscillator off a small Tesla transformer instead of directly from the coil, and to immerse the whole of the oscillator and the loop L in petroleum. The pitting of the spark-balls, which is the chief source of trouble in high-frequency work, is thus much reduced.

By this and other methods it has been found that the dielectric constant for substances like alcohol and glycerine depends markedly on the wave-length, but that for a particular wave-length K and μ^2 agree fairly well with one another.

198. Electric oscillations in rods and solenoids. We now come to a somewhat debatable (though extremely important) subject, in which the mathematical difficulties are such as to preclude any rigid treatment in the vast majority of cases.

Consider first the simple Hertzian rod-shaped oscillator, represented in Fig. 272. If we leave the damping out of account we shall have a set of stationary waves, with the condition $i = 0$ at both ends of the rod; hence it is reasonable to suppose that the various types of vibration are given approximately by $i = A \sin \frac{2\pi z}{\lambda} \sin \frac{2\pi ct}{\lambda}$, where λ is the wave-length. The general equation $\partial e/\partial t = -c\, \partial i/\partial z$ then shows that

$$e = A \cos \frac{2\pi z}{\lambda} \cos \frac{2\pi ct}{\lambda}.$$

If the two ends of the wire are $z = 0$ and $z = l$, the boundary conditions give $2\pi l/\lambda = n\pi$, where n is an integer: thus the admissible wave-lengths are given by $\lambda = 2l/n$ and the general expression for current and charge becomes

$$i = \Sigma A_n \sin \frac{\pi n z}{l} \sin \frac{\pi n ct}{l}, \quad e = \Sigma A_n \cos \frac{\pi n z}{l} \cos \frac{\pi n ct}{l}.$$

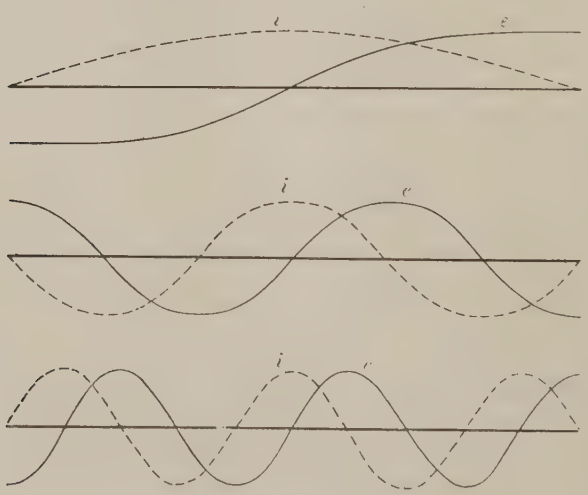


Fig. 286

Suppose that initially the current is zero and equal and opposite charges are distributed symmetrically on either side of the middle point of the rod. The first condition is satisfied by the above solution; the second gives

$$\Sigma A_n \cos \frac{\pi n z}{l} = -\Sigma A_n \cos \frac{\pi n (l - z)}{l}$$

for all values of z. Hence n must be odd, so that *only the odd harmonics are present.* The distribution of current and charge along the wire at any given time is shown in Fig. 286, in which only the first two odd harmonics are included.

These considerations make it very probable that the wave-length in air corresponding to the fundamental oscillation is approximately twice the length of the rod. The relation of wave-length to length of rod for a number of small oscillators has been examined very carefully by Ives, using an interference method to determine λ. The results were as follows:

Length of oscillator l (cm.)	Wave-length λ (cm.)	λ/l
4·93	10·42	2·10
7·49	15·24	2·04
9·85	19·86	2·03

The wave-length is therefore somewhat greater than $2l$, and in addition the damping is considerable (δ = about 0·2).

The natural oscillations of finite straight solenoids have been examined experimentally by Drude, who has given tables for calculating the fundamental frequency of any given coil from its dimensions. The coil to be examined was excited inductively by a small Blondlot oscillator with *variable* capacity, the wave-length corresponding to any particular position of which had been determined by a separate experiment with a pair of parallel wires. Resonance occurred in certain positions, and was recognised by the lighting up of a small vacuum tube attached to one end of the coil. The order of magnitude of the fundamental wave-lengths of ordinary coils will be seen from the fact that for

a coil 30 cm. long and 1·7 cm. in diameter, having 100 windings of thick bare copper wire, $\lambda = 554$ cm.

The constants necessary to specify a coil completely are

n = number of turns of wire,
g = distance between two consecutive turns,
h = axial length of coil,
$2r$ = diameter, supposed circular,
l = total length of wire,
δ = diameter of wire used;

of which only four are independent, since $h = (n-1)g$ and $l = 2\pi rn$.

The thickness and nature of the insulation and the dielectric constant K of the core will also have some effect. Drude set himself the problem of finding how the fundamental wave-length* λ depends on these quantities. In the first place we notice that for geometrically similar coils λ is proportional to the linear dimensions. For the differential equation $\partial^2\theta/\partial t^2 = c^2\Delta\theta$, satisfied by all the components of electric and magnetic force in the free ether, still holds good when x, y, z, t are replaced by kx, ky, kz, kt, k being any constant. Thus with a coil k times as great the natural period is k times as large as before, and the result still holds when dielectrics are present. Hence we may write

$$\lambda = lf\left(n,\ \frac{h}{2r},\ \frac{g}{\delta},\ K\right),$$

where f remains to be determined. Drude concluded from his experiments that the variation with n was, *caeteris paribus*, unimportant, so that

$$\frac{\lambda}{l} = f\left(\frac{h}{2r},\ \frac{g}{\delta},\ K\right).$$

The accompanying table gives the values of f for copper wires (1) wound on solid ebonite cores, (2) with no core. If the solenoid had been completely imbedded in ebonite the value of λ would have been increased in the ratio $\sqrt{K}$ to unity: the increase due to the core alone is naturally less.

* By this we mean the wave-length in air which corresponds to the natural period T of the coil, so that $\lambda = cT$. The distance between two consecutive nodes, measured along the axis of the solenoid, is of course very much smaller.

$\frac{h}{2r}$	Solid ebonite			Coreless		
	g/δ = 1·09	g/δ = 1·24	g/δ = 2·4	g/δ = 1·09	g/δ = 1·24	g/δ = 2·4
6·0	1·48	1·46	1·44	1·37	1·35	1·33
4·0	1·65	1·63	1·60	1·52	1·50	1·48
3·0	1·86	1·82	1·78	1·69	1·66	1·62
2·0	2·25	2·21	2·14	1·99	1·95	1·92
1·0	3·11	2·97	2·85	2·66	2·54	2·45
0·8	3·48	3·30	3·14	2·94	2·78	2·68
0·6	3·98	3·68	3·50	3·34	3·09	2·97
0·4	4·61	4·20	3·95	3·85	3·51	3·35
0·2	5·60	5·22	4·67	4·64	4·32	3·94
0·1	6·72	6·20	5·37	5·58	5·14	4·48
0·05	7·20	6·36	5·17	5·96	5·28	4·38

Drude found that the higher oscillations were not strictly harmonic, nor were the nodes exactly in the positions required by a simple theory. The fundamental wave-length of a circular coil of *one* turn was found to be 6·7 times the diameter, which is somewhat smaller than the number obtained by Sarasin and de la Rive.

The oscillations on solenoids can be applied to demonstration purposes by a method due to Seibt. One plate of the condenser in an electric oscillation circuit (Fig. 287) is joined to the end of a long helix HH containing a large number of turns of fine wire wound close together. Parallel to this is placed an exhausted tube T, and on the side remote from the helix is a strip of tinfoil extending along the whole length of the tube inside, which is connected to the other plate of the condenser. On exciting the oscillatory circuit forced waves of period $T = 2\pi\sqrt{(LC)}$ pass up and down the helix, and alternate bright and dark patches appear in the vacuum tube. The number of nodes may be increased or diminished by altering the self-inductance of the circuit, and the whole forms a very striking lecture experiment.

In all these cases the most powerful effects are obtained for certain resonance positions, in which the natural period of the oscillatory circuit coincides with one of the periods of the helix. The same is true of Tesla discharges, for which it is very advantageous to calculate out beforehand the natural frequency of the

secondary, so that it may be used with a suitable primary circuit. Drude's tables enable us to do this.

The natural frequency of a rod-oscillator may be considerably reduced, and the wave-length lengthened, by inserting a self-inductance in it. This principle is extensively applied in wireless telegraphy, to obtain resonance between a short aerial and waves much longer than its fundamental oscillation. A given coil does not, however, add the same wave-length to all aerials.

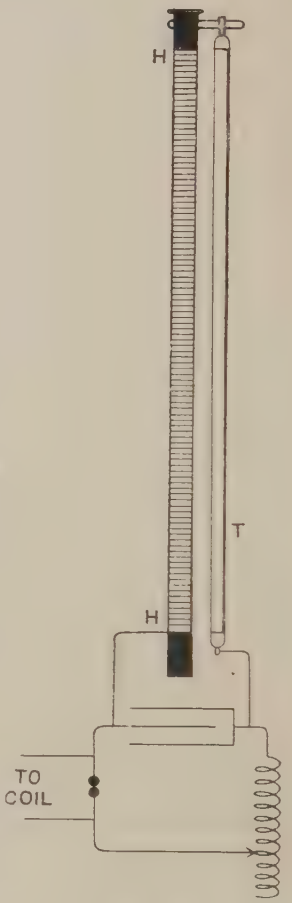


Fig. 287

199. Wireless telegraphy. It was recognised shortly after Hertz's discovery that electric waves might be used for signalling across space without wires, the difficulty being to produce them with sufficient intensity and detect them far enough from the source. The attempts to do so were first pushed to a really practical conclusion by Marconi in 1895—6. The two modifications introduced into Hertz's oscillator by Marconi were (1) to replace one half by a simple earth-connexion, (2) to use as the other half a long vertical wire, with or without a capacity at the summit. Marconi's transmitter is shown in the left-hand diagram of Fig. 288. The long wire AA, now generally known as the *aerial*, is connected to a large metal plate embedded in the soil, and interrupted at a point just above the ground by a spark-gap S. A Morse key inserted in the primary of the induction coil I enabled the operator to keep on the spark for longer or shorter intervals at a time, and thus make the recognised signals of the Morse code. In the receiver the spark-gap was

replaced by a coherer K in parallel with a battery and telephone T or other detecting apparatus*. When electric waves fall on the

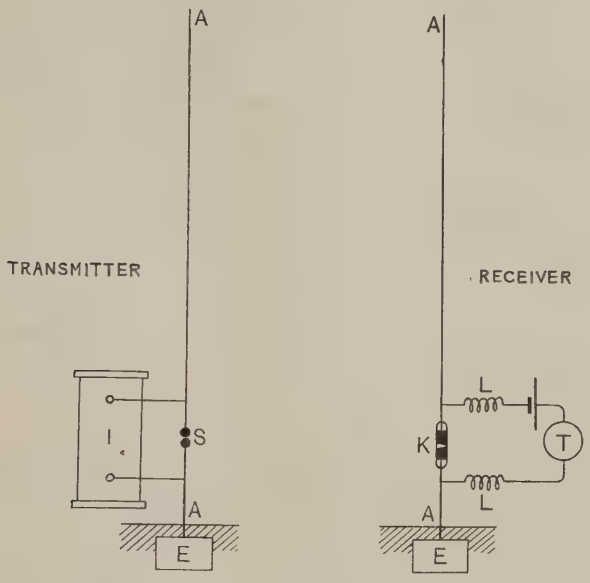


Fig. 288

receiver the coherer begins to conduct, and a sharp click is heard in the telephone corresponding to the first wave-train. An arrangement may be added by which the coherer may be tapped back to its sensitive state every time a current passes through it, in which case the receiver will act during the whole time that the spark-gap S is working. The inductances L, L serve to keep the high-frequency current out of the shunt circuit.

The advantage of the earth-connexion in Marconi's apparatus has been much discussed; but it is easy to see what its effect would be if the earth could be regarded as a perfect conductor. Since the lines of electric force of a Hertzian oscillator in a diametral plane $z = 0$ are perpendicular to that plane, the condition of cutting all the lines of electric force at right angles is satisfied. Hence we may imagine the plane to be replaced by the surface of a perfect conductor and the lower half of the oscillator taken away, when the phenomena go on precisely as before. It is

* Marconi actually used an automatic registering device.

probable, therefore, that the earthed aerial will act approximately like an isolated rod-oscillator of double the length: so that the fundamental wave-length of a simple Marconi aerial is about four times the height. This rule is useful in rough calculations.

We have now to consider the possible improvements in Marconi's apparatus. No real resonance is obtained between the transmitter and receiver with this arrangement, for before the coherer breaks down the oscillations excited are those of the aerial alone, and not of the aerial-earth system. Further, the aerial, being insulated, is very liable to disturbance from atmospheric electricity. For these reasons Marconi at an early stage removed the coherer from the aerial and put it in a shunt circuit, on which the aerial acted inductively. A common form of coupled receiver on this principle, for use with a crystal, is shown in Fig. 289.

The idea of using coupled circuits in the *transmitter* is due to Braun. The high damping of the simple aerial (δ about 0·2) prevents resonance and also lowers the efficiency by distributing the energy over a large range of wave-lengths. Braun's plan was to couple the aerial inductively with a condenser circuit, the damping of which can be made comparatively low. This method combines to some extent the advantages of small damping and large radiation from the aerial.

Fig. 289 shows a station arranged on this principle. For success it is essential that proper resonance should be obtained between all the circuits. The practical method of procedure is as follows. A temporary spark-gap is first inserted in the aerial AA, which is excited directly off an induction coil or alternating current transformer. The wave sent out will not be very homogeneous, but its wave-length can be ascertained approximately with a wave-meter held near the aerial. The auxiliary circuit is then adjusted to have the same frequency, and coupled up. If the coupling is not too close the wave-length is now the same as before, but there is a marked improvement in the purity of the wave. The receiving aerial may next be adjusted to the required frequency by means of the variable inductance L; and finally the variable condenser C is altered until there is a maximum sound in the telephone.

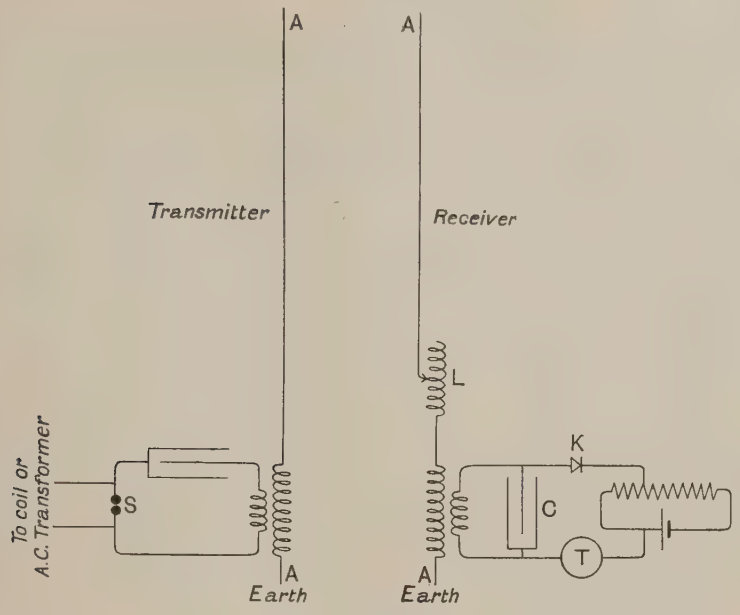


Fig. 289

It is not necessary that the two circuits of a coupled transmitter should be insulated from one another. Very often two points on the aerial inductance are joined to the condenser and spark-gap respectively, as shown in Fig. 290; and similarly for the receiver. This method is known as *direct coupling*.

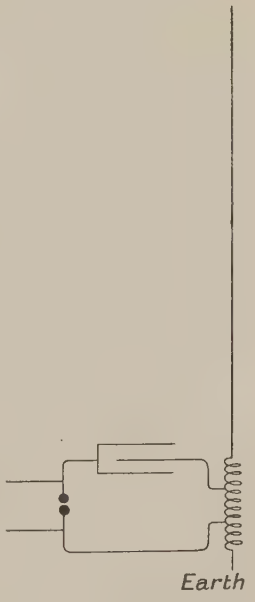


Fig. 290

200. Use of quenched sparks in wireless telegraphy. The most important progress in wireless telegraphy has been in the direction of *tuning* and *efficiency*; that is, confining the waves as far as possible to a single wave-length, and sending them out with the least expenditure of power.

The original idea of Braun's system seems to be as follows. If we have two

oscillation circuits C_1, C_2 of decrements δ_1, δ_2 loosely coupled together, δ_1 being less than δ_2, it can be shown that the oscillations ultimately subside with the smaller decrement δ_1. Hence it is possible to use an aerial of high radiating power and still keep the damping down by coupling it with a condenser circuit of small damping. But with weak coupling the current in the aerial is small, and more energy is wasted in the primary (particularly in the spark-gap) than in radiation from the aerial. The efficiency is therefore low. Practical stations on the Braun system usually worked with quite tight coupling. This, however, has the disadvantage of sending out *two* waves of different frequency.

In Wien's system the difficulty is avoided by the use of a *quenched spark-gap* at S (Fig. 289). The degree of coupling is then only important in that it affects the efficiency of the quenching (Art. 185) and the amount of energy transferred from the primary to the secondary. When the quenched spark has ceased the aerial continues to oscillate by itself, and emits a wave of one frequency only.

The solution at which we have arrived is somewhat surprising, as it practically means a return to the simple aerial, with the spark-gap eliminated. It is, however, an essential part of the quenched spark system to place inductances in the aerial until its total damping is small. If δ_1 is the part of the decrement arising from radiation and δ_2 that arising from other causes, then the efficiency is only high as long as δ_1 is large compared with δ_2. The damping is only low as long as $\delta_1 + \delta_2$ is small. Hence to satisfy both conditions it is necessary to have δ_2 very small, and this is impossible with a spark-gap in the aerial. It is found by experience that the total decrement of the aerial can be advantageously reduced to 0·05 or so.

One type of quenched spark-gap suitable for wireless telegraphy has already been described in Art. 185. The apparatus used by Marconi for long distance wireless telegraphy, shown in Fig. 291, depends on the same principle. Power is furnished at 12,000 volts by three direct-current generators in parallel with a battery of 6000 accumulators. The spark-gap is formed of two discs D_1, D_2, kept in slow rotation, and a third studded disc D_3 rotating very rapidly. The distances are so calculated

that a spark can take place when the studs are in between the discs, but not otherwise. An oscillatory discharge of the condenser C therefore begins whenever the studs are in this position, but it is rapidly quenched as the spark-gap gets longer and longer, leaving oscillations in the aerial to go on undisturbed. The self-inductances L, L prevent the condenser discharge from circulating in the generators or accumulators, and signals are made with a Morse key K in the main circuit.

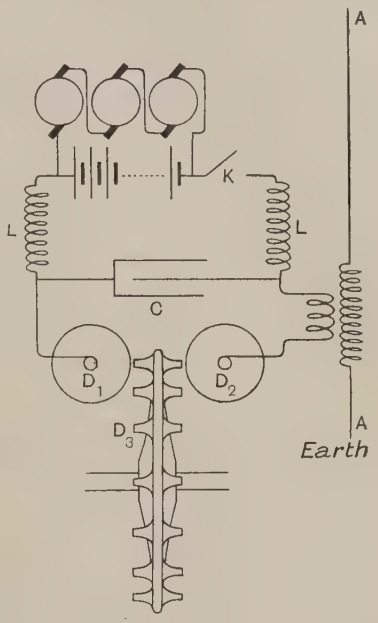


Fig. 291

The frequency of the sound heard in the telephone of the receiving station is equal to the number of wave-trains sent out by the transmitter per second. Hence it is advantageous to increase the number of sparks per second until a note of medium pitch is heard. The sensitiveness can be further increased by using a "tuned" telephone; i.e. one whose diaphragm has a natural frequency of vibration equal to that of the sequence of sparks.

201. Use of undamped oscillations. With the singing arc as ordinarily used oscillations cannot be obtained of frequency higher than about 10,000 to the second, so that some modification is required before the arc can be of any use in wireless telegraphy. The conditions of success were discovered by Poulsen in 1902. He found that it was either necessary or advantageous:

(1) to make the arc burn in hydrogen, or in a gas containing hydrogen, such as alcohol vapour;

(2) to make the positive electrode of copper, cooled with a stream of water, and only retain carbon for the negative electrode;

(3) to apply a transverse magnetic field;

(4) to rotate the carbon electrode slowly during the whole time.

In order to see the effect of conditions (1) and (2) it is necessary to refer back to Figures 266 and 267. At high frequencies there appears to be a great tendency of the arc to become conducting in the opposite direction when the voltage is reversed (point E) which destroys the regularity of the oscillations. With the ordinary carbon arc the most important factor in maintaining the current is the stream of electrons from the incandescent negative electrode (Art. 226). Now the positive electrode is even hotter than the negative, so that there is every reason why the arc should light in the opposite direction if sufficient negative voltage is at hand. This may be prevented by keeping the positive electrode comparatively cool, and it is also advantageous if the intervening gas is a good conductor of heat. Poulsen's condition (3) is necessary if large quantities of energy are to be emitted in the form of waves, while (4) is a practical device for increasing the steadiness of the arc.

An undamped oscillation is nothing more or less than a high-frequency alternating current, and many attempts have been made to produce the oscillations with a specially-designed alternator. Unfortunately the frequency required (30,000 per second or more) is very high and not easy to reach with a large machine. In Goldschmidt's alternator the problem is greatly simplified by the ingenious device of transforming the frequency automatically inside the machine. A description of the method would however occupy too much space.

A special difficulty arises when we attempt to receive undamped oscillations with a crystal and telephone, which does not occur with trains of damped oscillations. The telephone membrane is moved from its position of equilibrium by the first few oscillations, and a sharp click is heard in the telephone. Subsequent waves, however, only serve to keep the membrane in its displaced position, and there is no further sound till the waves cease, when a second click is heard. Thus a dash on the Morse code is heard as two clicks, and the middle of the signal is not utilised at all. One way out of this difficulty is to place an interrupter in the receiving circuit, which cuts out the telephone periodically and gives a

note of its own frequency during the whole time that the waves are acting. Goldschmidt's recent detector, which works independently of any crystal or rectifying device, is a rotary interrupter whose frequency is of the same order as that of the oscillations. The total charge let through in a given direction while a tooth is in contact with a brush clearly depends on the phase of the oscillations at the moment when contact begins. If the frequency of interruption is m and that of the oscillations n, where $n > m$, the same phase will recur periodically $n - m$ times a second, and a sound of that frequency will be heard in the telephone. Thus if $n = 40{,}000$ ($\lambda = 7500$ metres) a frequency of interruption of 39,000 will give a note of frequency 1000. One of the advantages of this detector is that the note can be easily changed at the receiving station to suit particular telephones.

At the time when the Poulsen arc was introduced it was thought that it would quickly revolutionise wireless telegraphy and oust the spark methods altogether. This belief has not been justified, and it still remains to be seen whether damped or undamped oscillations are most advantageous in the long run. The future of undamped oscillations seems to lie in *wireless telephony*. The principle of this is very simple. In ordinary telephony the resistance of a circuit is varied, in time with the sound-waves falling on a diaphragm, by a loose-contact device (microphone transmitter). In wireless telephony the aerial contains a microphone capable of carrying large currents, and the amplitude of the waves varies, more or less exactly, in accordance with the fluctuations of the voice. The waves are received in the ordinary way with a telephone and an oscillation valve, or suitable crystal. Communication has already been carried on over considerable distances.

202. Theory of resonance curves and measurement of the decrement of electric oscillation circuits. In taking off a resonance curve the only thing that is varied is the capacity of the secondary circuit. The quantity $\mu = R/2L$ is therefore a constant, and it is convenient to work with this quantity at first instead of the actual decrement δ, which is a function of the capacity.

Using the notation of Art. 184, the equations for two coupled oscillation circuits in general are

$$\left.\begin{aligned} L_1C_1\frac{d^2e_1}{dt^2} + MC_1\frac{d^2e_2}{dt^2} + R_1C_1\frac{de_1}{dt} + e_1 = 0 \\ MC_2\frac{d^2e_1}{dt^2} + L_2C_2\frac{d^2e_2}{dt^2} + R_2C_2\frac{de_2}{dt} + e_2 = 0 \end{aligned}\right\}.$$

They may be written

$$\frac{d^2e_1}{dt^2} + 2\mu_1\frac{de_1}{dt} + n_1{}^2e_1 + k_1\frac{d^2e_2}{dt^2} = 0 \quad \ldots\ldots\ldots(37)$$

and

$$k_2\frac{d^2e_1}{dt^2} + \frac{d^2e_2}{dt^2} + 2\mu_2\frac{de_2}{dt} + n_2{}^2e_2 = 0 \quad \ldots\ldots\ldots(38).$$

where $n_1{}^2 = 1/L_1C_1$, $n_2{}^2 = 1/L_2C_2$, $\mu_1 = R_1/2L_1$, $\mu_2 = R_2/2L_2$, $k_1 = M/L_1$, $k_2 = M/L_2$. The coefficient of coupling k is thus equal to $(k_1k_2)^{\frac{1}{2}}$. The initial conditions are

$$\left.\begin{aligned} e_1 = C_1V_0, \quad de_1/dt = 0, \quad e_2 = 0, \quad de_2/dt = 0 \\ \text{when } t = 0 \end{aligned}\right\} \ldots(39).$$

The solution of the differential equations is impracticable as it depends on that of an equation of the fourth degree; but, as has been pointed out by Mandelstam and Papalexi, the mean values required can be obtained by direct treatment of the equations. We require the thermal effect in the secondary for a single train of waves, namely

$$I_2{}^2 = \int_0^\infty i_2{}^2dt = \int_0^\infty \left(\frac{de_2}{dt}\right)^2 dt.$$

Write also $\quad I_1{}^2 = \int_0^\infty \left(\frac{de_1}{dt}\right)^2 dt, \quad K = \int_0^\infty \frac{de_1}{dt}\frac{de_2}{dt}\,dt.$

Then we have

$$\int_0^\infty e_1\frac{d^2e_1}{dt^2}\,dt = -I_1{}^2, \quad \int_0^\infty e_2\frac{d^2e_2}{dt^2}\,dt = -I_2{}^2,$$

$$\int_0^\infty \frac{d^2e_1}{dt^2}e_2dt = \int_0^\infty e_1\frac{d^2e_2}{dt^2}\,dt = -K,$$

and also

$$\int_0^\infty e_1\frac{de_2}{dt}\,dt = -\int_0^\infty \frac{de_1}{dt}e_2dt = P \text{ (say)},$$

$$\int_0^\infty \frac{de_1}{dt}\frac{d^2e_2}{dt^2}\,dt = -\int_0^\infty \frac{d^2e_1}{dt^2}\frac{de_2}{dt}\,dt = Q \text{ (say)}.$$

Multiply equations (37) and (38) by $n_2^2 e_2$ and $n_1^2 e_1$ respectively, integrate and equate. Then using the above results we have

$$n_2^2 K + 2\mu_1 n_2^2 P + k_1 n_2^2 I_2^2 = k_2 n_1^2 I_1^2 + n_1^2 K - 2\mu_2 n_1^2 P \quad \ldots(40).$$

Similarly, by multiplying both equations by de_1/dt and de_2/dt and integrating, we have

$$2\mu_1 I_1^2 - \tfrac{1}{2} n_1^2 C_1^2 V_0^2 + k_1 Q = 0 \quad \ldots\ldots\ldots(41),$$
$$-Q + 2\mu_1 K + n_1^2 P = 0 \quad \ldots\ldots\ldots(42),$$
$$Q + 2\mu_2 K - n_2^2 P = 0 \quad \ldots\ldots\ldots(43),$$
$$-k_2 Q + 2\mu_2 I_2^2 = 0 \quad \ldots\ldots\ldots(44).$$

The equations (40), (41), (42), (43), (44) enable us to eliminate I_1, K, P and Q and determine the required quantity I_2. We thus find

$$I_2^2 = \frac{n_1^4 C_1^2 V_0^2 M^2}{4 L_2^2} \times \frac{\mu_1 n_2^2 + \mu_2 n_1^2}{\mu_1\mu_2(n_1^2 - n_2^2)^2 + 4\mu_1\mu_2(\mu_1+\mu_2)(\mu_1 n_2^2 + \mu_2 n_1^2) + k^2(\mu_1 n_2^2 + \mu_2 n_1^2)^2}.$$

This formula is perfectly general and holds for loose or tight coupling, and even when the discharge is non-oscillatory. A special case of great importance is that in which *k is small and* n_1 *nearly equal to* n_2. Writing I for I_2, n for n_1 and $n + \Delta n$ for n_2, we then find

$$I^2 \propto \frac{\mu_1 + \mu_2}{\mu_1\mu_2\{(\Delta n)^2 + (\mu_1 + \mu_2)^2\}}.$$

We can now introduce the wave-lengths λ and $\lambda + \Delta\lambda$ and the damping coefficients δ_1, δ_2 of the circuits, the latter being taken for the frequency corresponding to λ. We have with sufficient accuracy

$$n = \frac{2\pi c}{\lambda}, \quad \Delta n = -\frac{2\pi c}{\lambda^2}\Delta\lambda, \quad \mu_1 = \frac{c\delta_1}{\lambda}, \quad \mu_2 = \frac{c\delta_2}{\lambda},$$

so that

$$I^2 \propto \frac{\delta_1 + \delta_2}{\delta_1\delta_2} \frac{1}{\left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(\frac{\delta_1 + \delta_2}{2\pi}\right)^2} \quad \ldots\ldots\ldots(45),$$

showing how the effective value of the secondary current depends on the decrements of the primary and secondary circuits and the wave-length to which the secondary is tuned. This formula,

which gives the form of the resonance curve near the peak, is due to Bjerknes.

We have now to show how δ_1 and δ_2 may be determined by experiment. Let y_0 be the maximum ordinate of the resonance curve, y a neighbouring ordinate corresponding to wave-length $\lambda + \Delta\lambda$. Then

$$\frac{y^2}{y_0^2} = \frac{\left(\frac{\delta_1 + \delta_2}{2\pi}\right)^2}{\left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(\frac{\delta_1 + \delta_2}{2\pi}\right)^2},$$

so that

$$\frac{\delta_1 + \delta_2}{2\pi} = \frac{\Delta\lambda}{\lambda}\left(\frac{y^2}{y_0^2 - y^2}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (46).$$

Thus the *sum* of the decrements δ_1, δ_2 can be found by measurement of the resonance curve. To find δ_1 and δ_2 separately, suppose that the maximum ordinate y_0 of the curve is reduced to y_1 by inserting a non-inductive resistance R in the secondary circuit. The effect of this is to increase δ_2 to $\delta_2 + \delta'$, where δ' has the known value $\pi R\,(C_2/L_2)^{\frac{1}{2}}$, C_2 being the capacity of the secondary in electromagnetic units. Thus

$$\frac{y_0^2}{y_1^2} = \frac{(\delta_2 + \delta')\,(\delta_1 + \delta_2 + \delta')}{\delta_2\,(\delta_1 + \delta_2)} \quad \ldots\ldots\ldots\ldots (47),$$

which gives δ_2 since $\delta_1 + \delta_2$ is known from (46). Hence both δ_1 and δ_2 can be found.

In carrying out this method it is necessary to have a very sensitive thermal detector, otherwise the value of k necessary to give readable deflexions may be so great as to vitiate the results altogether. Many interesting results have been deduced from observations of resonance curves by M. Wien and others. Thus it is found that the effect of the spark-gap is to distort the resonance curve as well as flattening it out, so that the value of δ_1 thus obtained is different when different points on the curve are used, and greater than when no spark-gap is present. The latter case can be realised experimentally by the method of quenched sparks (Art. 185). Those metals, such as silver and copper, which show this effect in the greatest degree, also affect the frequency of the oscillations quite appreciably, the periodic time being sometimes

one or two per cent. greater than that calculated from Kelvin's formula. The following table gives the mean decrement and percentage increase of wave-length for 5 mm. spark-gaps of various metals.

Metal	Decrement	Increase of wave-length due to spark, per cent.
Magnesium ...	0·059	—
Cadmium	0·065	0·04
Zinc	0·071	0·08
Aluminium ...	0·073	0·08
Platinum	0·084	0·16
Copper	0·090	0·25
Silver	0·116	0·35

In these experiments care was taken to reduce the other sources of energy-loss, and the decrement of the circuit without spark-gap was less than 0·01.

203. Resistance of wires for high-frequency currents. The exact theory of the distribution of high-frequency currents over the cross-section of round wires was first given by Lord Kelvin. The rate of alternation is not supposed to be great enough for ether currents to come into play, so that we have at every point $j = \sigma E$ where σ is the conductivity of the material of the wire.

Let j, H be the current-density and magnetic force at time t at a point in the wire distant r from the centre. Applying the "work law" to a circle of radius r with its centre on the axis and its plane at right angles to it (Fig. 292) we have

$$2\pi r H = 4\pi \int_0^r 2\pi r j \, dr.$$

Differentiating, we obtain

$$\frac{\partial H}{\partial r} + \frac{H}{r} = 4\pi j \quad \ldots\ldots\ldots\ldots\ldots\ldots (48).$$

Now apply the law of electromagnetic induction to the rectangle bounded by the axis, a parallel line distant r from it, and two

radii vectores at unit distance apart (Fig. 293). Let j_0 be the value of j along the axis. Then there are electric forces j/σ and

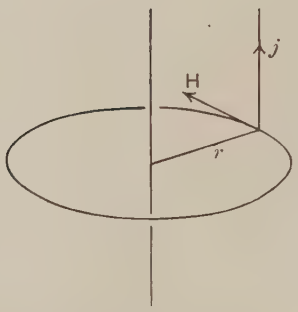


Fig. 292

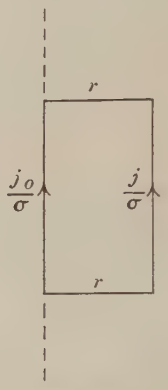


Fig. 293

j_0/σ acting along two sides of the rectangle, while the flux of magnetic force through it is $-\int_0^r H\,dr$. Hence

$$\frac{j-j_0}{\sigma}=\int_0^r \frac{\partial H}{\partial t}\,dr, \quad \text{or} \quad \frac{\partial j}{\partial r}=\sigma\frac{\partial H}{\partial t} \quad \ldots\ldots\ldots(49).$$

Eliminating H we have

$$\frac{\partial^2 j}{\partial r^2}+\frac{1}{r}\frac{\partial j}{\partial r}=4\pi\sigma\frac{\partial j}{\partial t}.$$

The solution corresponding to an alternating current of period $2\pi/p$ is got by writing $j=Re^{ipt}$, where R is a function of r only, and it is understood that the real part is to be taken when the quantities come to be interpreted. On substitution we find

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}=4\pi i\sigma pR,$$

so that $R=AJ_0(kr)$ where A is a constant and $k^2=-4\pi i\sigma p$ (cf. Art. 7). Lord Kelvin defines two new functions $\mathrm{ber}\,x$ and $\mathrm{bei}\,x$ by the equation

$$J_0(x\sqrt{-i})=\mathrm{ber}\,x+i\,\mathrm{bei}\,x,$$

so that
$$\left.\begin{aligned} \text{ber } x &= 1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \cdots \\ \text{bei } x &= \frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \cdots \end{aligned}\right\}.$$

Writing $2r\,(\pi\sigma p)^{\frac{1}{2}} = x$, we have therefore

$$j = A\,(\text{ber } x + i\,\text{bei } x)\,e^{ipt},$$

or
$$j = A\,(\text{ber } x \cos pt - \text{bei } x \sin pt).$$

It follows that the currents vary across the cross-section in phase as well as amplitude. If J is the effective value of the current-density at distance r from the axis,

$$J^2 = \tfrac{1}{2}A^2\,(\text{ber}^2 x + \text{bei}^2 x) \ldots\ldots\ldots\ldots (50).$$

The functions ber and bei have been tabulated and thus J can be found numerically. Fig. 294 shows the distribution of J over

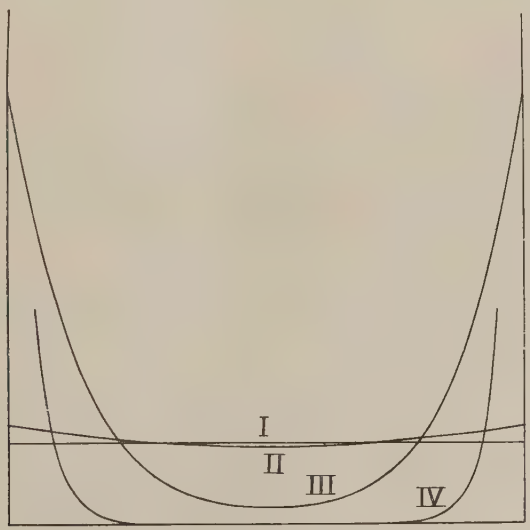


Fig. 294

the cross-section of a copper wire 2 mm. in diameter, curve I being for a frequency of 10^3, curve II 10^4, curve III 10^5 and curve IV 10^6 per second. The total current i in the wire at time t is obtained

by integration over the cross-section: thus $i = \int_0^a 2\pi r j\, dr$, where a is the radius of the wire. Hence

$$i = 2\pi A e^{ipt} \int_0^a r J_0(kr)\, dr$$
$$= -\frac{2\pi a A}{k^2} e^{ipt} \left[\frac{dJ_0(kr)}{dr}\right]_{r=a}$$

from Art. 7. Hence

$$i = -\frac{2\pi a A}{k^2} e^{ipt}\, 2(\pi\sigma p)^{\frac{1}{2}} (\mathrm{ber}'\, x + i\, \mathrm{bei}'\, x)_{x=a},$$

or writing $\xi = 2(\pi\sigma p)^{\frac{1}{2}} a$ for the value of x when $r = a$, the current is

$$\frac{A e^{ipt}}{2i\sigma p}\, \xi\, (\mathrm{ber}'\, \xi + i\, \mathrm{bei}'\, \xi).$$

If I is the effective value of the current,

$$I^2 = \frac{A^2 \xi^2}{8\sigma^2 p^2} (\mathrm{ber}'^2\, \xi + \mathrm{bei}'^2\, \xi) \quad \ldots\ldots\ldots\ldots (51),$$

giving A in terms of I.

The instantaneous rate of development of heat is j^2/σ per cubic centimetre per second. Hence in a volume-element $d\tau$ distant r from the axis the mean rate of development of heat is $J^2 d\tau/\sigma = A^2 d\tau\, (\mathrm{ber}^2\, x + \mathrm{bei}^2\, x)/2\sigma$. Putting

$$d\tau = 2\pi l r\, dr = l x\, dx/2\sigma p,$$

where l is the length of the wire, the heat developed per second in the whole wire is found to be

$$\frac{l A^2}{4\sigma^2 p} \int_0^\xi x\, (\mathrm{ber}^2\, x + \mathrm{bei}^2\, x)\, dx.$$

To evaluate this integral we notice that

$$\mathrm{ber}^2\, x + \mathrm{bei}^2\, x = J_0(x\sqrt{i})\, J_0(x\sqrt{-i}),$$

so that the integral can be found immediately (Art. 7) and becomes

$$-\tfrac{1}{2} i\xi \left[J_0(\xi\sqrt{i})\, \frac{dJ_0(\xi\sqrt{-i})}{d\xi} - J_0(\xi\sqrt{-i})\, \frac{dJ_0(\xi\sqrt{i})}{d\xi}\right]$$
$$= \xi\, (\mathrm{ber}\, \xi\, \mathrm{bei}'\, \xi - \mathrm{ber}'\, \xi\, \mathrm{bei}\, \xi).$$

Thus the heat developed per second is

$$\frac{lA^2\xi}{4\sigma^2 p}(\text{ber}\,\xi\,\text{bei}'\,\xi - \text{ber}'\,\xi\,\text{bei}\,\xi) \quad \ldots\ldots\ldots(52).$$

The *apparent resistance* R of the wire is defined so that the rate of heat-development is RI^2. Hence by comparison with (51) we find

$$R = 2lp\,\frac{\text{ber}\,\xi\,\text{bei}'\,\xi - \text{ber}'\,\xi\,\text{bei}\,\xi}{\xi\,(\text{ber}'^2\,\xi + \text{bei}'^2\,\xi)}.$$

The resistance for low frequencies is $R_0 = l/\pi\sigma a^2$. Hence

$$\left.\begin{aligned} \frac{R}{R_0} &= \frac{\xi\,(\text{ber}\,\xi\,\text{bei}'\,\xi - \text{ber}'\,\xi\,\text{bei}\,\xi)}{2\,(\text{ber}'^2\,\xi + \text{bei}'^2\,\xi)} \\ \text{where} \qquad \xi^2 &= 4\pi\sigma p a^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(53).$$

The quantity R/R_0 is that measured by calorimetric experiments (Art. 180). Some idea of its magnitude will be obtained from the following table, for copper wires.

Frequency	R/R_0	
	Diameter 2 mm.	Diameter ·2 mm.
10	1·000	1·000
10^2	1·000	1·000
10^3	1·000	1·000
10^4	1·144	1·000
10^5	8·67	1·000
10^6	84·2	1·144
10^7	839	8·67

The formula (53) seems to be in good agreement with experiment. Thus for a certain copper wire Fleming found $R/R_0 = 6{\cdot}62$, while theory gave 6·64. The smaller the wire the less the fractional increase of resistance for a given frequency. Moreover, the increase is less for materials of low conductivity than for materials of high conductivity like copper. These results follow from the fact, which is easily proved, that R/R_0 is a function of n/ρ only, where n is the frequency and ρ the resistance of the wire per unit length.

CHAPTER XII

CONDUCTION OF ELECTRICITY THROUGH GASES

204. Discharges in rarefied gases. It has already been pointed out that the insulation of the air separating two charged conductors will break down when the difference of potential between the conductors exceeds a certain amount, and a discharge will then pass between them. The electric discharge in air and other gases is most conveniently studied at a pressure of 5 mm. of mercury or less, as the potential required to produce the discharge is then much less than at atmospheric pressure, unless the electrodes are very close together. The necessary arrangements for manipulating air at low pressures are shown in Fig. 295. It is assumed that dry air has first been admitted and then pumped out to a pressure of about 1 cm. of mercury with an ordinary air-pump. Further exhaustion is obtained very simply by means of the Töpler mercury pump. Suppose that the mercury reservoir A, connected to the upright glass tube B by flexible rubber tubing, is first of all lowered until the mercury in the tube is at the level of C. The chamber D is then in connexion with the main apparatus, but on raising the reservoir A this connexion is cut off as soon as the mercury reaches the level of E. A little higher up a glass float F begins to be raised by the mercury, as shown in the figure, and is subsequently pressed against a constriction G in the tube, which jams it tight. On further raising the reservoir the air in D is forced down a long capillary tube into a test tube H, where it escapes into the air. When the reservoir is lowered the space above the mercury in D becomes a Torricellian vacuum, and by the time the mercury is again at C air has diffused from the main apparatus into D, and the process

can be repeated again and again. In this way it is an easy, though slow, process to reduce the pressure to $\frac{1}{100}$ mm. of mercury or less.

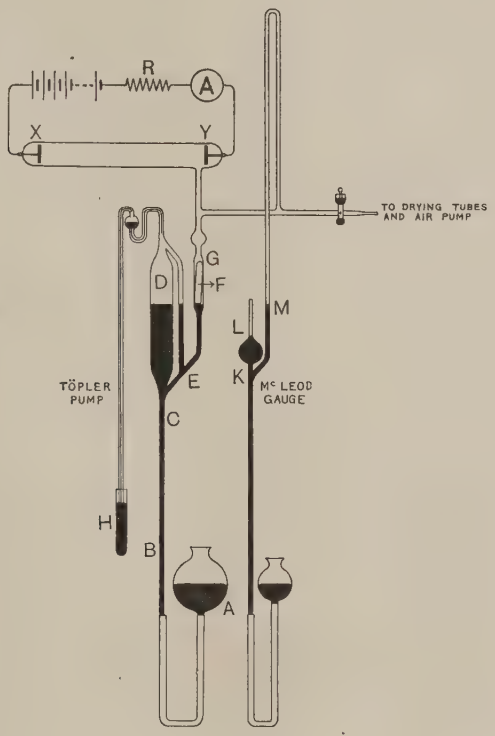


Fig. 295

In order to measure these low pressures the McLeod gauge is used. The level of the mercury is slowly raised until it stands at the branching point K of the two tubes. At this point the pressure in both branches is the pressure p in the main apparatus. As the mercury rises the air in the left-hand tube is compressed while the pressure on the right remains practically the same, so that the level in the two arms becomes different. Let v be the volume above a fixed mark L in the left-hand tube, V the volume of bulb and tube down to the branching point K. Then when the mercury stands at L the pressure in the left-hand tube is pV/v and that in the right-hand tube p. Hence the difference of

level of L and M is $p\left(\frac{V}{v}-1\right)$ millimetres, where p is the pressure in millimetres of mercury. The volumes V and v having been determined beforehand, a graduated scale can be attached to the tube M giving the pressure in fractions of a millimetre. A ratio $V/v =$ about 100 is most generally convenient, giving a difference of level of about 100 millimetres for every millimetre pressure.

The discharge tube may be about 4 cm. in diameter and about 20 cm. long, containing two aluminium discs X, Y to serve as electrodes. The necessary potential is furnished by about 500 small accumulators*, in series with a resistance R of the order of 50,000 ohms and a milliammeter A reading up to about $\frac{1}{100}$ ampere. In some cases the discharge does not take place when the battery is connected to the electrodes, since the potential required to start a discharge may be much greater than that required to maintain it. In order to get over this difficulty the discharge may be started with an induction coil.

The appearance of the vacuum-tube discharge is very striking, and has several well-marked features, of which the most prominent are the following (cf. Fig. 297). The colours refer to air.

A a bright pink layer of gas at the surface of the anode, or positive electrode.

B the positive column, a column of luminous gas extending for a considerable distance l from the positive electrode and terminating abruptly at a certain distance d from the cathode. For a given current and pressure d is constant, and l increases with the distance between the electrodes.

C the Faraday dark space, merging into

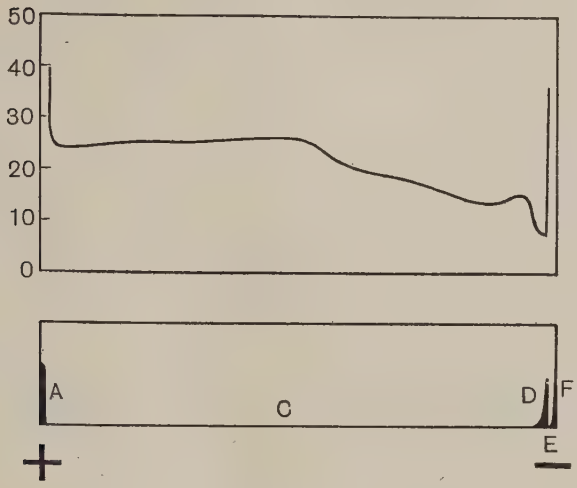
D the negative glow (pale violet).

E the Hittorf (or Crookes) dark space.

F a red or orange glow adjacent to the cathode.

The four main stages of the discharge, as the pressure is reduced, are admirably illustrated by Graham's experiments with nitrogen, which were made primarily in order to determine the electric force at different points of the gas. This was done by means of two wires placed close together along the axis of the tube. In

* A small direct-generator giving about 2000 volts would be extremely useful for this class of work.

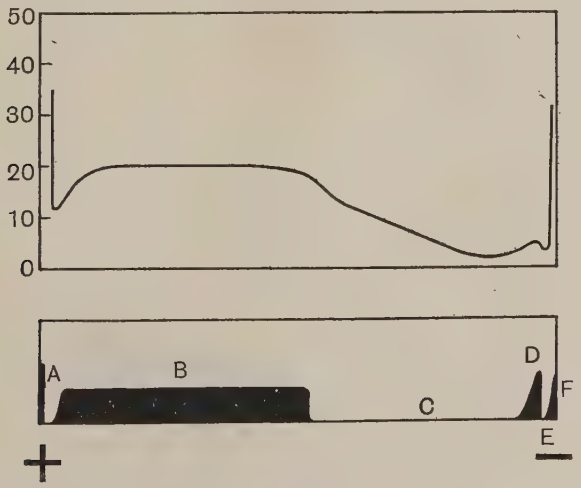


Pressure 2·5 mm. Current 2·39 × 10^{-3} amp.

Fig. 296

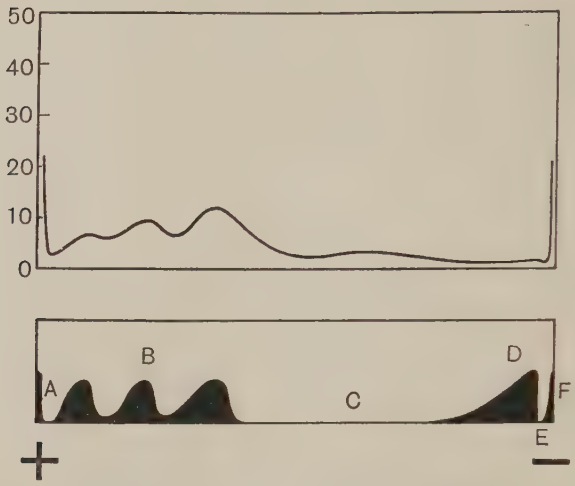
the accompanying figures the upper diagram gives the electric force in volts per centimetre at any point, the lower showing the approximate distribution of the intensity of light along the tube. The tubes were about 20 cm. long.

At the highest pressure (Fig. 296) there is no light at all



Pressure 0·95 mm. Current 2·46 × 10^{-3} amp.

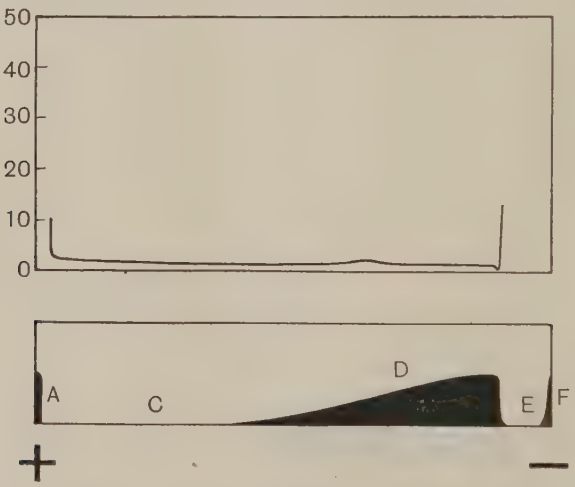
Fig. 297



Pressure 0·63 mm. Current $1{\cdot}16 \times 10^{-3}$ amp.

Fig. 298

except near the electrodes, and the Hittorf dark space is very narrow. As the pressure is reduced (Fig. 297) the positive column appears, perfectly uniform except near the ends. Further



Pressure very small (probably about 0·02 mm.)
Current $3{\cdot}3 \times 10^{-4}$ amp.

Fig. 299

rarefaction makes it break up into striations, or alternate bright and dark portions concave to the anode (Fig. 298), the process beginning at the end nearest the cathode. If the tube is long its appearance is exceedingly striking, as the striated column fills the greater part of the tube. The negative glow and Hittorf dark space expand continually as the pressure is reduced, until they push back the positive column and ultimately occupy the greater part of the tube (Fig. 299). The light in the tube becomes extremely faint and the electric force low, except near the electrodes. From this point the potential required to produce a discharge increases rapidly, and it becomes necessary to use an influence machine or induction coil instead of a battery of accumulators. As these low pressures are somewhat difficult to measure it is more generally convenient to specify the state of the tube by the potential required to produce the discharge.

205. Cathode rays. A remarkable change takes place in the appearance of the discharge in an ordinary-sized tube when the pressure is reduced still further, until the discharge potential exceeds about 3000 volts. The cathode glow becomes concentrated more and more at the centre of the cathode, until finally a narrow pencil of light leaves the surface at right angles. Striking experiments on this subject were made by Hittorf in 1869, using as cathode a wire with a flat end, enclosed in a tightly fitting glass

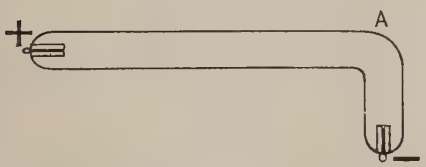


Fig. 300

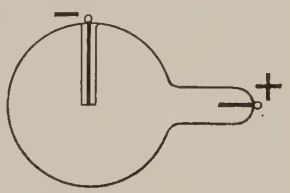


Fig. 301

tube, so that the available area was very small. With a tube of the form shown in Fig. 300 he found that the cathode rays, as they are called, did not bend round at the angle, but passed straight on and gave rise to a fluorescent spot on the glass near A. An obstacle in front of the cathode cast a shadow at A. In another experiment Hittorf used a tube like that shown in Fig. 301,

and showed that the rays were deflected by a magnet, sometimes into circles but more generally into spirals.

Before explaining these results let us consider what would happen to a charged particle projected in a uniform magnetic field of strength H. We have already shown (Art. 95) that it can reasonably be supposed to be acted on by a force equal to $\frac{e}{c}$ times the vector product of v and H, where v is the velocity and e the charge in electrostatic units. Taking the direction of the field as axis of z, the equations of motion of the particle are

$$m\frac{d^2x}{dt^2} = \frac{eH}{c}\frac{dy}{dt}, \quad m\frac{d^2y}{dt^2} = -\frac{eH}{c}\frac{dx}{dt}, \quad m\frac{d^2z}{dt^2} = 0.$$

The last equation shows that the velocity along the axis of z is uniform, and the first two equations are satisfied by

$$\frac{dx}{dt} = A\sin(\omega t + \epsilon), \quad \frac{dy}{dt} = A\cos(\omega t + \epsilon),$$

where $\omega = He/mc$ and A, ϵ are constants. We may suppose, without loss of generality, that the particle starts from the origin at time $t = 0$ with velocity $(v, 0, v')$. Thus $A = v$ and $\epsilon = \frac{1}{2}\pi$. Integrating, we have

$$x = \frac{v}{\omega}\sin\omega t, \quad y = -\frac{v}{\omega}(1 - \cos\omega t), \quad z = v't \ldots . (1).$$

The path in general is a helix, reducing to a circle if the initial velocity is at right angles to the magnetic field, and to a straight line if the two are parallel. The direction of description of the curves is right- or left-handed with reference to the axis of z according as ω is negative or positive.

These facts are, qualitatively, in exact agreement with Hittorf's observations, and in addition he found that the spirals were always described in a right-handed direction with respect to the magnetic field. The explanation of Hittorf's experiments which is now universally accepted is that *the cathode rays are negatively charged particles emanating from the cathode, which, by the time they have travelled some distance from it, have acquired a considerable velocity.* Their origin and velocity of emission remain thus far unknown.

In 1895 Perrin proved directly that the cathode rays carry a negative charge. In his discharge tube the anode A (Fig. 302),

which was connected to earth, was perforated so as to let the rays through, and nearly closed at the back in order to protect the space behind from electrostatic action. An insulated cylinder B

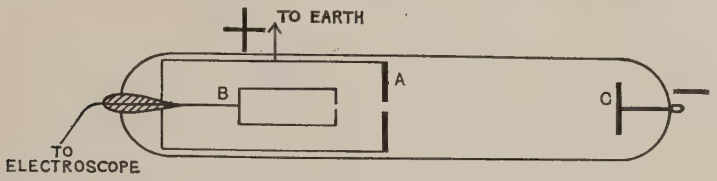


Fig. 302

is placed behind the anode and connected to an electroscope. The cathode rays from C, after passing through A, enter B through a small hole and charge the electroscope negatively. The effect is very large, as a single interruption of the coil gave enough charge to raise the potential of the insulated system to 300 volts. By bringing a magnet near the tube the cathode stream could be deflected, without stopping the discharge, until no electricity entered B.

206. Measurement of e/m for the cathode particles. Electrons. The path of a particle initially moving with velocity v along the axis of x is a circle of radius

$$\rho = \frac{v}{\omega} = \frac{mcv}{He}.$$

This follows at once from equation (1), but can be proved independently by elementary methods. For the force on the moving charge is always at right angles to its path, so that $mv^2/\rho = Hev/c$. Hence

$$\frac{e}{m} = \frac{cv}{\rho H} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It follows that if the velocity of the cathode particle is measured in any way the ratio e/m of its charge to its mass is also known, since ρ may be found from the magnetic deflexion produced by a known force H. This was first done by Wiechert in 1897. In his experiments the cathode rays, after passing the ring-shaped anode A (Fig. 303), fell on a sheet of glass G placed behind a diaphragm, forming a fluorescent spot. They were deflected near the anode by a loop of wire L_1 in an electric oscillation circuit,

so that they were broadened out into a fan-shaped beam. By means of an auxiliary magnet the beam could be further deflected and its edge brought precisely on the screen once more. A second

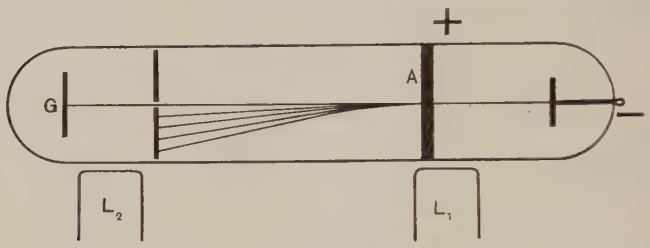


Fig. 303

loop of wire L_2 in the electric oscillation circuit was now brought near the beam between the screen and diaphragm, and the additional deflexion observed on the screen.

In order to see what is to be expected, suppose that the current in the oscillatory circuit at time t is $A \sin pt$, where $p\ 2\pi$ is the frequency of the circuit, and let l be the distance between the loops L_1 and L_2. The only rays that get through the diaphragm are those which are most deflected by the first loop; those, say, which passed it at the time t given by $pt = \frac{1}{2}\pi$, etc. The first ray passes the loop L_2 at time $t = \frac{\pi}{2p} + \frac{l}{v}$, by which time the current in the loop has become $A \sin p\left(\frac{\pi}{2p} + \frac{l}{v}\right) = A \cos \frac{pl}{v}$. Similarly for all the other beams. It follows that the deflexion produced by the loop L_2 will be in the same direction as that produced by L_1, as long as pl/v is less than $\frac{1}{2}\pi$. With $l = 20$ cm., Wiechert found that this was the case for frequencies up to 2×10^7, showing that v was greater than $1{\cdot}6 \times 10^9$ cm. per second. For these rays ρH was equal to 150. Hence from (2) we find

$$\frac{e}{m} > 3{\cdot}2 \times 10^{17}.$$

With the frequency 4×10^7 the deflexion seemed to be much reduced, and Wiechert concluded that the value of e/m did not much exceed 5×10^{17}.

Wiechert's experiments lead to one extremely important conclusion. We have seen (Art. 166) that if e is the charge on a univalent ion in electrolysis (the smallest charge known in electrolysis) and m_H the mass of a hydrogen atom, then

$$\frac{e}{m_H} = 2{\cdot}87 \times 10^{14}.$$

The value of e/m for cathode particles is about a thousand times as great. Among the infinity of possible ways of explaining this we may mention two limiting hypotheses as being most probable:

(1) The charge on the cathode particle is the same as that on a univalent ion in electrolysis, and its mass very much smaller.

(2) The mass of the cathode particle is comparable with that of a molecule, and its charge about a thousand times as large as that on an ion.

Wiechert adopted the first hypothesis, and was thus led to assert the existence of *particles smaller than the atom.* Exact experimental evidence will be given later as to the probable equality of the charges in the two cases.

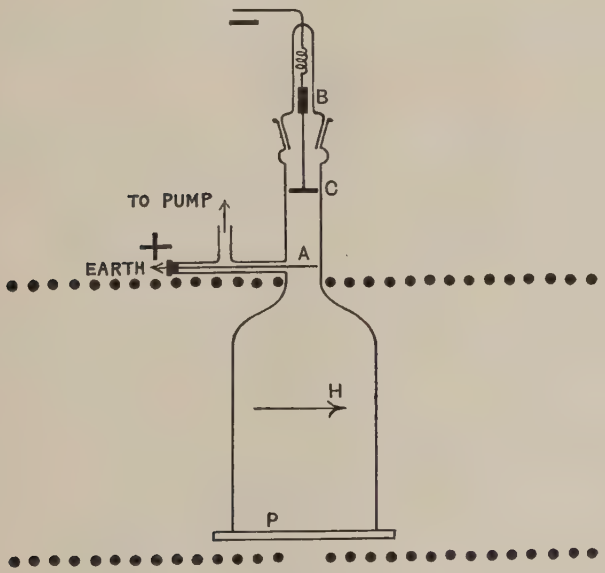


Fig. 304

Shortly after Wiechert's discovery, Kaufmann made a more accurate determination of the value of e/m for the cathode particles. The cathode C in his discharge tube (Fig. 304) was a plate of aluminium or copper, the anode A a platinum wire connected to earth. There was a large space behind the anode, screened to protect it from electrostatic action and enclosed between two solenoids placed so as to give an approximately uniform field H over a considerable length. The tube was closed by a prepared glass plate P which fluoresced under the action of the rays, the anode A casting a sharp shadow on it. The position of the cathode could be varied within certain limits by moving the iron cylinder B up and down with a magnet. A Wimshurst machine was used to produce the discharge, and the difference of potential V between A and C was measured with an electrostatic voltmeter.

Suppose for simplicity that the magnetic field is uniform and the deflexion small. On entering the field the particle has fallen through a potential V, so that its velocity v is given by

$$\tfrac{1}{2}m(v^2 - v_0^2) = eV,$$

where v_0 is the velocity on leaving the cathode. Kaufmann assumed provisionally that v_0 was negligible in comparison with v. Thus

$$mv^2 = 2eV \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where V is supposed to be measured in electrostatic units. If l is the distance travelled in the magnetic field before reaching the plate P, and δ the deflexion, regardless of sign, then equation (1) gives approximately

$$l = \frac{v}{\omega} \cdot \omega t, \quad \delta = \frac{v}{\omega} \cdot \frac{\omega^2 t^2}{2}.$$

Eliminating t, and remembering that $\omega = He/mc$, we find

$$\delta = \frac{He}{mc} \frac{l^2}{2v} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Substituting for v from (3), we have

$$\delta = \frac{Hl^2}{2c}\left(\frac{e}{2mV}\right)^{\frac{1}{2}},$$

or

$$\frac{V^{\frac{1}{2}}\delta}{H} = \frac{l^2}{2c}\left(\frac{e}{2m}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

This equation may be tested by seeing whether $V^{\frac{1}{2}}\delta/i$ remains constant under various conditions, i being the current in the solenoids. The following results were found by Kaufmann.

Gases	Cathode	Voltage	Distance between electrodes	$\frac{V^{\frac{1}{2}}\delta}{i}$ (mean)
Air	Aluminium	5920–11600	6	396
Air	Copper	4350–10630	6	393
Air	Copper	—	3·95–6	406
Coal-gas ...	Copper	6410–11850	5·8	401·5
Carbon dioxide	Copper	3870–11100	5·8	398
Hydrogen ...	Copper	4230–14400	5·8	404

The possibility of appreciable initial velocity is therefore excluded, and in addition we see that the value of e/m is *independent of the nature of the gas in the tube and of the material of the cathode.*

Equation (5) can easily be extended to small deflexions in a non-homogeneous field. Since $x = vt$ the equation of motion in the y direction becomes

$$mv^2\frac{d^2y}{dx^2} = \frac{Hev}{c},$$

or

$$\frac{d^2y}{dx^2} = \frac{eH}{mcv},$$

H being the magnetic field at distance x from the origin, which is taken at a point where the potential is nearly zero and H small.

Hence

$$\delta = \frac{e}{mcv}\int_0^l dx\int_0^x H\,dx$$

$$= \frac{1}{c}\left(\frac{e}{2mV}\right)^{\frac{1}{2}}\int_0^l dx\int_0^x H\,dx.$$

Using this formula Kaufmann found that

$$\frac{e}{m} = 5{\cdot}31 \times 10^{17}.$$

In what follows we shall adopt the value

$$\frac{e}{m} = 5{\cdot}30 \times 10^{17} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

which is the mean of the best recent determinations.

Since the cathode particles, as has been seen, are the same whatever their origin, we have evidence of a new fundamental unit or carrier of negative electricity, disproportionately light in comparison with its charge. These particles are called *electrons*.

The part played by electrons in physical phenomena will become more and more clear as we go on, but as far as we know the ratio of their charge to their mass is always that given by equation (6), from which we conclude that all electrons are the same. No evidence has yet been found of the existence of positive charges of less than atomic mass, so that there appears to be no such thing as a positive electron*. This justifies the mode of treatment that has already been adopted in the elementary parts of the subject.

The cathode stream has the valuable property of following almost instantaneously any change in the applied magnetic field. This has been made use of in Braun's cathode ray tube, which somewhat resembles Wiechert's tube in Fig. 303. After passing through a diaphragm the rays fall on a fluorescent screen of willemite or barium platinocyanide, producing a bright spot. An alternating current in a neighbouring coil makes the spot of light move rapidly to and fro, and with a revolving mirror, or other device, the actual form of the alternating current can be photographed.

207. Röntgen rays. We shall now give a brief account of the properties of the Röntgen rays, which, as is well known, originate in a discharge tube at very low pressures. A common form of Röntgen ray bulb is shown in Fig. 305. The cathode is of aluminium, hollow in order to focus the cathode rays on the metal plate T, known as the *target*. This also acts as anode, but it is found advantageous to have a second anode of aluminium on the axis of the tube behind T. As it will appear shortly that the target is the source of the Röntgen rays, it is the most important part of the tube. It is made of a metal of high atomic weight, such as platinum, rhodium or tungsten. Since cathode rays have a very considerable heating effect the melting point must be high, and for heavy discharges (10^{-3} ampere or more) some cooling

* See Art. 225.

arrangement should be added. With prolonged use Röntgen ray bulbs become more and more exhausted, probably on account of the occlusion of gas by the walls of the bulb. More gas can be

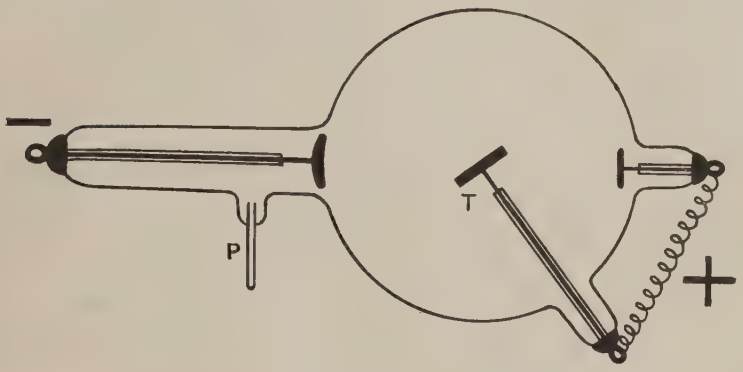


Fig. 305

admitted at any time by means of a palladium tube P closed at one end and sealed into the glass. Such a tube is quite air-tight at ordinary temperatures, but lets hydrogen through freely when hot. On heating the tube very gently in a Bunsen burner free hydrogen, which always exists in the flame, passes into the bulb and raises the pressure. The pressure in the bulb is an important factor in determining the quality of the rays. If it is high (discharge potential 20,000 volts or less) the rays are *soft*; that is, they are easily absorbed by matter. If it is low the rays are *hard* or penetrating; but below a certain point it becomes difficult to get the discharge to pass through the bulb at all.

The photographic action of Röntgen rays is well known. For demonstration purposes it is more convenient to make use of their power of exciting fluorescence in various substances. Thus a screen of barium platinocyanide will light up brightly in the path of the rays, even when covered at the back with black paper. If the hand is held at the back of the screen the bones are seen clearly, and the absorption of various substances may be observed by the shadows that they cast on the screen. Using a bulb of moderate hardness, we thus find that while $\frac{1}{2}$ mm. of lead is opaque to the rays, 4 cm. of aluminium still lets through about one-third. A sheet of aluminium $\frac{1}{3}$ mm. thick is absolutely

transparent, and invisible on the screen. Glass and quartz are about as opaque as aluminium, and ebonite is almost transparent in a sheet 1 mm. thick. It is clear, therefore, that while the absorption of Röntgen rays in various substances is very different from that of light, it does not depend markedly on whether the substance is a conductor or not. It is much more of a mass-phenomenon, depending on whether the substance is light or heavy.

The sharpness of the shadows shows that the source of the rays is very nearly a point. A sheet of lead perforated with a small hole will give rise to quite a small spot, even when the screen is some distance behind the hole. With two such sheets we can find the position of the source within the bulb, and with three we can prove the rectilinear propagation of the rays. We thus find that the rays emanate from the *centre of the target*, i.e. the point of impact of the cathode particles.

A fluorescent screen shielded from the direct action of the rays may be made to glow faintly when a sheet of metal or other substance is placed near it in the path of the beam. This may be due either to rays diffusely reflected from the surface or to some analogous radiation excited in the substance. For the present we may call them *secondary rays*, without stopping to inquire further what they are.

The Röntgen rays show no trace of regular reflexion at polished surfaces, or of refraction by prisms, etc. Nevertheless, it has long been suspected that they are actually waves in the ether, emitted by the target under the impact of the cathode particles. The absence of regular reflexion and refraction is then to be ascribed to the shortness of the wave-length. This theory, which is due originally to Schuster, has recently been completely vindicated by Friedrich and Knipping, who, acting on a suggestion of von Laue, observed the effects produced by the transmission of Röntgen rays through crystals. Laue's idea was that the regular arrangement of atoms in a crystal might make it act like a diffraction grating, each atom scattering the rays and the resulting wavelets interfering so as to reinforce each other in certain directions and destroy each other in others. A narrow beam of rays, after passing through a plate of crystal, should

therefore give rise to a diagram of discrete spots. The exact theory will be given in Art. 227. Fig. 306 shows the position of the strongest spots on a photographic plate behind a crystal of

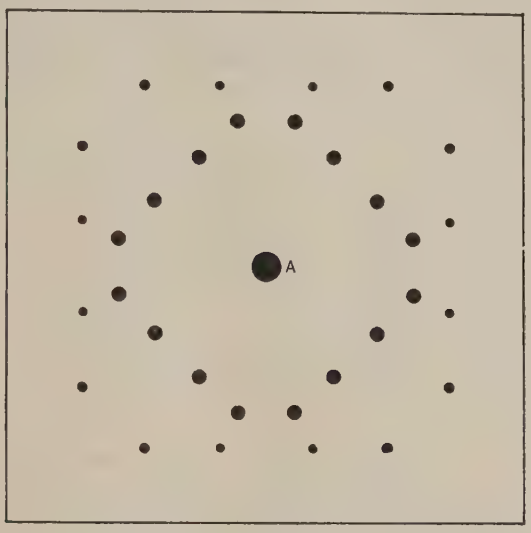


Fig. 306

zinc blende, cut parallel to one of the natural cube faces. The central spot A arises from rays that have not suffered any diffraction. Other crystals, such as quartz, give different but very beautiful patterns. The wave-lengths involved, as we shall see, are of the order of 10^{-8} cm.; but we are here chiefly concerned with the fact that it definitely proves the wave-like nature of Röntgen rays, since the spots cannot be simply accounted for on any other hypothesis.

Crystal diffraction may be demonstrated on a fluorescent screen, when the rays are sufficiently strong. For this purpose the Coolidge bulb (see Art. 226) is very suitable. It is advisable to use fairly large apertures (5 mm. diameter) and to rest the eye by remaining some minutes in the dark before observing. The experiment in this form is very striking, as the configuration of the spots changes rapidly when the crystal is turned about a vertical axis.

208. Normal and abnormal conductivity of the air. Giese's theory. If an ordinary electroscope is left to itself for some days, the leaves sink gradually; but there is nothing to show whether this is due to passage of electricity through the air or leakage along the supports. After allowing for the latter it has been shown beyond doubt that atmospheric air conducts appreciably, but very slightly. Fig. 307 shows an arrangement,

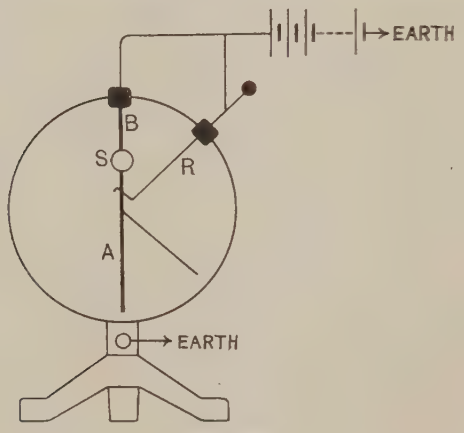


Fig. 307

due to C. T. R. Wilson, by which the leak over the insulators is entirely eliminated, and unequivocal results obtained. The gold leaf is attached to a metal rod A, insulated from the leading-in wire B by a small sulphur bead S. The leaf is charged up to a high potential (of the order of 100 volts) by a moveable rod R connected to one end of a battery of accumulators, the other end of which, together with the case of the electroscope, is earthed. The leading-in wire B is maintained throughout at the same potential as R. If now the potential of A falls on account of leakage through the air, electricity would tend to pass, if at all, from B to A and diminish the rate of fall of the leaves. Initially this effect is zero since A is at the same potential as B.

With this apparatus the amount of electricity lost in a given time is the same whether the charge is positive or negative, and independent of the potential of the battery from about 50 volts

upwards. This fact we shall find to be significant. The current is normally of the order of 10^{-8} electrostatic units per cubic centimetre of the vessel.

It will appear shortly that there are several ways of obtaining larger currents in air. The conductivity of air in the neighbourhood of flames is especially interesting, as it first led Giese, in 1882, to the modern theory of conduction in gases. It has long been known that the air which rises from a flame conducts electricity, so that a current may be made to pass between two electrodes on either side of the stream, whether the electrodes are warm or not. The main facts about this conduction are:

(1) The conductivity persists for some little time after the gas has left the flame, since currents may be detected in a stream of air which has risen to a considerable height above the flame.

(2) The current between a pair of electrodes is diminished by applying an electric force between two other electrodes lower down the stream.

(3) Ohm's law is not obeyed, the current being too small for the higher forces.

Giese tried to assimilate the conduction in gases as far as possible to that in electrolytes; that is, to explain it as a convection by charged particles, called *ions*, not necessarily identical with those occurring in the electrolysis of liquids. Giese supposed that ions are being constantly produced in a flame, either on account of the high temperature or as a result of chemical action. Only very few are present in air at ordinary temperatures, and give rise to the normal conductivity already described. When a mass of gas rises from a flame, it becomes cool and the formation of new ions practically ceases. At the same time those formed in the flame tend to *recombine*; that is, when a positively charged ion collides with a negatively charged one it coalesces with the formation of a neutral atom or molecule. The conductivity of the gas, if left alone, will therefore diminish more or less rapidly, but the last traces remain for some time, since the chance of a collision between the residual ions ultimately becomes very small. Any cause tending to remove ions from a stream of gas, such as a transverse electric force, will clearly weaken the conductivity obtainable at further points in the stream. As regards (3), it is

clear that there must ultimately be a limit to the rise of current with electric force, which occurs when all the ions which enter the space between the electrodes are taken away before they have time to recombine. The smaller the applied electric force the more time there is for recombination, and the less the current.

209. Ionisation by Röntgen rays. Shortly after Röntgen's discovery Benoist and Hurmuzescu found that Röntgen rays would rapidly discharge positively or negatively charged bodies in their neighbourhood. This affords the most generally convenient method of ionising air or other gases. Since the current does not often exceed 10^{-10} amperes, it cannot be measured with an ordinary galvanometer: but it can be measured with a quadrant electrometer as shown in Fig. 308. The Röntgen ray bulb,

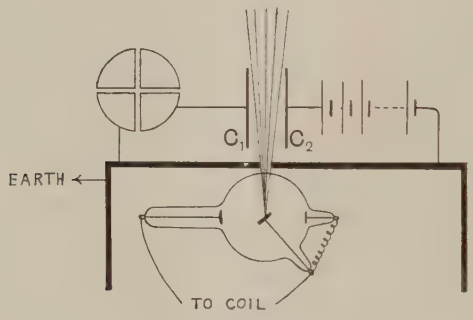


Fig. 308

worked off an induction coil, is placed inside a wooden box covered with lead about 2 mm. thick in order to prevent rays from escaping in all directions. The box also acts as an electrostatic screen. A pencil of rays passes through a hole in the lid and afterwards traverses the space between two metal plates C_1, C_2, placed parallel to one another a centimetre or two apart. To avoid considering the secondary radiation we may suppose that the rays touch neither of the plates. The plate C_1 is connected to one pair of quadrants of an electrometer, the other pair of which is earthed, and C_2 is maintained at a potential of the order of 40 volts by means of a battery of small accumulators (Art. 58).

In making an experiment the electrometer is first insulated and the zero read. The rays are then put on for a measured time t by pressing down a key in the primary circuit of the induction coil. The charge received by the plate C_1 is shown by the deflexion of the electrometer, and the final position of rest of the needle is observed after cutting off the rays. If C is the capacity of the insulated system, the current is $i = CV/t$, where V is the potential required to give the same deflexion of the needle. As an example, suppose that $C = 50$ electrostatic units and that a deflexion of 200 scale divisions is obtained in 10 seconds, 80 divisions corresponding to 1 volt. Then $i = 1{\cdot}4 \times 10^{-11}$ amperes; from which it is evident that much smaller currents might be detected in this way.

A form of apparatus which has been found useful for many experiments on the conductivity of gases is shown in Fig. 309.

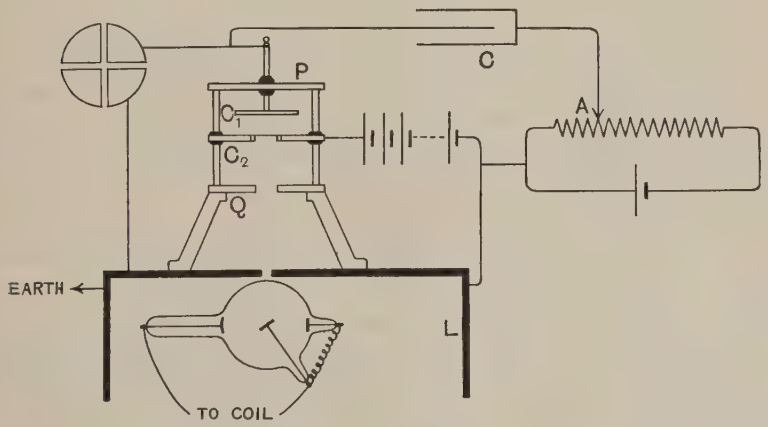


Fig. 309

A perforated brass plate Q is fixed on a solid metal tripod, of which two legs are shown, and rests on the earthed lead box L. Three vertical brass rods serve to support a parallel plate P, the whole frame thus being kept automatically at zero potential. The method of supporting and insulating the plates C_1, C_2 will be evident from the figure. The Röntgen rays pass through a thin sheet of aluminium occupying the centre of the plate C_2,

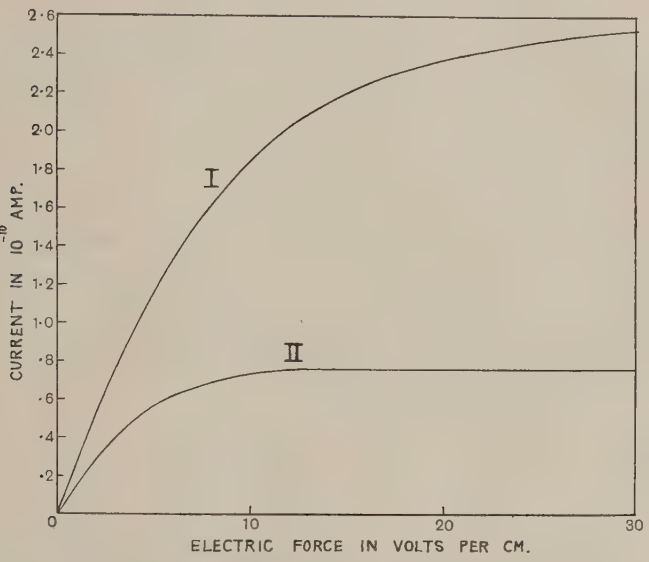
and afterwards fall on C_1*. However great the difference of potential between the plates C_1 and C_2 may be, the insulation of C_1 has only to withstand the potential to which C_1 may rise, which is of the order of 1 volt. This is the important principle of split insulation. If C_1 was supported directly from C_2 the leak over the supports might vitiate the experiments.

The currents may be found in absolute measure by a method due to Townsend. The inside plate of a condenser C is connected with the electrometer, the outside plate being joined to a moveable point A on a sliding potentiometer (Fig. 88). A suitable form of potentiometer is one having 50 junctions and a total resistance of about 1000 ohms. While the current is flowing the slider is moved so as to keep the electrometer permanently near the zero, and after it has ceased the position for zero deflexion is found accurately by the method of proportional parts. Let V be the corresponding potential of the outside plate. Then since neither the electrometer nor the plate C_1 has received any access of charge, the whole of the electricity has gone into the inside plate of C_1, and is of amount CV, where C is the capacity of the condenser. The current is therefore given by $i = CV/t$. Since the capacity of the insulated system does not appear in this equation, the method is valuable in experiments in which the distance between the plates is altered.

The relation of current to applied potential in a particular case is shown in Fig. 310. Here curve I is the ordinary curve obtained with an unscreened beam of Röntgen rays, curve II that obtained when a sheet of aluminium $\frac{1}{2}$ mm. thick is placed in the path of the beam. These curves, which were first obtained by Sir J. J. Thomson and Sir E. Rutherford, are called *saturation curves* from their external resemblance to the magnetisation curves of iron. Their explanation on Giese's theory has already been outlined (Art. 208). If n_0 positive ions, say, each carrying a charge e, are produced between the plates per second, the current for large forces is $n_0 e$, since all the ions are taken away before they have time to recombine. This current is called the *saturation current*. It should be noticed that *the weaker the ionisation*,

* For the effect of this see Arts. 230, 231.

the less the electric force required for saturation. It is easy to see why this should be, since the rate at which ions recombine is proportional to the square of the number present per c.c., and



Distance between plates 4 cm., cross-section of beam of Röntgen rays 1 sq. cm.

Fig. 310

with weak ionisation the ions can be left for a longer time in the gas without fear of their recombining. With very weak ionisation the current for the lower electric forces is reduced not so much by recombination as by diffusion, by which they may come into contact with the opposite electrode from that towards which they are directed by the force, and henceforth cease to act as carriers of the current.

210. The motion of ions in gases. The motion of ions in liquid electrolytes having already been discussed fully (Art. 174), we have only to quote the results, with the remark that here it is much easier to justify the application of the kinetic theory of gases.

Taking all the free paths of the ions equal to l, and all the velocities of agitation equal to V, the *mobility* of an ion, or its velocity under unit electric force, is

$$u = \frac{el}{2mV} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

The *coefficient of diffusion* is

$$K = \tfrac{1}{6} lV \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and

$$\frac{u}{K} = \frac{Ne}{\Pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

N being the number of molecules of a gas at 15° C. and pressure Π, e the charge on the ion.

The general equation regulating the motion of the ions is

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv_x) + \frac{\partial}{\partial y}(nv_y) + \frac{\partial}{\partial z}(nv_z) = K\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right) \quad \ldots (10),$$

$v\,(v_x, v_y, v_z)$ being the velocity in the electric field of the ions in the neighbourhood of (x, y, z), n the number of ions per c.c. at the point (x, y, z).

These formulae only apply when the velocity of drift is small in comparison with the velocity of agitation.

When ions come into contact with a metal surface they give up their charge to it, and do not return to the gas as charged ions. There is thus a certain condition to be satisfied at the boundary of the metal, which we proceed to find. Let the element dS of the surface of the metal be at the origin of co-ordinates, and take the axis of z normal to the surface and pointing inwards towards the metal. Suppose first of all that the metal above $z = 0$ is removed, allowing free motion to the ions. It can be shown that the number of ions crossing dS per second from either side on account of the motion of agitation is $\frac{1}{4} nV\,dS$. When diffusion and the motion under the electric force are included, let the fluxes become $(\frac{1}{4} nV + \theta)\,dS$ from below to above and $(\frac{1}{4} nV - \theta)\,dS$ from above to below. The difference of these two fluxes must

be equal to $\left(-K\frac{\partial n}{\partial z}+nv\right)dS$, where v is the velocity under the electric force: hence

$$\theta=\tfrac{1}{2}\left(-K\frac{\partial n}{\partial z}+nv\right).$$

If now the *régime* is such that the flux from above to below vanishes, the space above $z=0$ may be replaced by metal without altering anything. Hence the boundary condition at a metal surface is $\frac{1}{4}nV=\theta$, or $\frac{1}{2}nV+K\frac{\partial n}{\partial z}-nv=0$, and the actual number of ions falling on dS per second is $\left(-K\frac{\partial n}{\partial z}+nv\right)dS$. These expressions can be simplified by considering the order of magnitude of the terms. Since v is small in comparison with V, $K\,\partial n/\partial z$ is comparable with nV, so that nv is small in comparison with $K\,\partial n/\partial z$. Hence the flux across the boundary is approximately $-K\frac{\partial n}{\partial z}\,dS$. Again, the boundary condition becomes

$$n+\tfrac{1}{3}l\frac{\partial n}{\partial z}=0$$

on substituting for K its value $\frac{1}{6}lV$. On account of the smallness of l this may be written $n=0$. It follows that *if no ions are generated in the region considered, the number per unit volume in the neighbourhood of a metal boundary is negligible in comparison with that at some distance away.*

211. Measurement of Ne by the diffusion of ions. Townsend has shown how to determine the quantity Ne by observing the lateral diffusion of a stream of ions moving in a uniform electric field, and hence to compare the charges on ions in liquids and gases. His apparatus is shown in Fig. 311. The ions are generated by Röntgen rays in the space between the plates A and B. In order to collect, say, the positive ions, an electric force E is applied which drives them through the grating in B. They next pass through a circular aperture in the plate C. The plate D is composite, consisting of a central disc R_1 (the same size as the aperture) and a ring R_2, surrounded by an earthed guard-ring R_3.

The ions move throughout in a uniform electric field, so that they have acquired a uniform drifting motion before they arrive at C. As they move from C to D they diffuse out laterally, some

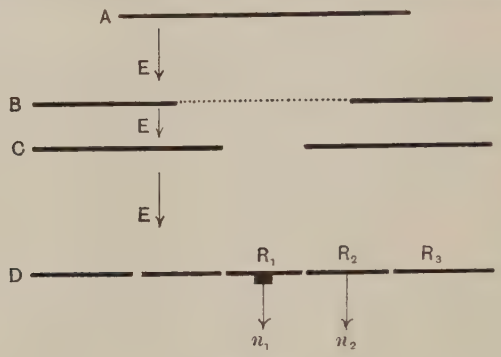


Fig. 311

being caught on the disc R_1 and some on the ring R_2. The disc and ring are insulated from one another and from the case, and the ratio of the charges n_1, n_2 arriving on them in a given time is measured accurately by a compensation method, the disc and ring being kept as nearly as possible at zero potential during the process.

It is obvious that, in the absence of diffusion, all the ions would fall on R_1 and none would be received on R_2. In order to calculate the actual value of n_2/n_1, take as origin the centre of the circular aperture in the plate C, the axis of z being directed from C towards D. The equation (10) becomes in this case

$$K\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}+\frac{\partial^2 n}{\partial z^2}\right)=uE\frac{\partial n}{\partial z},$$

or remembering (9),

$$\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}+\frac{\partial^2 n}{\partial z^2}=\frac{NeE}{\Pi}\frac{\partial n}{\partial z} \quad \ldots\ldots\ldots\ldots(11).$$

Since the whole motion is symmetrical about the axis of z, we may transform (11) to cylindrical co-ordinates r, z, and obtain

$$\frac{\partial^2 n}{\partial r^2}+\frac{1}{r}\frac{\partial n}{\partial r}+\frac{\partial^2 n}{\partial z^2}=2\gamma\frac{\partial n}{\partial z} \quad \ldots\ldots\ldots\ldots(12),$$

where

$$\gamma=\frac{NeE}{2\Pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Let a be the radius of R_1 and the aperture in C, b the outside radius of R_2 and h the distance between C and D. If b is sufficiently great very few ions reach R_3, and n may be taken as zero, for all values of z, when r is equal to b. This simplifies the numerical calculation without leading to serious error. We have thus to find the solution of the differential equation (12) subject to the following boundary conditions:

(i) when $z = h$, $n = 0$ for all values of r;

(ii) when $r = b$, $n = 0$ for all values of z;

(iii) when $z = 0$, $n = n_0$, a constant*, for $r < a$,
$= 0$ for $a < r < b$.

To solve equation (12) put $n = Re^{\theta z}$, where R is a function of r only and θ a constant. On substitution we find

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (\theta^2 - 2\gamma\theta)\, R = 0.$$

Hence $R = AJ_0(kr)$, where $k^2 = \theta^2 - 2\gamma\theta$.

Solving for θ in terms of k, we have therefore the particular solutions $n = AJ_0(kr)\, e^{(\gamma \pm \sqrt{k^2+\gamma^2})z}$, which may be combined into the single solution

$$n = J_0(kr)\, e^{\gamma z} \{A \cosh(z\sqrt{k^2+\gamma^2}) + B \sinh(z\sqrt{k^2+\gamma^2})\}.$$

If this solution is to satisfy condition (i) it can clearly be put in the form

$$n = CJ_0(kr)\, e^{\gamma z} \frac{\sinh\{(h-z)(k^2+\gamma^2)^{\frac{1}{2}}\}}{\sinh\{h(k^2+\gamma^2)^{\frac{1}{2}}\}} \quad \ldots\ldots(14),$$

where C is a constant.

The boundary condition (ii) shows that

$$J_0(kb) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Hence k must be one of the roots of this equation; and generalising (14) we have the solution

$$n = \Sigma C J_0(kr)\, e^{\gamma z} \frac{\sinh\{(h-z)(k^2+\gamma^2)^{\frac{1}{2}}\}}{\sinh\{h(k^2+\gamma^2)^{\frac{1}{2}}\}} \quad \ldots\ldots(16),$$

* This assumption is nearly, but not quite, correct, since some ions are lost by diffusion to C near the edge of the aperture.

the summation extending over all the positive roots of equation (15).

The coefficients C can now be determined so as to satisfy the condition (iii). From (16) we have to make $\Sigma C J_0(kr) = f(r)$, where

$$\left.\begin{aligned} f(r) &= n_0 \text{ for } r < a \\ &= 0 \text{ for } a < r < b \end{aligned}\right\}.$$

The problem of determining the coefficients has already been considered in Art. 7. We thus find

$$n = -\frac{2an_0 e^{\gamma z}}{b^2} \Sigma \frac{J_0'(ka)\, J_0(kr)}{kJ_0'^2(kb)} \frac{\sinh\{(h-z)(k^2+\gamma^2)^{\frac{1}{2}}\}}{\sinh\{h(k^2+\gamma^2)^{\frac{1}{2}}\}} \quad \ldots(17).$$

The number of ions falling on the disc R_1 per second is

$$n_1 = -2\pi K \left[\int_0^a r \frac{\partial n}{\partial z}\, dr\right]_{z=h},$$

and the number falling on the disc and ring together is

$$n_1 + n_2 = -2\pi K \left[\int_0^b r \frac{\partial n}{\partial z}\, dr\right]_{z=h}.$$

The integrals occurring can be evaluated by the formulae of Art. 7. We find

$$\frac{n_1}{n_1+n_2} = \left[\frac{\int_0^a r \frac{\partial n}{\partial z}\, dr}{\int_0^b r \frac{\partial n}{\partial z}\, dr}\right]_{z=h} = \frac{a}{b} \cdot \frac{\Sigma \frac{J_0'^2(ka)}{k^2 J_0'^2(kb)} \frac{(k^2+\gamma^2)^{\frac{1}{2}}}{\sinh\{h(k^2+\gamma^2)^{\frac{1}{2}}\}}}{\Sigma \frac{J_0'(ka)}{k^2 J_0'(kb)} \frac{(k^2+\gamma^2)^{\frac{1}{2}}}{\sinh\{h(k^2+\gamma^2)^{\frac{1}{2}}\}}} \quad \ldots\ldots(18).$$

It is evident from equation (11) that, if the dimensions of the apparatus are given, the ratio $n_1/(n_1+n_2)$ is a function of the quantity NeE/Π only. Its value can be calculated accurately from equation (18), only about four terms of each series being necessary in practice. The results for the particular apparatus used are shown graphically in Fig. 312, from which the value of Ne can be obtained immediately when $n_1/(n_1+n_2)$ is known, since $\Pi = 1{\cdot}013 \times 10^6$. Some of Townsend's results for negative ions are given in the following table:

Pressure of air in millimetres	Electric force in volts per centimetre	Ne
3	1·45	$1·23 \times 10^{10}$
6	0·98	1·23
6	1·47	1·25
6	1·96	1·15
6	2·96	1·27
12	1·00	1·20
12	0·98	1·20
12	1·46	1·25
12	2·75	1·26
	Mean	$1·23 \times 10^{10}$

This is identical, within the limits of experimental error, with the number $1·22 \times 10^{10}$ obtained in Art. 166, *e* there denoting the charge on a univalent ion in electrolysis. Hence *the two charges are equal.* Experiments were made with different gases and various modes of ionisation, with identical results. Experiments with positive ions generally gave the same result, but under

$a = 0·75$, $b = 2·5$, $h = 7$ cm.

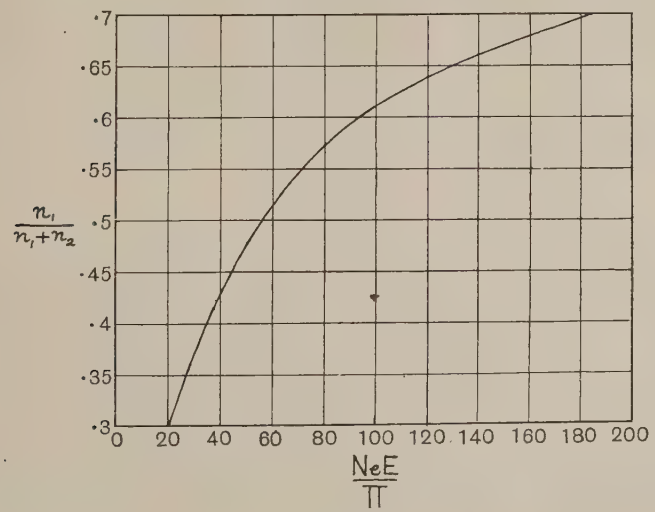


Fig. 312

certain circumstances higher values, up to $2{\cdot}41 \times 10^{10}$, were obtained, indicating that ions with double charges were present. If we accept this conclusion we may say that *the charge on a gaseous ion is either equal to, or an exact multiple of, the charge on a univalent ion in electrolysis.* There is no evidence of the existence of charges less than this elementary charge. The bearing of this result on the nature of the cathode ray particle has already been mentioned.

212. Measurement of the charge on a gaseous ion. The first determinations of the elementary charge e in absolute measure were made by Townsend, J. J. Thomson, and H. A. Wilson. The method has since been improved and made very accurate by Millikan, who has used it to give direct experimental proof of the discontinuous or atomic structure of electricity. The principle of the experiment, as far as it need be stated at present, is as follows. Let v_1 be the rate of fall, under gravity, of a small oil-drop of mass m. If the air in the neighbourhood is ionised by Röntgen rays, one or more ions collide with the drop, and it acquires a charge ne, where e is the elementary charge and n an integer. To fix ideas suppose that the charge is positive. Then a vertical electric field E of sufficient strength will stop the fall of the drop and convert it into a rise with uniform velocity v_2. Thus we have

$$\left.\begin{aligned} mg &= \lambda v_1 \\ neE - mg &= \lambda v_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where λ is a certain constant, which may however depend on the size of the drop and the pressure of the gas. Hence

$$\frac{v_1 + v_2}{v_1} = \frac{neE}{mg}.$$

It follows that if the ratio $(v_1 + v_2)/v_1$ is observed by allowing a particular drop to rise and fall many times through a measured distance, the values will, with sufficiently weak ionisation, be small multiples of a common unit.

Millikan's apparatus for observing the velocities is shown in Fig. 313. The oil-drops are formed by a sprayer at A in the figure (not shown), and fall slowly until one enters a small aperture B in the plate C_1 of a parallel-plate condenser. This plate can be kept at a positive or negative potential by means of a battery

of about 2500 accumulators, the other plate C_2 being earthed. The condenser is illuminated by light entering through the window D, and ionised by Röntgen rays entering through E. A third

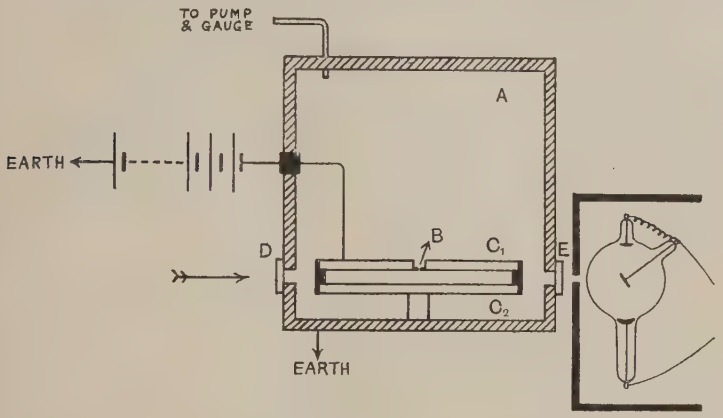


Fig. 313

window (not shown) allows the motion of the oil-drop to be observed through a short-focus telescope with a graduated eyepiece, so that the time of falling through a distance of 1·021 cm. could be accurately measured. The outer case was immersed in a constant-temperature bath in order to avoid convection currents, and in addition the condenser $C_1 C_2$ was edged with an ebonite strip containing only three holes opposite the windows. The following table shows one set of observations:

$E = 6{\cdot}98$ E.S.U., $p = 16{\cdot}95$ mm., temp. 22·98° C.

v_1	v_2	$\frac{v_1 + v_2}{v_1}$
·020241	·022972	$2{\cdot}1349 = 4 \times 0{\cdot}5337$
·020251	·033748	$2{\cdot}6664 = 5 \times 0{\cdot}5333$
·020110	·033668	$2{\cdot}6742 = 5 \times 0{\cdot}5348$
·020348	·044577	$3{\cdot}1908 = 6 \times 0{\cdot}5318$
·020235	·033590	$2{\cdot}6600 = 5 \times 0{\cdot}5320$
·020234	·012196	$1{\cdot}6028 = 3 \times 0{\cdot}5343$
·020372	·001284	$1{\cdot}0630 = 2 \times 0{\cdot}5315$

The constancy of v_1 shows that the drop is not evaporating appreciably during the experiment. The submultiples in the last

column are constant within the limits of experimental error, so that we have a very accurate proof of the existence of the elementary charge.

In order to find e in absolute measure the earlier experimenters used Stokes' theorem* that the force required to drive a sphere of radius a with velocity v through a fluid of viscosity μ is $6\pi\mu av$. Thus $\lambda = 6\pi\mu a$, where a is the radius of the drop. The weight mg in equations (19) must be corrected for the buoyancy of the air, so that if ρ is the density of the oil and σ that of air, we must write $mg = \frac{4}{3}\pi a^3 (\rho - \sigma) g$. Hence we have, for a drop carrying a single charge,

$$\tfrac{4}{3}\pi a^3 (\rho - \sigma) g = 6\pi\mu a v_1 ,$$

$$eE - \tfrac{4}{3}\pi a^3 (\rho - \sigma) g = 6\pi\mu a v_2 .$$

The first equation may be solved for a, and then the second equation gives e, all the other quantities being known ($\mu = 1{\cdot}824 \times 10^{-4}$ at 23° C.).

Millikan, however, found that the values of e thus obtained were not the same in all cases, being larger for the smaller drops. As it is inconceivable that the charge on an ion should depend on the size of a drop with which it collides, the experiments show that Stokes' law ceases to be accurate for very small drops. The proof of the law, in fact, being purely hydrodynamical, fails when the mean free path l of the air molecules becomes comparable with the radius of the drop. This is usually the case in Millikan's experiments. The most natural generalisation of Stokes' formula is to assume a retarding force due to viscosity of the form $6\pi\mu a v f\left(\frac{l}{a}\right)$, where f has the value unity for small values of l/a. Since l is inversely proportional to the pressure p, this becomes $6\pi\mu a v \phi\left(\frac{1}{ap}\right)$. Millikan found that the observations could be reconciled over a large range, and the same value of e obtained in all cases, by taking

$$\phi\left(\frac{1}{ap}\right) = 1 + \frac{b}{ap},$$

the constant b having the value $6{\cdot}254 \times 10^{-5}$.

* See Lamb, *Hydrodynamics*, p. 553.

In this way Millikan found the value

$$e = 4{\cdot}774 \times 10^{-10} \text{ electrostatic units,}$$

estimated correct to 1 part in 500. The radius of the drops used was of the order 2×10^{-4} cm.

213. Fundamental physical constants. We have now all the data for determining the quantities N, e and m separately. Adopting the values

$$Ne = 1{\cdot}22 \times 10^{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

and

$$e = 4{\cdot}75 \times 10^{-10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

we find, for the *number of molecules in a gas at* 15° C. *and* 760 *mm. pressure*,

$$N = 2{\cdot}57 \times 10^{19} \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Again, since

$$e/m = 5{\cdot}30 \times 10^{17} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

we find, for the *mass of the electron*,

$$m = 8{\cdot}96 \times 10^{-28} \text{ grams} \ldots\ldots\ldots\ldots (24).$$

The mass of the hydrogen atom has already been estimated at $1{\cdot}65 \times 10^{-24}$ grams, so that

$$\frac{\text{mass of } H \text{ atom}}{\text{mass of electron}} = 1840 \ldots\ldots\ldots\ldots (25).$$

According to the principles of the kinetic theory, the kinetic energy of a molecule of a gas at absolute temperature θ is $a\theta$, where a is a certain constant which we shall call *Boltzmann's constant*. Its value can now be found. For the pressure p is given by $p = \frac{1}{3}mN\Omega^2$, where $\frac{1}{2}m\Omega^2 = a\theta$. Hence

$$p = \tfrac{2}{3}Na\theta.$$

Taking $\theta = 15° \text{ C.} = 288°$ absolute, $p = 760$ mm. of mercury $= 1{\cdot}013 \times 10^6$ dynes per sq. cm., $N = 2{\cdot}57 \times 10^{19}$, we have

$$a = 2{\cdot}05 \times 10^{-16} \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

The kinetic energy of a gaseous molecule at ordinary temperatures is therefore about $5{\cdot}9 \times 10^{-14}$ ergs.

These numbers, which give an idea of the order of molecular magnitudes, will be useful in subsequent calculations.

214. Velocity of ions in gases. The velocity of ions in gases was first measured by Rutherford, and afterwards by Zeleny and Langevin, by different methods. Fig. 314 shows a method used by Lattey, which is due in principle to Rutherford.

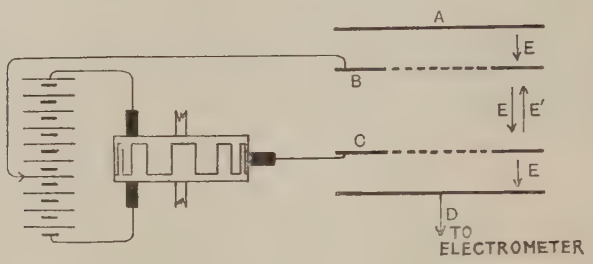


Fig. 314

Four plates A, B, C, D are placed parallel to one another inside an earthed metal case. The plates B and C carry central gauzes, and D is connected to one pair of quadrants of a sensitive electrometer. The ions are generated by Röntgen rays in the space between A and B. To fix ideas, suppose that it is desired to measure the velocity of the positive ions. An electric field E is applied in the space between A and B, which drives the ions through the gauze. The electric force between B and C can be made to alternate between the values $+E$ and $-E'$ by means of the rotating commutator shown in the figure, E' being numerically somewhat greater than E.

During the interval of the reverse force E' the ions are thrown back upon the gauze; but at other times they proceed from B towards C in the same field as before. Let t be the interval of time during which the direct force E is acting, l the distance between B and C, and v the velocity of the ions in the field E. If t is less than l/v the ions are overtaken by the reversed field before they reach C; but if t exceeds l/v some of them pass through C and are received on D. Hence the electrometer will remain unaffected when the commutator is rotating very rapidly, but as it slows down a deflexion will begin to occur. On account of diffusion the transition is not absolutely abrupt, but quite well marked, and the critical value of t can be determined to within a

few per cent. The velocity of the ions under the electric force E is then given by $v = l/t$.

The following table gives the *mobilities*, or velocities of the ions under an electric force of 1 volt per centimetre, at atmospheric pressure, where E is so small that the velocity is proportional to it.

Gas	u_+	u_-	Observer
Air	1·36	1·87	Zeleny
	1·40	1·70	Langevin
	1·37	1·80	Franck and Pohl
Hydrogen ...	6·70	7·95	Zeleny
	6·02	7·68	Franck and Pohl
Carbon dioxide	0·76	0·81	Zeleny
	0·86	0·90	Langevin
Oxygen ...	1·36	1·80	Zeleny

The theoretical formula $v = eEl/2mV$ shows that the velocity is proportional to the electric force as long as the constitution of the ion, whatever that may be, remains the same. The experimental results give a clue to this. Let us calculate the mobility of a negative ion in hydrogen on the hypothesis that it is (1) a charged molecule, (2) an electron. The mean free path of a molecule at atmospheric pressure is about 2×10^{-5} cm.; since the electron offers a smaller target its mean free path is longer, say 10^{-4} cm. The velocity of agitation can be calculated from the equation $\frac{1}{2}mV^2 = a\theta$. Thus in case (1) we have $m = 3{\cdot}3 \times 10^{-24}$, $V = 1{\cdot}9 \times 10^5$ and $u = 25$. In case (2), $m = 9 \times 10^{-28}$, $V = 1{\cdot}1 \times 10^7$ and $u = 8000$. The results therefore show that in a weak field the ion is not a free electron, or even a charged molecule, but a group of molecules clustered round a positive or negative charge. The present theory does not allow us to calculate the number of molecules in a group, because when a particle becomes much larger than a molecule it is less deflected by a single collision, and the effective free paths become longer in a way which it would be beyond the scope of this book to determine. It is, however, clear that the positive ion in any gas is larger than the negative ion.

Lattey's experiments were not restricted to small electric forces. Working at low pressures, he found that when the velocity

v is plotted against E/p (E being the electric force in volts per centimetre and p the pressure in millimetres of mercury) all the points lie on a single curve. The results for air are shown in Fig. 315. For positive ions in dry air we have $v = 1120\, E/p$,

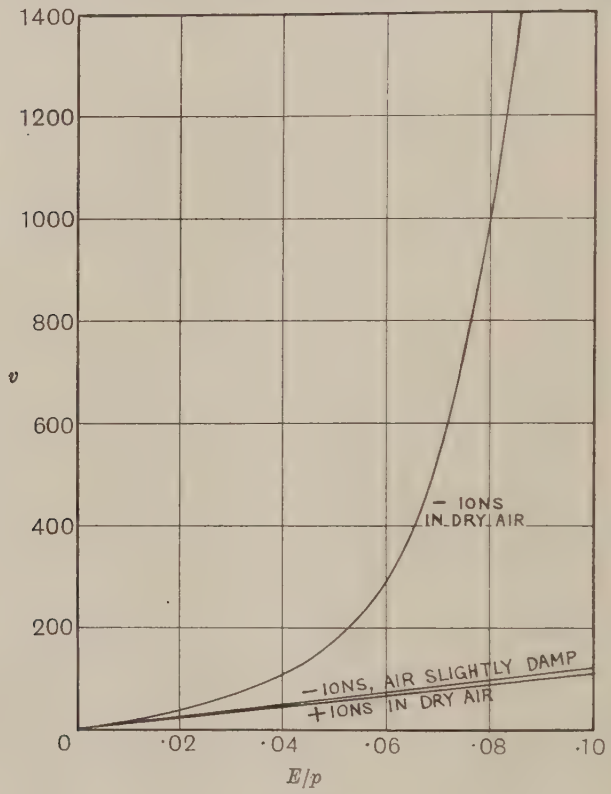


Fig. 315

from the lowest forces up to $E/p = 0{\cdot}1$, corresponding to a mobility of 1·47 at atmospheric pressure. The velocity of the negative ions ceases to be proportional to the electric force when E/p exceeds ·01, and rises rapidly for the larger forces. This effect is not obtained with ordinary air freshly admitted into the apparatus. It requires careful drying over phosphorus pentoxide before the velocity in a given field attains its final maximum value, and the addition of a small quantity of water vapour makes the

velocity of the negative ion once more comparable with that of the positive.

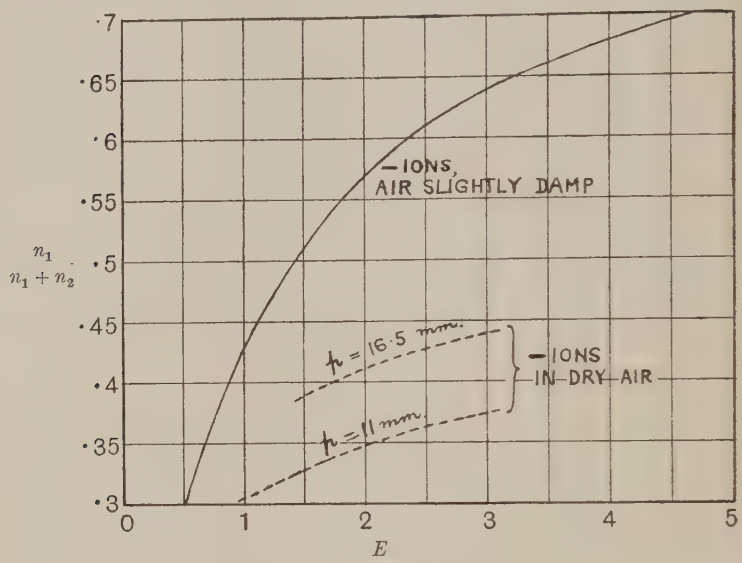
The relation $v = f(E/p)$ between velocity, electric force and pressure, thus proved experimentally, is what would be expected from theoretical considerations. Consider a number of ions in a gas, having assigned velocities immediately after a collision. If the pressure is increased from p to λp and the electric force from E to λE, the free paths of the ions are reduced in the ratio λ to 1, but the same velocity is acquired during a free path. Again, since the velocities just before a second collision are the same in both cases, the average effect of collisions is also the same. It follows that if the first *régime* is permanent the second one is too, and the average velocity of drift depends on E/p only. The same is true of the average constitution of the ions; that is, the ratio of the numbers of electrons, charged molecules and molecular clusters at any time.

For $E/p = \cdot 08$ the velocity in Lattey's experiments is intermediate between what would be expected for electrons and charged molecules respectively, showing that the molecular cluster is disappearing under the high electric force and free electrons are becoming frequent. This process goes on until only free electrons are present in the gas, as we shall show in the following article.

In pure argon and nitrogen Franck obtained very high velocities with quite small electric forces, which suggests that many of the ions in these gases are free electrons even under normal conditions.

215. Diffusion and velocity of ions under high electric forces. In his experiments on the diffusion of ions (Art. 211) Townsend observed an abnormal behaviour of negative ions in dry gases, which will be understood by reference to Fig. 312. This is re-drawn in the continuous curve in Fig. 316, on the hypothesis that Ne has the value $1 \cdot 22 \times 10^{10}$, E being expressed in volts per centimetre. Under ordinary conditions all the points determined experimentally were found to lie on this curve, as is obvious from what has been said already. In dry air, however, such as that used in Lattey's experiments, the lateral diffusion became abnormally great in comparison with the velocity under the electric force, so that fewer ions were received on the central disc. This

is shown by the two dotted curves in the figure. There is no reason to suspect any diminution in the velocity under the electric force—quite the contrary. Hence we conclude that the coefficient



$a=0{\cdot}75$, $b=2{\cdot}5$, $h=7$ cm.

Fig. 316

of diffusion of negative ions in dry gases, and therefore also their velocity of agitation, is abnormally great for high values of E/p.

Townsend assumed that the mean kinetic energy of agitation of an ion in this case was k times as great as that of a molecule of the gas, that is $k\alpha\theta$, where the factor k depends on the electric force and the pressure. This was explained by supposing that the kinetic energy gained by an ion during a free path was not all lost on collision with a molecule, so that energy might accumulate for some time. On this theory it is clear that k should be a function of E/p only. Referring back to equations (7) and (8), we have no longer $\frac{1}{2}mV^2 = \alpha\theta$, but $\frac{1}{2}mV^2 = k\alpha\theta$. Hence (9) becomes

$$\frac{u}{K} = \frac{Ne}{k\Pi},$$

and the equation (11) regulating the motion of the ions is replaced by

$$\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2} = \frac{NeE}{k\Pi} \frac{\partial n}{\partial z} \quad \ldots\ldots\ldots\ldots (27).$$

This equation being of the same form as before, it follows that the same value of the ratio $n_1/(n_1 + n_2)$ is obtained with the electric force $E' = kE$ that was previously obtained with the force E. For example, in moist air at 16·5 mm. pressure, $n_1/(n_1 + n_2) = 0{\cdot}4$ when $E = 0{\cdot}87$ volts per cm., but a force of 1·75 volts per cm. is required to produce the same ratio in dry air. In this case $k = 2$ very nearly ; and similarly k can be calculated for any other force and pressure.

More recently, Townsend and Tizard have made measurements with higher forces and lower pressures, and have at the same time measured the velocity of the ions by deflecting them in a magnetic field. For this purpose a modified form of apparatus was used, shown in Fig. 317. The plate B in Fig. 311 was removed and the

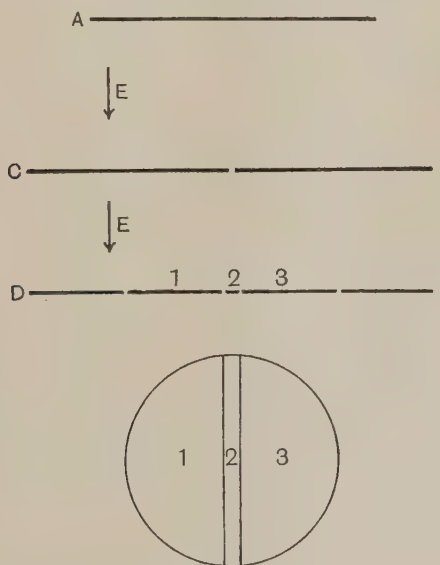


Fig. 317

hole in C replaced by a narrow slit, 1·5 cm. long and 2 mm. wide. The insulated electrodes within the guard-ring D were three in

number, as shown. If 1 and 3 are connected together and the total charge received by them compared with that received by 2, the calculation of k proceeds on much the same lines as for a disc and ring. To measure the velocity a magnetic field H is applied at right angles to the plane of the paper, and adjusted till the charge received on 1 is equal to that received on 2 and 3 together, so that the centre of the stream falls on the line of separation of 1 and 2. The stream is thus deflected through a known small angle θ. The force exerted by the magnetic field per unit charge is Hv/c in a direction nearly perpendicular to the original direction, so that $\theta = Hv/Ec$ approximately. Hence

$$v = Ec\theta/H.$$

The continuous curves in Fig. 318 represent the results of these

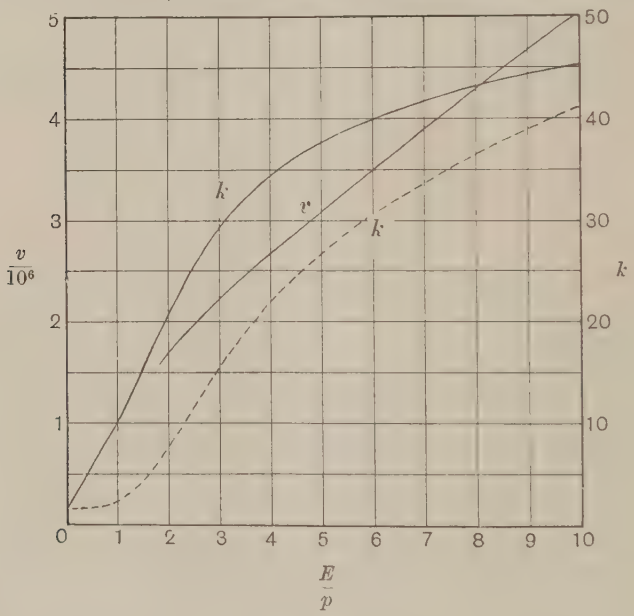


Fig. 318

experiments. In order to see what the ions are for these forces, consider the lowest result given by Townsend and Tizard, namely $v = 5 \times 10^5$ and $k = 2{\cdot}8$ for $E/p = 0{\cdot}2$, corresponding to a force of 150 volts per cm. at atmospheric pressure. In comparing it

with the theoretical formula $v = eEl/2mV$ it must be remembered that the velocity of agitation V is to be multiplied by the factor $\sqrt{k}$. The velocity of a free electron in this field should then be roughly 7×10^5 cm. per second, which is sufficiently near to the above value of v. The ions therefore seem to be free electrons throughout.

To recapitulate, we have the following information as to the constitution of the negative ion in dry air:

(1) When E/p is less than about 0·01 the ion consists of a group of molecules;

(2) A transition stage occurs between the limits 0·01 and 0·2, in which the average mass of the ion continually diminishes;

(3) When E/p exceeds 0·2 all the ions are free electrons.

The behaviour of positive ions is of much less importance, as the stage (3) is absent and the most that can happen is the transition from a small group of molecules to a single atom or molecule.

The dotted curve in Fig. 318 refers to a sample of air which had not been specially dried. The effect of a trace of water vapour, which is very considerable when $E/p = 1$, becomes less and less as the force increases, and above $E/p = 20$ ions move as free electrons in moist gases, or even in water vapour.

Since molecular clusters are not present in gases in their normal state, the process of ionisation must consist in the detachment of one or more electrons from the molecule. Both parts then move freely, or act as nuclei, according to the circumstances already described. The nature of the ions being thus settled, we proceed to describe some of their properties, and some other means of producing them.

216. Measurement of the coefficient of recombination. If there are n positive ions and n' negative ions per c.c. of a gas at any time, the rate at which they recombine is proportional to nn'. Thus

$$\frac{dn}{dt} = \frac{dn'}{dt} = -\theta nn',$$

where the constant θ is known as the *coefficient of recombination*. If equal numbers of ions are present initially, $n = n'$ at all

subsequent times, and $dn/dt = -\alpha n^2$. Hence n falls from n_1 to n_2 in time t, where

$$\frac{1}{n_2} - \frac{1}{n_1} = \theta t \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

The coefficient of recombination can be measured as shown in Fig. 319. A stream of air passes down a wide tube T kept at a high potential by means of a battery of small accumulators. It

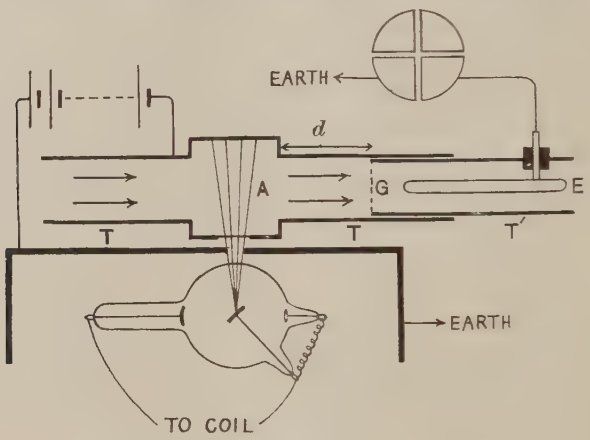


Fig. 319

is ionised at A by a beam of Röntgen rays entering through an aluminium window. A second tube T', fitting tightly into T, carries a fine gauze G at one end, and can be placed so that the distance d between G and a fixed point on the tube T has any desired value between certain limits. The ions which pass through G come into the field of force between the tube and the electrode E, and are collected on the electrode. The charges e_1, e_2 received per second when $d = d_1$, d_2 respectively are measured with an electrometer.

The velocity v of the stream of air differs at different points of the cross-section, but in an approximate calculation the stream may be considered as moving with its average velocity. Its value can be found by observing the total amount of gas passing through the tube per second. Let A be the cross-section of the tube, $\pm e$ the charges on the ions, and n_1, n_2 the number of ions

per c.c. in the neighbourhood of the gauze when $d = d_1$, d_2 respectively. Then if all the ions which pass through the gauze are collected on E, $e_1 = An_1ev$ and $e_2 = An_2ev$. Since the time taken by the ions in travelling through the distance $d_2 - d_1$ is $(d_2 - d_1)/v$, equation (28) gives

$$\frac{1}{n_2} - \frac{1}{n_1} = \theta \frac{d_2 - d_1}{v}.$$

Substituting for n_1 and n_2 from the above, we have

$$\frac{Aev}{e_2} - \frac{Aev}{e_1} = \theta \frac{d_2 - d_1}{v},$$

or

$$\frac{\theta}{e} = \frac{Av^2}{d_2 - d_1} \frac{e_1 - e_2}{e_1 e_2}.$$

In this experiment it is necessary to use wide tubes and moderately strong ionisation; otherwise the loss of ions will be mainly due to diffusion and not to recombination. The values of the coefficients of recombination in various gases at atmospheric pressure are given in the following table:

Gas	θ/e	θ
Air	3320	$1{\cdot}58 \times 10^{-6}$
Oxygen ...	3380	$1{\cdot}60 \times 10^{-6}$
Hydrogen ...	3000	$1{\cdot}42 \times 10^{-6}$
Carbon dioxide	3500	$1{\cdot}66 \times 10^{-6}$

It is remarkable that these numbers should be so nearly equal.

217. The photoelectric effect. In 1887 Hertz showed that ultra-violet light facilitated the passage of a spark between metallic electrodes when it fell on the negative electrode. Following on this, Hallwachs was able to show that negatively charged metals lost their charge on exposure to ultra-violet light, but that positively charged bodies did not. This experiment can be made very easily with an electroscope and an arc lamp.

Fig. 320 shows a convenient laboratory method of measuring the leak under various conditions. The source of light is a zinc or aluminium spark-gap S in a Leyden jar circuit, with which it is necessary to use at least a medium-sized induction coil. The

whole is placed in an earthed metal box in order to guard against electrostatic influence. The rays emerge through a hole in the lid, and after passing through a wire grating in the plate C_2 fall

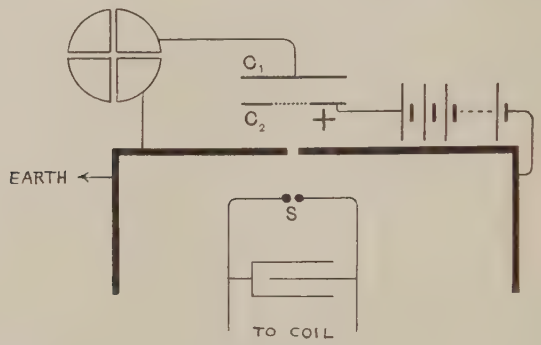


Fig. 320

on the zinc plate C_1. C_1 is connected to an electrometer and C_2 to the *positive* pole of a battery of small accumulators. On putting on the light a deflexion of the electrometer is obtained, in a direction showing that negative electricity has left the plate C_1. The current through the gas can be measured by the deflexion, or by the compensation method described in Art. 209.

There are several ways of showing that the leak is caused by ultra-violet light. The most striking is to mount the parallel-plate condenser C_1C_2 on the moveable arm of a spectrometer and to analyse the light from the spark-gap with a quartz prism. As the spectrometer arm is moved round in the direction of increasing deviation (so that red, blue, violet and ultra-violet light fall on the aperture in succession) no deflexion is observed until the visible spectrum is passed and the more refrangible rays are reached.

Another way is to observe the effect of placing various substances in the path of the light. Thus a glass plate laid over the aperture in Fig. 320 entirely suppresses the leak, while a quartz plate hardly reduces it at all; and generally all substances known to absorb ultra-violet light cut down the photoelectric effect, while substances which are transparent to it do not.

The order of sensitiveness of the common metals is somewhat as follows:

Aluminium, zinc, magnesium, lead, copper, silver, iron.

It must however be borne in mind that these lists give but little information. Ultra-violet light is a comprehensive term, and the absolute and relative behaviour of the metals is very different for different wave-lengths. Moreover, the state of the surface is an important factor here as well as with contact potentials. A zinc plate is very sensitive when freshly cleaned, but the deflexions fall off quickly at first, and more slowly afterwards. For many purposes all that is wanted is a copious and steady supply of negative ions: in these cases heterogeneous light can be used and is of course easier to produce. It is also generally found that the deflexions from a zinc plate become steadier after a time, and it is not advantageous in ordinary laboratory work to use a freshly cleaned surface.

Fig. 321 illustrates the way in which the current between the plates depends on the applied electric force. The two curves correspond to light of different intensity. Comparing them with

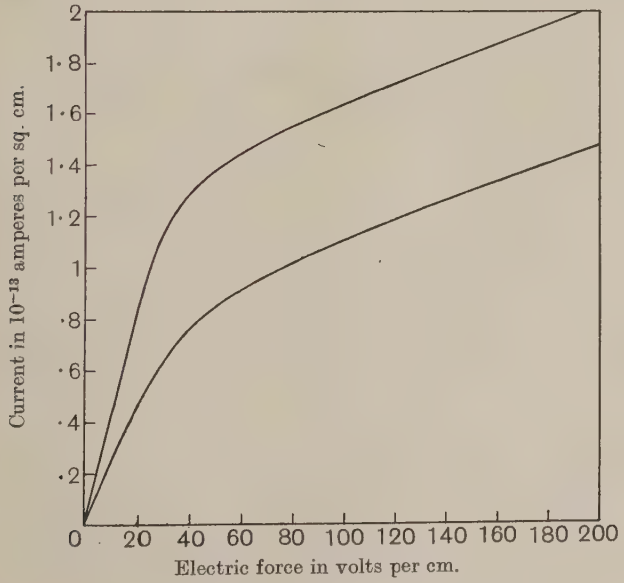


Fig. 321

the Röntgen ray characteristics shown in Fig. 310, two remarkable differences are observed. The curves for the different intensities are similar, and also the part corresponding to the higher forces is by no means horizontal. The circumstances are, in fact, entirely different. In the ultra-violet light curve there can be no recombination, as no positive ions are present. Diffusion must be operative at the lower forces: some ions diffuse back and become discharged by the negative plate in spite of the electric force. The final slope of the curve, again, seems to show that the number of ions actually emitted increases with the force, at any rate up to a certain point. At atmospheric pressure an approach to saturation is reached, but at much higher forces (of the order of 10,000 volts per cm.).

Nearly all substances are photoelectric to some extent: solids, liquids and, as Lenard showed, gases. The latter case is important, as the volume-effect in gases is very small in comparison with the effect on the metallic electrodes, and might seem to be absent altogether. Lenard and Ramsauer have recently made careful experiments on the conditions under which gases can be ionised by ultra-violet light. It was found impossible to avoid the photo-electric effect on the sides of the containing vessel altogether, so that there were always rather more negative than positive ions present. The percentage of the latter was taken as a measure of the volume-effect. With the longer-waved ultra-violet light ($\lambda > 1{\cdot}8 \times 10^{-5}$ cm.), such as is transmitted by quartz, positive ions were obtained, but only in the presence of impurities such as condensible vapours or the emanations from rubber tubing. Extraordinary precautions were taken to remove these, and it was concluded that there was no volume-effect whatever in pure gases. On the other hand, the shorter-waved ultra-violet light (Schumann rays of wave-length 10^{-5} cm.) ionised air even when slightly impure.

218. Emission of electrons under the action of ultra-violet light. In the above we have implicitly adopted the view that under ordinary conditions the photoelectric current is carried exclusively by negative ions sent out by one electrode. In this connexion it is interesting to notice that a "non-conducting"

gas does not resist the passage of electricity when it is once in it, but that the difficulty may be to get the ions from the metal into the gas.

The phenomena here, as elsewhere, are simplest when the pressure is very low. In 1899 Lenard showed that the photo-electric effect *in vacuo* consists in the *emission of electrons from the negative electrode.* His apparatus is shown in Fig. 322. Light

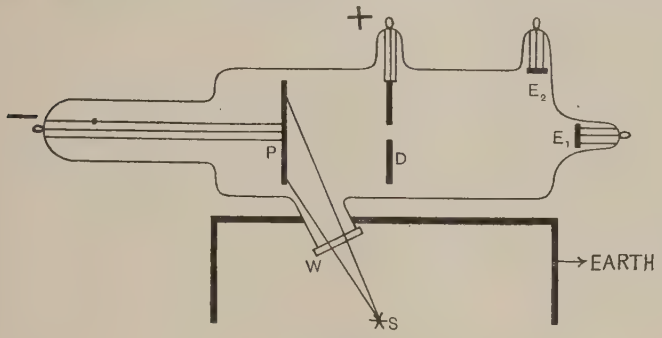


Fig. 322

from the spark-gap S passes through the quartz plate W and enters a highly exhausted glass vessel, falling on the aluminium plate P. The aluminium disc D opposite to it is earthed, and the whole of the tube at the back is covered with tinfoil so as to form a chamber practically at zero potential. E_1, E_2 are two insulated electrodes which can be connected to an electrometer. When P is kept at a negative potential of 600 volts or more a stream of negative ions passes through the aperture in D and falls on the electrode E_1. A magnetic field is then applied perpendicular to the plane of the paper in the space DE_1E_2. Two curves are taken off showing the electrometer deflexions at E_1 and E_2 for various currents in the coils producing the magnetic field. The difference in the positions of the maxima gives the current required to deflect the ions from E_1 to E_2; and knowing also the potential of the plate P, the values of e/m and v can be found as in Kaufmann's experiment.

The initial velocity of the ions, though small, is quite appreciable. Lenard determined the maximum velocity by insulating the plate P and observing the positive potential to

which it would rise *in vacuo* under the action of the light. If this is V, then $\frac{1}{2}mv^2 = eV$; for as long as the kinetic energy exceeds eV some ions will arrive on D and tend to raise the potential.

Lenard found that $e/m = 3{\cdot}5 \times 10^{17}$. Recently Alberti has obtained the value $5{\cdot}28 \times 10^{17}$. There is no doubt that the particles are electrons. The potential V was found to be 2·1 volts, showing that the velocity of emission is of the order of 10^8 cm. per second. The velocity is independent of the intensity of the light.

Many experiments have since been made to find how the velocity of emission varies with the wave-length of the incident light. The results have not been very consistent, partly because the state of the surface of the metal exercises such a pronounced effect. Hughes distilled his metals *in vacuo* in order to avoid the formation of surface-layers as far as possible, and obtained a number of results, of which we select those for cadmium.

Wave-length in Ångström units	+ potential V acquired in volts	Maximum velocity of emission
1849	2·480	$9{\cdot}36 \times 10^7$
2257	1·427	$7{\cdot}08 \times 10^7$
2537	·897	$5{\cdot}63 \times 10^7$
2967	·148	$2{\cdot}29 \times 10^7$
3126	0	—
3340	0	—

The velocity of emission is always greatest for the shortest wave-lengths. Hughes found that the relation between the positive potential V and the frequency n of the incident light was given approximately by the linear equation $V = kn - V_0$. Expressed in terms of the kinetic energy $\frac{1}{2}mv^2 = eV$ of the fastest particles, this becomes

$$\frac{1}{2}mv^2 = hn - eV_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (29).$$

The values of h for the various metals are nearly equal, and do not differ much from $5{\cdot}6 \times 10^{-27}$. On this view the emission of electrons ceases altogether when n is less than eV_0/h; that is,

there is no photoelectric effect whatever beyond a certain wave-length λ_0 characteristic of the metal.

219. Normal and selective photoelectric effect. The general rule is for metals to be more sensitive to short-waved ultra-violet light than to longer-waved or visible light. The measure of the sensitiveness may be taken as the photoelectric effect divided by the energy of the incident light; and with this understanding there is no exception to the above rule as long as the light is incident normally on the metal. With oblique incidence, however, Elster and Geitel found that rubidium had a maximum in the visible spectrum. Further, the position of the plane of polarisation had a marked effect, the current being sometimes much greater when the electric force was in the plane of incidence, and therefore had a component perpendicular to the surface ($E \perp$), than when it was parallel to the surface ($E \parallel$).

Further experiments have been made by Pohl and Pringsheim, in which the colour of the light was varied as well as the angle of incidence and the plane of polarisation. Light from a mercury arc was analysed in a special spectrometer with a glass or quartz prism and "achromatic" lenses of quartz and fluor-spar. It then fell both on a thermopile and on the metal, and a double-image prism was used for bringing either plane of polarisation to bear at will. Fig. 323 shows their results for a sodium-potassium alloy

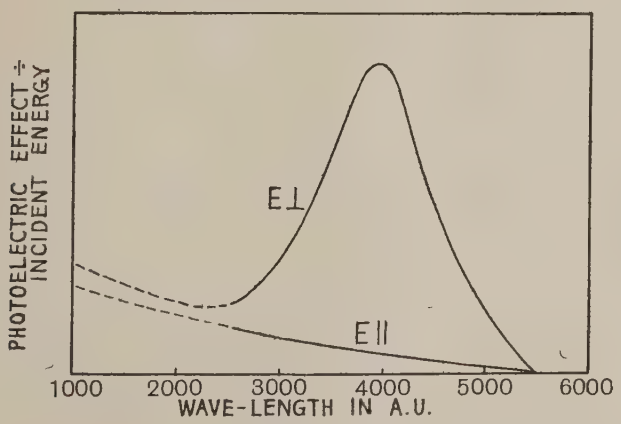


Fig. 323

at an angle of incidence 60°, the photoelectric sensitiveness being expressed in arbitrary measure. The dotted parts of the curves were not observed, but are conjectural extensions derived from the similar behaviour of other metals.

The existence of a long-waved limit to the photoelectric effect, referred to in the last article, is here brought out very clearly. While the $E \parallel$ curves show a continuous rise of sensitiveness with decrease of wave-length, the $E \perp$ curves for the alkali metals and alloys present well-marked peaks, in the following positions:

Sodium	3200	Ångström units
KNa alloy	ca. 4000	,,
Potassium	4400	,,
Rubidium	4800	,,

This is known as the *selective* photoelectric effect, and is regarded as superposed on a *normal* effect consisting of a continuous rise of sensitiveness with frequency. Pohl and Pringsheim consider that the two effects are independent of one another. It is easy to see how these experiments give an explanation of the results obtained by previous experimenters. Thus with normal incidence there is never any component of electric force perpendicular to the surface of the metal, and the selective effect cannot appear.

The selective effect has all the character of a resonance phenomenon. It is accompanied by high optical reflexion, so that the maxima become more pronounced (and the physical interpretation of the results more definite) if the curves are plotted in terms of the *absorbed* instead of the *incident* energy. Pohl and Pringsheim found that the charge liberated by the absorption of 1 calorie in the middle of the resonance region was 5×10^{-3} coulombs for potassium, and about twice as much for sodium. While the selective effect is not limited to the alkali metals, there are many metals (e.g. mercury, lead, tin, cadmium) which do not show it, at any rate above 2000 A.U.

220. Ionisation by collision of electrons with molecules. A remarkable change takes place in the Röntgen ray characteristic curve when the electric force exceeds about 20,000 volts per centimetre*. The complete curve is shown diagram-

* In air at atmospheric pressure with electrodes a centimetre or two apart.

matically in Fig. 324. The *ABC* part of the curve is the ordinary "saturation" curve already given. At some point *C*, which cannot be exactly specified as it depends on the amount of moisture present,

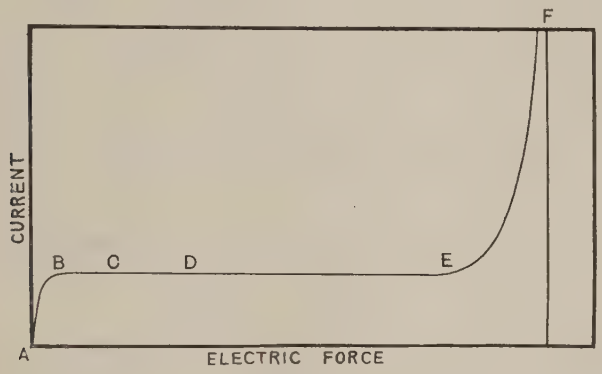


Fig. 324

the molecular clusters begin to break up and the ions to become electrons. At *D* (10,000 volts per cm.) this process is complete. The current of negative electricity is then carried entirely by electrons; but there is nothing to show on the curve, since the charge transported is the same as before. At *E*, however, the current begins to rise above the saturation value, increasing beyond measurable limits with further increase of the electric force. Finally at *F* a spark passes: the gas becomes luminous and the current continues to flow even after the Röntgen rays have been cut off.

These effects can be explained by Townsend's theory, now generally adopted, that the new ions arise by collision of the existing ions with the molecules of the gas: first the electrons, and then, as the sparking potential is approached, the positive ions also acquire the property of generating others by collision. We shall consider the former hypothesis first. The simplest conditions occur when the initial ionisation is produced by ultra-violet light falling on a zinc plate. In a given time let n_0 electrons be emitted by the plate in the field *E*. Assuming that they begin to move immediately with the velocity proper to the field, let α be the average number of ions formed by one electron in moving

through one centimetre in the direction of the electric force. Then if the number of electrons reaching a plane distant x from the zinc plate is n, $dn = \alpha n dx$, since all the new electrons arise from collisions in the length dx. By integration, $n = n_0 e^{\alpha x}$, and therefore the number of electrons reaching the positive electrode is $n_0 e^{\alpha d}$, where d is the distance between the plates.

In carrying out the experiments it is advantageous to work at low pressures, as a considerable increase in the current is then obtained with potentials of a few hundred volts. The results of a typical experiment are shown in the following table:

Air, 4 mm. pressure, electric force 700 volts per cm.
$\alpha = 8{\cdot}16$.

Distance d between plates in cm.	0·2	0·3	0·4	0·5	0·6	0·7	0·8
Current in arbitrary units	5·12	11·4	26·7	61	148	401	1500
$e^{\alpha d}$	5·11	11·6	26·1	59	133	301	680

The increase of the current is thus accurately exponential over a 12 to 1 ratio, after which the law ceases to hold. The unit of current has been chosen in order to exhibit this most clearly. The consideration of the larger currents is deferred for the present: meanwhile we may regard the figures as affording a satisfactory verification of the theory.

The value of α may thus be found for any electric force and pressure, and when the values of α/p are plotted against $E\ p$ for any gas, all the points are found to lie on a single curve. Certain of these curves are shown in Fig. 325, in which p is expressed in millimetres of mercury and E in volts per centimetre. It follows that α, E and p are connected by a relation of the form

$$\alpha/p = f(E/p).$$

This relation is what would be expected theoretically. For, as has already been explained in another connexion in Art. 214, the velocities with which the ions collide with molecules remain unaltered when E and p are changed to λE and λp respectively.

The number of molecules encountered by each ion in traversing a distance of 1 cm. is now λ times as great as before. Hence α becomes $\lambda\alpha$, and α/p is unchanged.

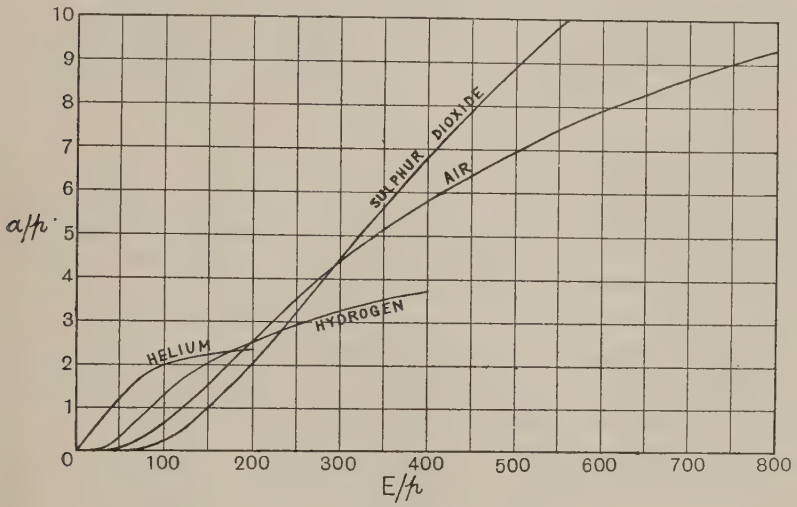


Fig. 325

The general form of these curves is also what one would expect from theoretical considerations. For small values of E/p few electrons move fast enough to ionise by collision, and α/p is small. As E rises (p remaining constant) more and more collisions result in ionisation. For very high velocities of the electron another stage sets in, in which the electron is capable of penetrating appreciable distances through gases after the electric force has ceased to act. In 1897 Lenard showed that cathode rays, emerging from a thin aluminium window in a discharge tube, traverse the air outside the tube for a short distance and generate ions along their paths. This was the first clearly recognised case of ionisation by collision. The work that has been done on these penetrating rays shows that the slowest of them ionise more molecules per unit length of their path than the faster ones. Hence there must be a certain critical speed (of the order 10^9 cm. per second) at which an electron has its maximum ionising power. As a general rule it is the lightest gases that are most easily ionised in weak

fields; but the molecular weight of the gas is not the only determining factor.

221. Average velocity required for ionisation by collision. The measurements of the diffusion and velocity of electrons in air have been carried much further than is described in Art. 215. Some of the results are shown in the following table:

E/p	20	40	60	80	100
k	55	85	115	140	160
$v/10^7$	0·9	1·5	2·0	2·4	2·7
$V/10^7$	8·5	10·6	12·3	13·8	14·5

The velocities of agitation V in the last row are calculated from the formula $\frac{1}{2}mV^2 = ka\theta$. Since v is approaching V it is clear that the limits of application of the theory are being reached, and the above numbers can only be approximately correct. In view of this fact it is not worth while to correct the formulae for the ions produced by collision, although it is not difficult to do so. The error is not so great as might be supposed, since the new ions, like the old ones, are most numerous in the centre of the field and scarce in the outer regions. Thus the velocity of agitation of the electrons for $E/p = 300$, when ions are produced freely by collision, is probably of the order 3×10^8 cm. per second, which suggests that this is about the average velocity required to ionise a molecule by collision. A more accurate estimate can be obtained as follows.

For the sake of simplicity let us assume that ionisation takes place at every collision when the velocity of the electron exceeds a certain definite value v_0, and that the velocities of agitation of the electrons are distributed according to Maxwell's law*. Thus the fraction of the total time during which the velocity of a particular electron lies between V and $V + dV$ is

$$f(V)\,dV \equiv \lambda V^2 e^{-\mu V^2}\,dV,$$

* Jeans, *Dynamical Theory of Gases*, pp. 11–28. To take all the velocities of agitation equal would, of course, lead to an absurd conclusion in this case.

where λ and μ are certain constants. They have to be chosen so that

$$\int_0^\infty f(V)\,dV = 1, \quad \int_0^\infty \tfrac{1}{2}mV^2 \cdot f(V)\,dV = kW,$$

where, to avoid confusion, W is written for $\alpha\theta$, the kinetic energy of a molecule at ordinary temperatures. This gives

$$\lambda = 4\left(\frac{\mu^3}{\pi}\right)^{\frac{1}{2}}, \quad \mu = \frac{3m}{4kW}.$$

Consider the collisions which occur while an electron drifts through 1 cm. in the direction of the electric force. The total time taken is $1/v$, where v as usual denotes the velocity under the electric force, and the time during which it is moving on a free path with a velocity intermediate between V and $V + dV$ is $f(V)\,dV/v$. Since the average time between collisions is l/V, where l is the mean free path, the number of collisions occurring with velocities between V and $V + dV$ is $Vf(V)\,dV/lv$. The number of collisions with velocities exceeding v_0 is therefore $\dfrac{1}{lv}\displaystyle\int_{v_0}^\infty Vf(V)\,dV$. But this, by hypothesis, is equal to α. Hence

$$\begin{aligned} l\alpha v &= \int_{v_0}^\infty Vf(V)\,dV \\ &= \lambda \int_{v_0}^\infty V^3 e^{-\mu V^2}\,dV \\ &= \frac{\lambda}{2\mu^2}\int_{\mu v_0^2}^\infty \xi e^{-\xi}\,d\xi. \end{aligned}$$

Carrying out the integration and substituting for λ its value in terms of μ, we have

$$\frac{1 + \mu v_0^2}{e^{\mu v_0^2}} = (\mu\pi)^{\frac{1}{2}} l\alpha v \quad \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

Since all the quantities except v_0 in this equation are known, the value of v_0 can be found by trial. Taking $E/p = 40$, $l = \cdot 03/p$, $\alpha = \cdot 019\,p$, $v = 1{\cdot}5 \times 10^7$ and $k = 85$, we find $\mu = 1{\cdot}34 \times 10^{-16}$. Equation (30) then gives $v_0 = 2{\cdot}9 \times 10^8$. This is the velocity that would be acquired by an electron in falling freely through a difference of potential of about 25 volts. It does not, however, follow that ions are never produced with smaller velocities. Thus

Lenard found evidence of ionisation by collision for potentials of about 10 volts, corresponding to a velocity of 2×10^8 cm. per second. This may be taken as the practical lower limit.

222. Ionisation by positive ions. The theory of Art. 220 can be extended to allow for the effect of positive ions. Take $x = 0$ as negative electrode, $x = d$ as positive electrode. Let n_0 be the number of electrons emitted by the negative electrode per second, n the number arriving at the positive electrode, and p, q the numbers produced by collision on either side of the plane x, as shown. Then $n = n_0 + p + q$. In one second $n_0 + p$ electrons pass the plane x from left to right, and q positive ions from right to left. If β is the number of electrons produced by a positive ion in travelling a distance of 1 cm. in the direction of the electric force,

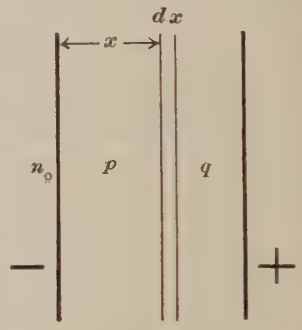


Fig. 326

$$dp = (n_0 + p)\,\alpha dx + q\beta dx,$$

since each represents the number of electrons generated per second in the strip dx. Hence

$$\frac{dp}{dx} = (n_0 + p)(\alpha - \beta) + n\beta,$$

or

$$\frac{d}{dx}(n_0 + p) - (\alpha - \beta)(n_0 + p) = n\beta.$$

The integrating factor is $e^{-(\alpha-\beta)x}$. Hence

$$n_0 + p = Ae^{(\alpha-\beta)x} - \frac{n\beta}{\alpha - \beta},$$

where A is an arbitrary constant. When $x = 0$, $p = 0$, so that $A = n_0 + n\beta/(\alpha - \beta)$. When $x = d$, $n_0 + p = n$. Making these substitutions, we have

$$n = \left(n_0 + \frac{n\beta}{\alpha - \beta}\right) e^{(\alpha-\beta)d} - \frac{n\beta}{\alpha - \beta},$$

or

$$n = n_0 \frac{(\alpha - \beta)\, e^{(\alpha-\beta)d}}{\alpha - \beta e^{(\alpha-\beta)d}} \quad \ldots\ldots\ldots\ldots\ldots (31).$$

When $\beta = 0$ this reduces to $n = n_0 e^{\alpha d}$, as before. Although few ions are produced by a positive ion in moving through a centimetre, the effect on the current is very great at the larger distances. This is shown in the following table, which is a completion of that in Art. 220.

Air, 4 mm. pressure, electric force 700 volts per cm.
$\alpha = 8{\cdot}16$, $\beta = {\cdot}0067$.

Distance d between plates in cm.	0·2	0·3	0·4	0·5	0·6	0·7	0.8
Current in arbitrary units	5·12	11·4	26·7	61	148	401	1500
$e^{\alpha d}$	5·11	11·6	26·1	59	133	301	680
$\frac{(\alpha - \beta)\, e^{(\alpha-\beta)d}}{\alpha - \beta\, e^{(\alpha-\beta)d}}$	5·11	11·6	26·5	62	149	399	1544

The theory has been verified* by this and many similar experiments. Like α/p, β/p is a function of E/p only, shown for certain gases by the curves, Fig. 327, which may be advantageously

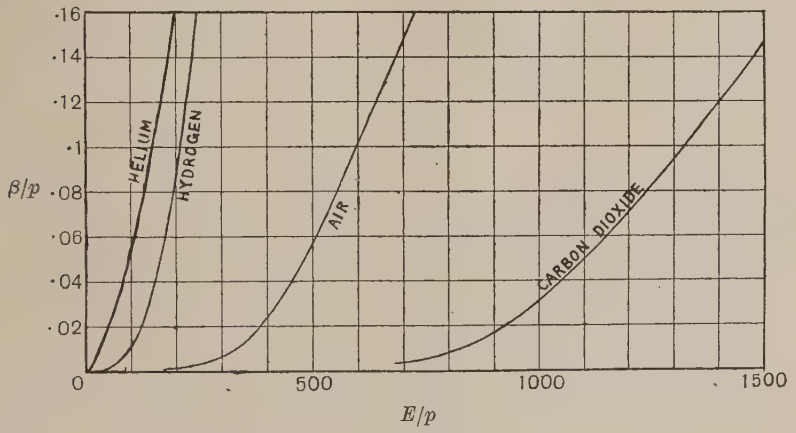


Fig. 327

* *Seven* currents are, in effect, accurately accounted for by the adjustment of *three* constants n_0, α, β.

compared with the curves giving α/p in terms of E/p (Fig. 325). The difference between light and heavy gases is very well marked here, as might be expected, since the positive ion under these forces is probably a charged atom of the gas, and the heavier ions move more slowly. The experiments therefore, as far as they go, support the conclusion of the absence of anything of the nature of a "positive electron." The ionising power of a positive ion is much less than that of an electron in the same field; but this is only because the velocity is so much smaller. We shall see later that the α particle from radioactive substances, which is a charged helium atom moving with a velocity of the order 2×10^9 cm. per second, produces more ions per centimetre than an electron travelling at any speed whatever.

223. Theory of the sparking potential. The equation (30) has one important consequence. If d is such that

$$\alpha - \beta e^{(\alpha-\beta)d} = 0,$$

the denominator vanishes and $n/n_0 = \infty$. The physical interpretation of this is that as large a current as desired can be obtained from indefinitely few ions by making the distance between the plates approach the above value. This, as is well known, is what happens when a spark passes. Hence according to Townsend's theory the sparking distance d, for the uniform field between two parallel-plate electrodes, is given by

$$d = \frac{\log_e \alpha - \log_e \beta}{\alpha - \beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

Since some ions are constantly being produced in a gas under normal conditions, it is easy to see how a spark can take place in the absence of external radiation. The above theory has been verified experimentally. Since α and β are known in terms of the electric force and the pressure, the plates can be set at a distance apart given by equation (32), and the least potential V required to produce a spark found by trial. It may then be compared with the value Ed which would be predicted by theory. The results of some experiments on air are given in the following table:

E in volts per cm.	p in mm. of mercury	d in cm.	Ed	V in volts	pd
1050	8	·765	803	803	6·12
1400	8	·431	601	603	3·45
1050	6	·572	601	604	3·43
700	4	·871	610	615	3·48
1050	4	·454	477	480	1·82
525	2	·910	481	488	1·82
700	2	·575	403	407	1·15
350	1	1·13	395	398	1·13
437	1	·832	364	365	·83
350	·66	·965	338	340	·64
437	·66	·766	335	336	·505

The above numbers also exemplify a remarkable law, first enunciated by de la Rue and Müller in 1880, that the sparking potential is a function of the product pd of pressure and sparking distance only. Thus V is practically the same when $p = 4$ and $d = ·454$ as it is when $p = 2$ and $d = ·910$. In order to prove this theoretically, put $\alpha = pf(E/p)$ and $\beta = p\phi(E/p)$ in equation (32). Then since $E = V/d$, the equation becomes

$$pd\left[f\left(\frac{V}{pd}\right) - \phi\left(\frac{V}{pd}\right)\right] = \log_e f\left(\frac{V}{pd}\right) - \log_e \phi\left(\frac{V}{pd}\right),$$

which involves V and pd only.

The curves connecting V and pd for certain gases are given in Fig. 328.

All the curves have much the same shape: the potential becomes infinite when pd is either very small or very large, and has a minimum value somewhere between. The minimum sparking potentials (in volts) for the various gases are approximately as follows:

Carbon dioxide	470	Nitrogen ...	290
Sulphur dioxide	455	Hydrogen ...	275
Oxygen ...	455	Argon ...	230
Air	340	Helium ...	155

In air the minimum occurs when $pd = 0·5$, corresponding to a sparking distance at atmospheric pressure of 1/150 mm. The

sparking potentials at atmospheric pressure would therefore normally be much above the minimum. There is still some uncertainty as to the exact values, but a rough rule is to allow

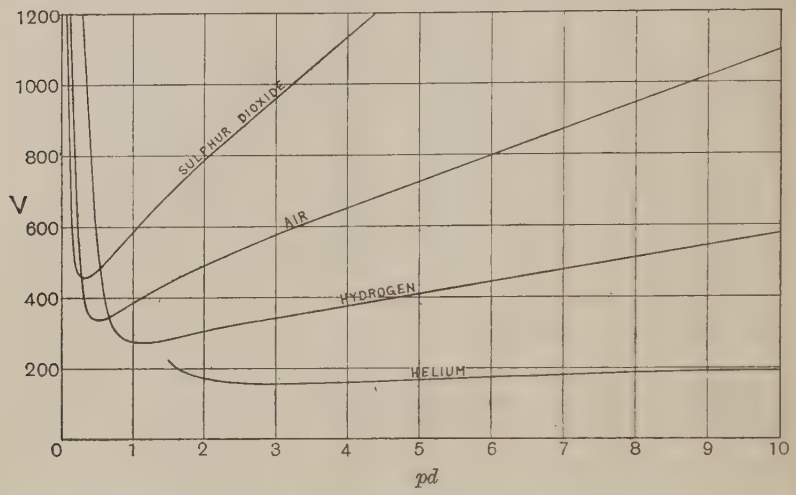


Fig. 328

30,000 volts to the centimetre for all distances above 1 mm. For values of *pd* below 0·5 there is a tendency for the discharge to choose a longer path round the back of the electrodes in preference to the straight path between them, since the potential in the latter case exceeds that in the former. In Carr's experiments this was prevented by embedding the electrodes in ebonite.

Another precaution is necessary in measuring sparking potentials, which is readily explained by the theory. Although, as has been said, a discharge takes place when the applied potential is equal to the sparking potential, it does not follow that it will do so at once, because the few ions present may not be favourably placed for multiplying themselves by collision. As much as a quarter of an hour may elapse if the applied potential is only a volt or so above the sparking potential. The "lag" is much reduced when care is taken to ionise the gas slightly during the experiment, most conveniently by allowing ultra-violet light to fall on the negative electrode, and consistent observations may then be made without waste of time. This is no doubt the true

explanation of Hertz's observation (Art. 217), which led to the discovery of the photoelectric effect.

224. The general similarity theorem. An electric discharge, when once started, goes on by itself as long as a sufficiently high potential is applied to the electrodes. The potential required to maintain a current is not in general the same as that required to start it, being greater, for example, in brush discharges from fine points and less in the discharges between parallel plates for the higher values of pd. The current itself maintains the supply of ions necessary for its own continuance. Nevertheless, it is very difficult to give any comprehensive theory, because there are several processes of ionisation at work, and their relative efficacy is uncertain. Thus it is generally held that the cathode rays are produced by the impact of positive ions which have acquired a high velocity in the Hittorf dark space. Positive ions may therefore liberate electrons from the negative electrode as well as from the molecules of the gas. This effect is negligible at the higher pressures, because the sparking potential has been almost universally found to be independent of the material of the electrodes; but it is certain that it comes into play at very low pressures.

The larger the linear dimensions of an apparatus, the lower the pressure required to exhibit a particular phenomenon. These and similar results are covered by a general theorem, due to Holm, on discharges in geometrically similar systems. For the sake of definiteness we shall consider two discharge tubes D_1, D_2 (Fig. 329), made of the same materials and filled with the same gas. Let the linear dimensions be in the ratio 1 to k, the pressures in the

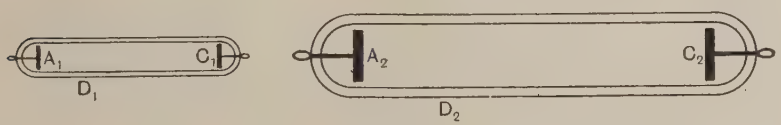


Fig. 329

inverse ratio k to 1. "Corresponding times" will be taken to be the times of describing a free path; that is, times in the ratio 1 to k. We shall suppose that at a particular time the distribution

and movement* of the ions in the tubes are precisely similar, but that the total numbers are in the ratio 1 to k. Then the electric force $E_1 = e_1/r_1^2$ at any point becomes $E_2 = ke_1/k^2r_1^2$ in the second system, so that $E_2 = E_1/k$. These correlations, together with some others easily deduced from them, are given here for reference:

	D_1	D_2
Linear dimensions	d	kd
Pressure	p	p/k
Free path of an ion	l	kl
Time	t	kt
Charge in corresponding regions ...	e	ke
No. of ions per c.c.	n	n/k^2
Electric force	E	E/k
Potential at corresponding points ...	V	V

The velocity acquired by an ion during a free path l is proportional to El, and therefore invariant. It follows that the equality of the distribution of velocities is unchanged until the next collision, and the velocities on collision are the same in the two cases; that is, for N ions colliding with velocities in the range (v_x, v_y, v_z), $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ in the first tube kN do so in the second. Corresponding stages are however only reached in corresponding times. Since the distribution of velocities on collision is the same, the effect of the collisions is also the same: the same fraction is deviated through a given angle and the same fraction of collisions results in ionisation. It follows that if the distribution of ions in the two tubes is similar at times t, kt respectively, it will be so at times $t + dt$, $k(t + dt)$, and therefore at all subsequent corresponding times. Diffusion, velocity under the electric force and ionisation by collision of both kinds are hereby taken into account.

This conclusion still holds in the presence of the electrodes and the walls of the tube. Consider the former. The number of positive ions, say, colliding with unit area of the cathodes C_1, C_2 per second are in the ratio $n : n/k^2 = k^2 : 1$. Hence the total numbers colliding per second are equal, and the numbers of ions

* There are no special assumptions here as to equality of free paths, velocities of agitation or otherwise. What is meant is that, at corresponding points, the fraction of the ions with velocities between (v_x, v_y, v_z) and $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ is the same in both cases.

disappearing from the gas in corresponding times are in the proper ratio 1 to k. Again, if any cathode rays are generated at all their numbers are in the same ratio, since the velocities of impact of the positive ions on the cathode are the same in the two cases. The effect of the walls of the tube is similar. In corresponding times corresponding numbers of ions collide with equal velocities, and the effect both in loss of ions and in disturbance of the electric field by the charge is the same.

These considerations suffice to show that, if the state of affairs in the first tube is a permanent one, that in the second tube is also*. The potential difference V required to send a current i through a discharge tube must be of the form $V = \phi(i, p, d)$, where d represents the linear dimensions and ϕ depends on the shape only. It follows from the above that V is the same as long as i and the product pd remain unaltered. Hence the relation must be of the form

$$V = f(i, pd) \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

in the two variables i, pd. When $i = 0$, V is the sparking potential, and we recover de la Rue and Müller's law.

The theory does not include the effect of Röntgen rays produced in the tube, unless their ionising power varies inversely as the square of the distance. Though there is some evidence for this it does not seem safe to rely on it in general. Nor is recombination allowed for. Nevertheless, it seems probable that, if equal currents were sent through two similar discharge tubes at pressures inversely proportional to their linear dimensions, the potentials would be the same and the distributions of light exactly similar.

225. Positive rays. We have seen (Art. 204) that at low pressures the glow nearest to the cathode in a discharge tube tends to spread out, and when the discharge potential is of the order of 10,000 volts it may extend to a distance of 2 cm. or so from the cathode. If the cathode is perforated the glow is seen also on the far side, taking the form of parallel streams of red or orange light. These rays were first observed by Goldstein in 1886, and called by him canal rays; but since, as we shall see

* If not, the permanent states are approached in corresponding times.

presently, they contain a proportion of positively charged particles moving with a high velocity, they are more appropriately called positive rays.

The magnetic deflexion of these rays is certainly small in comparison with that of the cathode rays; but in 1898 W. Wien succeeded in deflecting them by both electric and magnetic forces. His apparatus, which has been used with slight modifications by all subsequent observers, is shown in Fig. 330. The rays, after

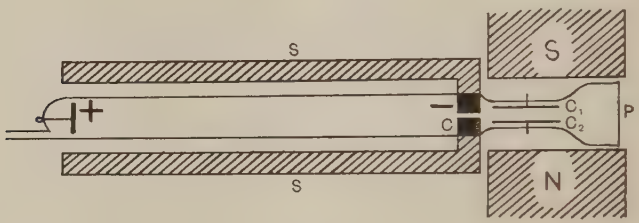


Fig. 330

passing through the perforated iron cathode C, fall on a glass plate P which fluoresces at the point of impact, so that the deflexions may be easily observed. An electric field can be applied between the pair of plates C_1, C_2, and a magnetic field in the same direction between the poles of a large electro-magnet. In order to screen the discharge tube from the direct action of the electro-magnet it is necessary to surround the tube with an iron cylinder S, as shown in the figure.

For simplicity suppose that the electric and magnetic fields are uniform over the whole distance l from the cathode to the plate P, and consider the effect on a charged particle moving with velocity v. The electric deflexion is $\frac{1}{2}\frac{eE}{m}t^2$, and the magnetic deflexion, supposed small, is $\frac{1}{2}\frac{evH}{mc}t^2$ in the perpendicular direction, where $t = l/v$ is the time occupied by the free flight of the particle. Calling these deflexions x and y respectively, we have

$$x = \frac{eEl^2}{2mv^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34)$$

and $$y = \frac{eHl^2}{2mcv} \quad \ldots\ldots\ldots\ldots\ldots\ldots (35).$$

Eliminating v, we have

$$y^2 = \frac{H^2 l^2}{2Ec^2} \cdot \frac{e}{m} \cdot x \quad \ldots\ldots\ldots\ldots\ldots\ldots (36).$$

Hence all particles having a definite value of e/m lie on a parabola. The electric deflexion is, *caeteris paribus*, inversely proportional to the kinetic energy of the particle. It is natural to suppose that the positive rays are the positive ions generated in the discharge tube, which move past the cathode with a high velocity. In this case no particle can have a kinetic energy greater than eV, where V is the discharge potential; hence there is a definite inferior limit to the electric deflexion, and the points corresponding to the different velocities should be spread over a parabolic arc, as shown in Fig. 331.

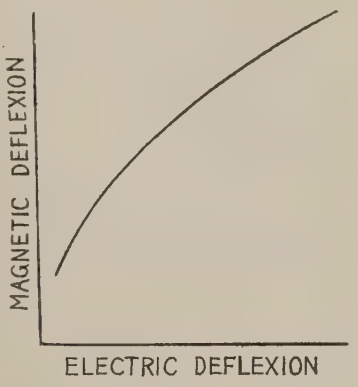


Fig. 331

All experiments on electric and magnetic deflexion agree in giving values of e/m of the order 10^{14}, corresponding to that of a charged atom or molecule, and velocities of the order 10^8 cm. per second; but the phenomena are not so simple as is represented above. Thus there is always an undeflected central spot, and no gap between it and the deflected curve, nor do the curves usually much resemble parabolas. This may be explained by supposing that some of the particles become discharged between the cathode and the fluorescent screen, and thus experience less than the normal deflexion. A complete explanation was found by Wien in 1908, namely that a stream of positive rays, as it advances, tends towards a limiting state in which the ratio of charged to uncharged particles is constant: thus a beam originally charged becomes partially discharged, and a beam from which the charged particles have been removed acquires charges in some way from the molecules with which they collide.

It is obvious, therefore, that the normal parabolas can only be obtained when the pressure in the space behind the cathode is so low that very few collisions occur there. The pressure in the main tube must at the same time be high enough to allow the discharge to pass with ease; but no separation of the chambers is possible since the rays cannot pass through even the thinnest membranes. Wien solved the difficulty by having a long capillary tube between the discharge tube and the tube in which the properties of the rays are examined. The latter is kept very highly exhausted, and gas is continually supplied to the discharge tube by a second fine capillary to make up the loss of gas by diffusion.

Using these methods von Dechend and Hammer were able to obtain parabolic bands like that shown in Fig. 331. In hydrogen the strongest band corresponds to a hydrogen atom with a single positive charge ($e/m = 2{\cdot}9 \times 10^{14}$), and the kinetic energy of the least deflected particles is that which would be acquired in falling through the whole discharge potential. Evidence was also obtained of charged hydrogen molecules (H_2+) and of singly and doubly charged carbon atoms ($C+$, $C++$), the latter arising from impurities present in the tube. The work of von Dechend and Hammer has been extended by Sir J. J. Thomson, who has devised a method of deciding whether the charge is single or multiple, and thus determining the atomic weight of the corresponding particle quite unequivocally. This will be understood by reference to Fig. 332, which shows (diagrammatically) two curves obtained with oxygen in the discharge tube. Here

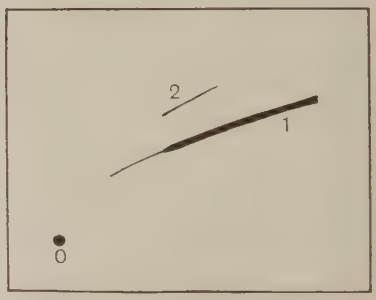


Fig. 332

O is the undeflected spot, 1 a parabolic arc corresponding to $m/e = 16$ in terms of the hydrogen atom. The thick part of the curve is no doubt caused by singly charged oxygen atoms: it has, however, a thinner prolongation extending to exactly half its distance from the vertical through O. This must be due to particles with

kinetic energy up to twice the normal amount; that is, to oxygen atoms which originally carried a double charge, but have lost one of them after passing the cathode. This view is confirmed by the appearance, on all plates showing the beaded parabola 1, of a curve 2 corresponding to $m/e = 8$, which is caused by atoms which have retained their double charge throughout.

The behaviour of positive electricity in gases at low pressures thus contrasts strikingly with that of negative electricity. Here e/m never approaches the value 5×10^{17} obtained for the cathode rays. There is, on the contrary, good reason to believe that the carrier is always an atom or molecule of a gas or impurity present in the tube. In this connexion we may mention Stark's interesting discovery of the "Doppler effect"; that is, that some of the spectral lines of the luminosity excited by the positive rays, when viewed end on, appear to be displaced towards the violet or red*, according as the spectroscope is placed so that the rays approach it or recede from it. This confirms the conclusion that the carriers of the positive current are at least atoms.

The above researches suggest that positive ions may be constantly getting charged and discharged as they travel in an electric field in ordinary experiments. But even if this is so it would not affect the developments of Art. 222, as there would always be a certain average number of electrons produced per centimetre of the path of the particle.

226. Emission of ions by incandescent solids. The leak of negative electricity from an incandescent metal has already been mentioned in Art. 179. It is found that the presence of gas, particularly hydrogen, complicates matters, the simplest results being obtained in a high vacuum. The unilateral nature of the discharge in high vacua was first clearly proved by Elster and Geitel in 1885. It is found that very few ions escape when the temperature is below 500° C., but the number rises rapidly with increase of temperature, and at 1200° C. an ordinary wire will give a current capable of deflecting a moving-coil galvanometer.

It is important to determine the value of e/m for the carriers of the negative current, so as to find out what they are. This

* Or more accurately doubled, since there is always a line in the original position.

was first done by Sir J. J. Thomson in 1899. He found the value $2{\cdot}6 \times 10^{17}$, which, though low, is sufficient to show that the particles are electrons.

Wehnelt found that the emission of electrons was much increased by coating the metal with calcium oxide or some similar substance, and introduced the cathode bearing his name. A Braun-Wehnelt

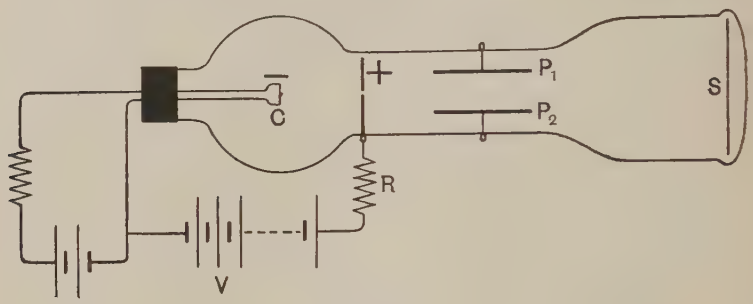


Fig. 333

tube, which affords the most generally convenient way of demonstrating the properties of moving electrons, is shown in Fig. 333. There is an important difference between this and an ordinary Braun's tube, in that the supply of electrons has no longer to be maintained by the impact of swift positive ions on the cathode*. The high potential required for the discharge at low pressures is no longer necessary, and a battery of 250 small accumulators suffices for most purposes. The Wehnelt cathode C is a strip of platinum heated to redness by means of a subsidiary battery, and carrying a small spot of calcium oxide. The stream of electrons passes through a small hole in the centre of the anode and produces a blue glow in the slight amount of gas remaining in the tube, which marks its path very clearly. An electric deflexion may be obtained by means of the small condenser $P_1 P_2$, a magnetic deflexion by a suitably placed coil, or pair of coils. Transparent squared paper or fabric should be used as backing for the fluorescent screen S, so that the movement of the spot of light may be followed. In this way it is easy to make a rough determination

* In Coolidge's Röntgen ray bulb the supply of electrons is likewise maintained by means of an incandescent cathode, and heavy currents can be passed without dangerous heating.

of the value of e/m. The main circuit must contain a resistance R of the order of 50,000 ohms, otherwise an excessive current may flow.

Positive electricity may escape from incandescent metals, but the carriers of the current are never of less than atomic mass.

The preceding facts are particularly interesting in throwing light on the phenomena of the electric arc. It is generally agreed that the essential part of an ordinary carbon arc is the incandescent negative electrode, emitting streams of electrons which fall on the positive electrode and keep it also very hot. No arc can be formed with a cold cathode, although, as we have seen, arcs with water-cooled anodes are quite possible. The only difference between an arc and a Wehnelt cathode discharge is that in the latter case the cathode is heated by an external agency, and in the former case by the impact of positive ions from the gas or positive electrode. The transition from one type of discharge to the other is well illustrated by an experiment described by Sir J. J. Thomson. If the resistance R in Fig. 333 is not too large, and the electromotive force V gradually increased, a time comes when the cathode becomes hotter by the impact of the residual positive ions in the gas, and the current rises from a small value to an ampere or so. At this stage the current used to heat the cathode may be cut off and the main current will maintain itself, the cathode remaining hot. The discharge is then indistinguishable from an arc.

227. Theory of the diffraction of Röntgen rays by crystals. We have now to consider in greater detail the diffraction phenomena mentioned in Art. 207. They are analogous to the effects produced when light falls on a diffraction grating: but whereas in the latter case the dimensions of the grating are known beforehand, here both the structure of the crystal and the nature of the incident "light" have to be found from the observations. It is essential to bear in mind the two problems involved.

To illustrate this more clearly take the case of a simple cubical array of identical atoms; that is, one which is such that axes can be chosen so that there is an atom at each of the points (pa, qa, ra), where a is a constant length and p, q, r are positive

or negative integers. Let a plane wave of Röntgen rays of wave-length λ pass through the crystal in the direction (l, m, n) referred to these axes. The rays are scattered in all directions by the atoms, and subsequently interfere. Consider the effect produced in the direction (l', m', n'). Let O, A, B, C be four consecutive atoms, O being taken as origin. Since the projection of OA on the normal to the wave-front is la, the wave reaches A at a time la/c after it reaches O, c being the velocity of light. By similar reasoning it follows that a scattered ray from A in the direction (l', m', n') is in advance of one from O by the amount $l'a/c$; hence the total phase-difference between the two rays after scattering is $(l' - l)\, a/c$. The directions of appreciable intensity of the scattered beam are those for which the phase-differences between rays scattered by O, A, B, C differ by integral multiples of the periodic time λ/c of the waves. Hence we have $(l' - l)\, a/c = h_1 \lambda/c$, or

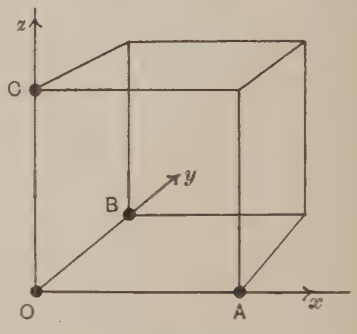


Fig. 334

$$\left.\begin{aligned} l' - l &= h_1 \frac{\lambda}{a} \\ m' - m &= h_2 \frac{\lambda}{a} \\ n' - n &= h_3 \frac{\lambda}{a} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

where h_1, h_2, h_3 are integers. Solving for l', m', n', squaring and adding, we have

$$\frac{\lambda}{a}(h_1^2 + h_2^2 + h_3^2) + 2(h_1 l + h_2 m + h_3 n) = 0 \ldots\ldots (38),$$

since $l'^2 + m'^2 + n'^2 = l^2 + m^2 + n^2 = 1$. This last equation is important, as it shows that λ can only have one of a certain set of values. If λ has not one of these values no spot is formed, and if it has the direction (l', m', n') is then given uniquely by

equations (37). Hence *even a heterogeneous beam of Röntgen rays will give a diagram of discrete points.*

If the rays fall normally on a crystal plate, $l = m = 0$ and $n = 1$, so that

$$l' = \frac{h_1\lambda}{a}, \quad m' = \frac{h_2\lambda}{a}, \quad n' = 1 + \frac{h_3\lambda}{a}.$$

Hence

$$\frac{l'}{h_1} = \frac{m'}{h_2} = -\frac{1 - n'}{h_3},$$

showing that there are simple numerical relations to be found between the direction-cosines corresponding to the various spots.

The following theorem applies to any crystal whatever. From the regular arrangement of atoms in a crystal, it is clear that many planes can be found containing large numbers of atoms. A cleavage plane is one such. If OA (Fig. 335) is one of these planes, another will be found parallel to it at a perpendicular distance d (say). The atoms in the second plane are not in general precisely behind those in the first; they may be sheared over through a distance a as shown in the figure. Let Röntgen rays fall obliquely on the plane OA, and consider only the waves diffracted at an angle equal to the angle of incidence i*. The condition of equality of phase from O and A is then satisfied identically. Proceeding as before, we find, for the condition of reinforcement of waves from O and B,

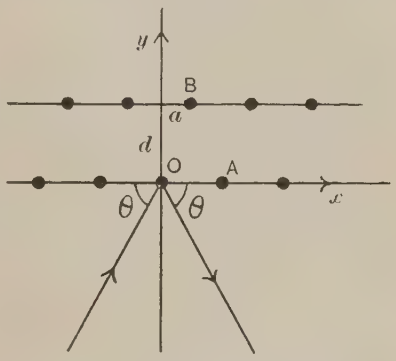


Fig. 335

$$\frac{d \sin\theta + a\cos\theta}{c} + \frac{d \sin\theta - a\cos\theta}{c} = n\frac{\lambda}{c},$$

or

$$2d \sin\theta = n\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots (39),$$

n being an integer. Since a has disappeared, we see that the shearing over of the atoms in the second plane is immaterial.

* As the angle of incidence is generally nearer 90° than 0° it is more convenient to use instead the *glancing angle* $\theta = \frac{1}{2}\pi - i$.

228. Analysis of crystal structure by Röntgen rays. The Laue diagrams were first used to determine the structure of crystals by W. L. Bragg. He found the deductions of the last article entirely verified for potassium chloride, and concluded that the diffracting centres were in simple cubical array. The behaviour of analogous crystals, such as sodium chloride and potassium bromide, showed that this is accidental, probably because the atomic weights of potassium and chlorine are so nearly equal. We shall not, however, follow out his reasoning, as a more powerful method of examining crystal structure has since been devised, and is described below. The structure adopted for all these crystals is shown in Fig. 336, in which the black spots represent (say) sodium atoms and the white spots chlorine atoms. The complete set of spots form what is called a simple cubical array: the black spots only, or the white spots only, a "face-centred lattice" of cubes with a point at the centre of each face. It will be quickest to assume this provisionally and verify the consequences afterwards.

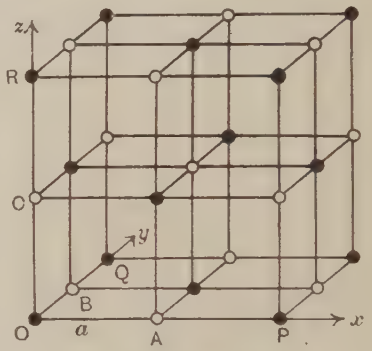


Fig. 336

The most effective method of examining the structure of crystals is one devised by W. H. Bragg, depending on equation (39). The spectral composition of the incident rays is examined at the same time. The beam of Röntgen rays from the bulb B (Fig. 337) is narrowed and made nearly parallel by two slits S_1, S_2 in thick lead sheets. The crystal K is mounted on the central table of a spectrometer, and the diffracted beam passes through a third slit S_3 into an ionisation chamber occupying the position of the telescope in an ordinary spectrometer. A sensitive electroscope is required to measure the weak ionisation current.

Observations of the strength of the diffracted pencil are made for different angles θ, setting a face of the crystal so as to make the

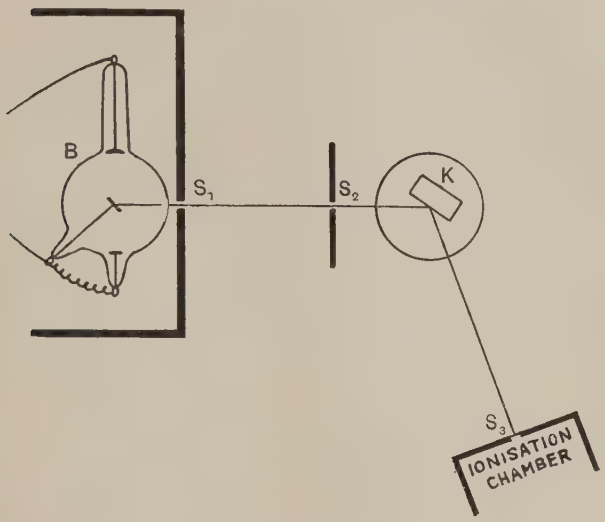


Fig. 337

same angle with both pencils. In this way the various wave lengths are examined in succession (though they overlap to some extent), and any spectral line in the original beam will stand out as a peak in the curve. In their earliest experiments W. H. and

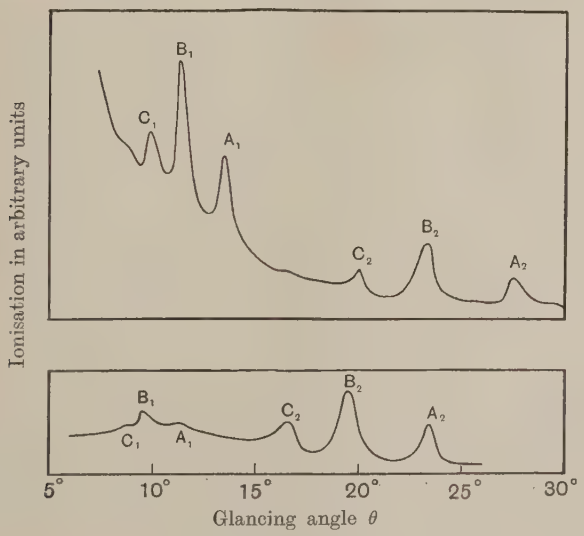


Fig. 338

W. L. Bragg used a bulb with a platinum target. The top curve in Fig. 338 was taken from a cleavage plane of rock-salt. In general, the intensity falls off as θ increases; but there are certain angles at which the intensity of the diffracted pencil is abnormally great. They undoubtedly correspond to three rays of definite wave-length emitted by the platinum target, the peaks A_1, B_1, C_1 forming the first order spectrum and the peaks A_2, B_2, C_2 the second. For after applying certain small angular corrections the peaks B_1, B_2 were found to occur at angles 11° 33′ and 23° 39′, and a third peak B_3 was found at 36° 39′. The sines of these angles are 0·200, 0·401 and 0·597, which are very nearly in the ratio 1 : 2 : 3.

Important as this discovery is, we shall at present consider it only as an aid to determining crystal structure, regarding the peaks as convenient landmarks. For the sake of brevity, a plane of atoms parallel to the plane $lx + my + nz = 0$, with any convenient axes, will be called an (lmn) plane. The curves for the three principal planes (100), (010), (001) of rock-salt were found to be identical. This is to be expected since the distance between successive planes of atoms is in each case equal to the side a of the elementary cube. Now consider the curve for the (111) planes. This is given in the lower half of Fig. 338. One such plane is the plane ABC in Fig. 336. The parallel plane PQR, at a distance $a/\sqrt{3}$ from it, contains only black atoms, and we have to move through a distance $2a\sqrt{3}$ before coming across an equivalent plane. Hence the relation of corresponding peaks in the curves for the planes (100) and (111) should, if this model is correct, be given by $a \sin\theta_{100} = 2a \sin\theta_{111}\sqrt{3}$, or $\sin\theta_{100} = 1{\cdot}15 \sin\theta_{111}$. The result of experiments on the B peak is $\sin\theta_{100} = 1{\cdot}16 \sin\theta_{111}$. Additional support is given by observations of the (110) planes, which show that $\sin\theta_{100} = 0{\cdot}718 \sin\theta_{110}$, while theory gives the ratio $1/\sqrt{2} = 0{\cdot}707$.

It is obvious, therefore, that we have a method of finding the structure of simple crystals, and of checking the accuracy of any proposed model of a more complicated crystal. In simple cases the evidence is very clear: thus a simple cubical array (potassium chloride) possesses the ratio $1/\sqrt{3} = 0{\cdot}57$ for $\sin\theta_{100}/\sin\theta_{111}$ instead of the ratio $2/\sqrt{3}$ mentioned above.

When the structure of a crystal is known, the absolute dimensions of the lattice formed by corresponding atoms can be found. Consider the case of rock-salt. The mass of a sodium atom is $3{\cdot}77 \times 10^{-23}$ gm., and that of a chlorine atom $5{\cdot}82 \times 10^{-23}$ gm. If a is the side of the elementary cube in Fig. 336, the number of atoms of either kind in 1 c.c. of crystal is $1/2a^3$. Hence the mass of 1 c.c. of the crystal is $4{\cdot}8/10^{23}a^3$. Equating this to the known density 2·15 we have $a = 2{\cdot}81 \times 10^{-8}$ cm. For potassium chloride, adopting the same structure (black spots = potassium atoms, white spots = chlorine atoms) we find $a = 3{\cdot}14 \times 10^{-8}$ cm. The peculiar circumstance of the black and white spots being practically identical as regards diffracting power has been referred to already. The atoms of the two constituents of sodium chloride are not so nearly equal: but even there there is a remarkable weakening of the odd order spectra in the (111) curves, which the reader who wishes to trace the effect of the transition from inequality to equality will find significant.

229. High-frequency spectra of the elements. The wave-lengths corresponding to the various platinum peaks may now be determined, since the constants of the crystal structure are known. W. H. Bragg found that the B and C peaks were really double, so that there are at least five homogeneous radiations from a platinum target in a Röntgen ray bulb. Accurate experiments showed that the mean B peak in Fig. 338 corresponds to an angle $\theta = 11° 18'$. Hence $\lambda = 2a \sin\theta = 1{\cdot}10 \times 10^{-8}$ cm.

In view of the complete analogy existing between Röntgen rays and light, these rays must be regarded as forming part of the spectrum of platinum. Other high-frequency spectra are examined by changing the material of the target. Fig. 339 shows part of the spectrum of palladium. It is remarkable for the smallness of the continuous spectrum, nearly all the energy being concentrated into two peaks, one of which is much stronger than the

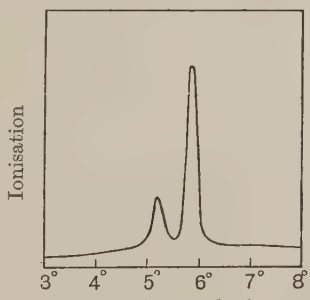


Fig. 339

other. The wave-lengths are about $5{\cdot}10 \times 10^{-9}$ and $5{\cdot}76 \times 10^{-9}$ cm. Several other elements were also examined.

Bragg's work was carried further by Moseley, who examined the high-frequency spectra of most of the elements between aluminium and gold, within a certain range of wave-length. The method used resembled the preceding, except that photographic registration was employed throughout. The spectral lines fall naturally into two sets, called the K and L sets respectively*. Of these the K set has the shorter wave-length, and is given out freely by the lighter elements. It consists of two lines K_1, K_2, of which K_2 is the stronger. The wave-lengths are shown in the following table:

Element	$10^8 \lambda$		Rydberg's number R
	K_1	K_2	
Aluminium	7·892	8·343	15
Silicon	6·712	7·124	16
Chlorine	—	4·738	19
Potassium	3·454	3·750	21
Calcium	3·086	3·360	22
Titanium	2·518	2·751	24
Vanadium	2·291	2·513	25
Chromium	2·088	2·295	26
Manganese	1·813	2·206	27
Iron	1·761	1·941	28
Cobalt	1·625	1·794	29
Nickel	1·501	1·658	30
Copper	1·398	1·545	31
Zinc	1·303	1·441	32
Yttrium	—	0·836	41
Zirconium	—	0·792	42
Niobium	—	0·748	43
Molybdenum	—	0·719	44
Ruthenium	—	0·636	46
Palladium	—	0·583	48
Silver	—	0·559	49

The numbers in the last column require some explanation. From considerations of a different kind Rydberg had previously

* The existence of homogeneous rays in general, and of the K and L sets in particular, was first shown by Barkla before the discovery of the diffraction phenomena, by considering the law of absorption of the rays in metals. The method is, however, not suited for distinguishing between rays of nearly equal wave-lengths.

been led to arrange the elements in order and assign a definite ordinal number R to each. The order is not exactly that of the atomic weights, but it is that of the chemical properties in the periodic law. Moseley found that these ordinals were applicable to the high-frequency spectra, and further that the wave-length λ of any particular line (say K_2) was connected with the ordinal by a relation of the form

$$\lambda = \frac{a}{(R-b)^2},$$

where a and b are the same for all the elements. This is clearly shown by plotting R against the square root of the oscillation frequency $\nu = c/\lambda$, as is done in Fig. 340. For the K_1 series we have $a = 1{\cdot}057 \times 10^{-5}$, $b = 3{\cdot}5$, and for the K_2 series $a = 1{\cdot}213 \times 10^{-5}$, $b = 3$.

The remarkable relation between the b's, pointed out by

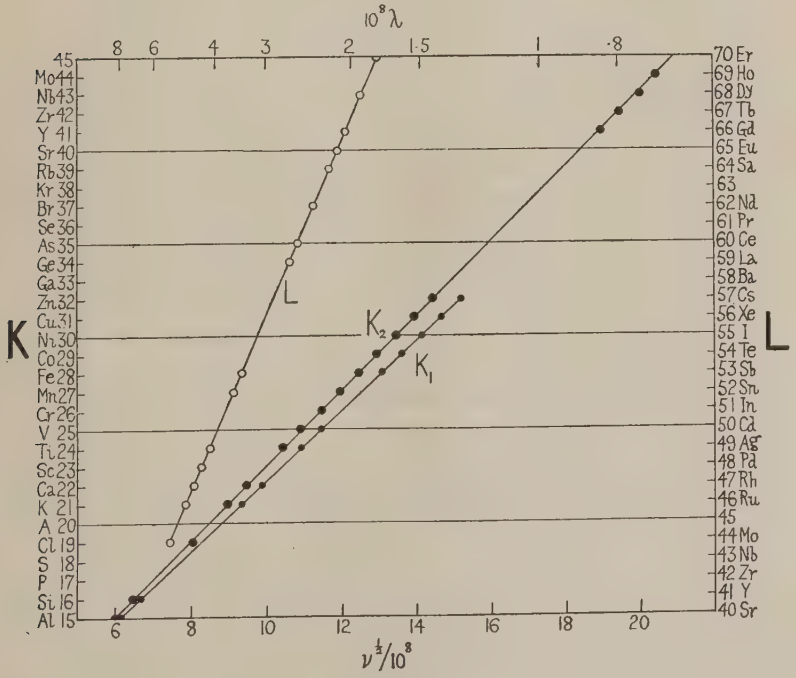


Fig. 340

Rydberg, can be paralleled in ordinary series spectra. These regularities have not yet been completely worked out. The L spectra are more complicated, and only the line of greatest wavelength is shown in the figure.

The preceding results show the use of high-frequency spectra in settling the position of elements in the periodic table. The spectroscopic evidence is much simpler than that given by ordinary spectra. Two or more substances can be examined together, as the spectra are simply superposed. This is strikingly illustrated by brass, which shows the spectra of zinc and copper, or by potassium chloride, which simply shows the lines of potassium and chlorine. The spectra therefore originate well within the atom, and are in no way affected by the arrangement of atoms in a molecule, or other aggregations.

The L radiations of the elements of high atomic weight continue the series shown in Fig. 340, but the K radiations are difficult to excite on account of the large voltage required.

230. Production and properties of homogeneous Röntgen radiation. The rays from an ordinary Röntgen ray bulb, as Fig. 338 shows, are of two kinds: a heterogeneous radiation and the homogeneous waves of known frequency. At this stage it is desirable that we should form some idea as to their origin. Both are disturbances in the ether set up by the impact of the cathode ray particles on the target. Some of these swiftly moving electrons will come so near to the centre of one of the atoms of the metal as to receive a violent acceleration and emit an ether wave. From each electron we have thus one or more irregular pulses, or disturbances of short period, lasting until the electron is finally stopped by the obstacle. At the same time, the atom of the target receives some shock or other, which may set its own interior electrons into vibration with definite frequencies characteristic of the metal only. This would account for the homogeneous radiation.

The effect of the heterogeneity of an ordinary beam of Röntgen rays is to give a complicated law of absorption in metals: the softest rays probably never emerge from the bulb at all. Homogeneous rays, however, are absorbed according to an exponential

law; that is, if the ionising power of the original beam is represented by I_0, it sinks to $I = I_0 e^{-Ad}$ after the rays have traversed a metal sheet of thickness d. The *absorption coefficients* A are much greater for heavy than for light metals: they become more comparable when divided by the density ρ. Fig. 341 gives the values of A/ρ for aluminium, copper and platinum over a certain

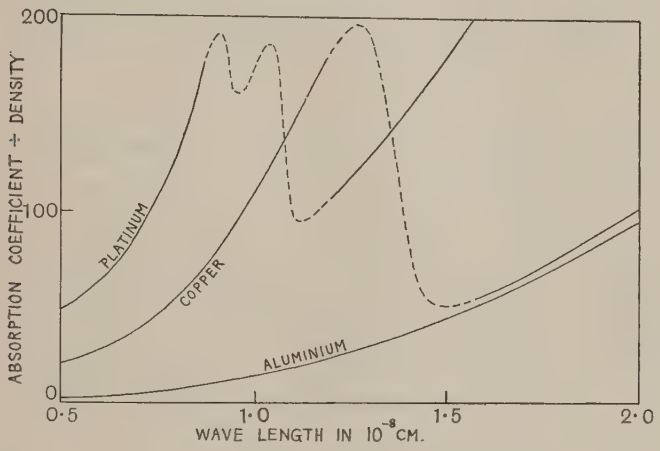


Fig. 341

range of wave-lengths. The numbers are only very approximate, and the shape of the dotted parts of the curves somewhat uncertain*.

As a general rule long waves are more easily absorbed than short ones. The intensity of the very soft K radiation from calcium, for example, is reduced to one-half after traversing a thickness of 1/150 mm. of aluminium, while 1 mm. is required for the K radiation of silver. The increase of absorption with wave-length, however, is by no means universal, being set back whenever the wave-length becomes equal to that of a characteristic radiation of the absorbing body. This is illustrated by the copper curve in Fig. 341. Since the wave-length of the K_2 line of copper is $1{\cdot}55 \times 10^{-8}$ cm., we see that the metal is exceptionally opaque

* The curves are plotted from the observations of Barkla and Sadler, in which the only test of homogeneity is the exponential law of absorption. The K radiation is here identified with its strongest line K_2.

to rays slightly shorter than those of its own high-frequency radiations.

The anomalous behaviour of platinum is similarly caused by its L radiation; but it is not so well-defined, since the radiation is more complex. Aluminium would behave in the same way if its natural wave-length was not so much longer. It is, of course, the inequality in the absorption in different substances that makes Röntgen ray photography possible: we now see that the amount of contrast attainable depends very much on the wave-length.

The question of the energy relations of cathode and Röntgen rays is, both theoretically and practically, very important. The quality of the rays, as we have seen, depends on the degree of exhaustion of the bulb as well as the material of the target. The higher the pressure the softer, in general, the radiation. The most striking fact, however, is that a certain definite velocity of the cathode rays is required to produce Röntgen rays of a given frequency. Using the exponential test of homogeneity, Beatty found that the cathode particles must have a velocity of $6{\cdot}25 \times 10^9$ cm. per second, or a kinetic energy of $1{\cdot}75 \times 10^{-9}$ ergs, to excite the K radiation of copper. The minimum energy is greater for smaller wave-lengths. It is, for example, difficult to excite the K radiations of elements higher than tin or silver, on account of the high discharge potential required.

We have now to consider the effects produced when Röntgen rays fall on matter. Sagnac was the first to show that an electroscope, screened from the direct action of a beam of Röntgen rays, might be rapidly discharged by placing a plate of a solid substance in the path of the beam near it. The plate sends out what may be called secondary Röntgen rays. Fig. 342 shows a simple apparatus for measuring the ionisation produced by them. The primary rays from the bulb pass upwards through a hole in the containing lead box, and fall on the plate R. G is a portable gauze cylinder containing a central rod connected to an electrometer, and sufficient potential V is applied to the cylinder to produce saturation between it and the rod. Wires leading to the electrometer should be screened as far as possible by passing them down the centre of metallic tubes connected to earth. Under these circumstances

the only ions reaching the electrometer are those generated inside the cylinder. Positive ions generated outside are repelled

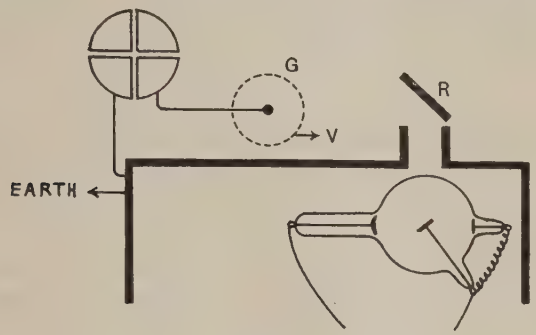


Fig. 342

by the gauze: negative ions, if attracted within the gauze, are then repelled by the central rod and fail to reach the electrometer. With an ordinary Röntgen ray bulb the greatest effect is generally obtained from zinc or copper radiators—less from either light elements like aluminium or heavy elements like silver.

The conditions under which secondary rays are produced have been carefully examined by Barkla. He found that the rays from substances of low atomic weight, such as air, water or carbon, have the same penetrating power as the primary rays, and concluded that they are merely the primary rays *scattered*. This could be verified by using, say, a bulb with a palladium target and seeing whether the palladium lines occurred in the secondary spectrum. From what we know now it is exceedingly likely that the characteristic radiation of the obstacle would also be present, though probably weak, and certainly very much absorbed by the air. For, with somewhat heavier elements such as zinc or copper, Barkla showed that most of the secondary rays were homogeneous and characteristic of the radiator; and de Broglie has found the corresponding spectral lines by analysis with a crystal of rock-salt.

We thus have the important result that an element may be made to yield its characteristic high-frequency spectrum either by the impact of swift electrons or under the action of Röntgen rays. This is paralleled by the various methods of exciting ordinary

spectra. Barkla showed that a characteristic radiation could only be excited by rays of higher penetrating power, i.e. shorter wave-length. For example, the K radiation of zinc falling on a copper plate is capable of exciting that of copper, but not *vice versâ*.

231. Emission of electrons under the action of Röntgen rays. Several experiments show that Röntgen rays, like ultra-violet light, cause negative electricity to be emitted from metals on which they fall. Thus Benoist and Hurmuzescu, in their original communication, state that a negatively charged body is discharged more rapidly by Röntgen rays than a positively charged body. In their experiments the rays fell on the body. Curie and Sagnac afterwards showed that a metal plate *in vacuo* becomes positively charged under the action of Röntgen rays, and Dorn showed that rays are emitted which are deflected by a magnet. Although the value of e/m has never been measured, there is no reason to doubt that the rays consist of electrons. Their velocity of projection, as deduced from Dorn's experiments, is then of the order 5×10^9 cm. per second. Such rays are really cathode rays, and we should expect them to penetrate some distance in atmospheric air and to produce considerable ionisation. Perrin, in fact, found a very considerable increase in the ionisation of a beam of Röntgen rays, over and above that arising from direct action on the air, when the rays fell on the electrodes as in Fig. 309. Townsend measured the saturation current for different distances between the plates. If a represents the ionisation due to the electrons ejected from one or both electrodes, and b that due to the primary and secondary Röntgen radiations per unit length, the saturation current for a distance x should be $i = a + bx$, provided that x exceeds the limit of penetration of the electrons. The curve connecting i and x was actually found to be a straight line for values of x exceeding 5 mm., which gives roughly the distance traversed by the electrons before they lose the power of causing ionisation.

The velocity of ejection of the electrons does not vary much, if at all, with the intensity of the rays. Innes, working with heterogeneous rays, showed that the velocity was greater for

hard rays than for soft ones, and Whiddington, by a somewhat indirect method, concluded that the velocity did not differ much, if at all, from that of the cathode rays in the original Röntgen ray bulb. These are very significant facts, to which we shall return later (Art. 258). Recent experiments by Robinson and Rawlinson show that there are usually two or more sets of electrons with different velocities. Thus when the K radiation of nickel fell on iron three strong bands occurred on the photographic plate as well as some weak ones, the maximum velocities in the three sets being 4·7, 5·9 and 6·4 $\times$ 10^9 cm. per second respectively. It is interesting to notice that at least four bands would be expected on the simple hypothesis that Röntgen rays of definite wave-length communicate a definite amount of energy to the electrons; for the K_1 and K_2 radiations of nickel almost certainly excite the corresponding iron radiations, which liberate electrons before leaving the metal. The experiments, which are not yet completed, promise to give valuable information on this and other points.

The above facts have an important bearing on the ordinary ionisation of gases by Röntgen rays. There is every reason to believe that the electrons ejected from the molecules of a gas have velocities of the same order as that given above; hence the main part of the ionisation is due, not to the rays themselves, but to the ejected electrons moving with a velocity sufficient to ionise by collision. Striking evidence of the accuracy of this view (due to W. H. Bragg) will be presented in the next article.

232. Condensation of water-vapour round gaseous ions. Observations which would now be interpreted in this sense were made by Coulier in 1875. Commenting on the known fact that clouds appear when air saturated with water-vapour is suddenly expanded, Coulier stated that the explanation then current of precipitation of water-vapour by the lowering of temperature was not altogether tenable, because the formation of clouds ceased after the expansion had been performed several times with the same air. This he showed to be due to the settling of the dust in the air, which forms nuclei on which water-vapour can condense. But clouds could be formed in the absence of dust

provided that the air was ozonised, or exposed for a short time to a red-hot platinum wire. Later Richarz, using a steam jet, showed that Röntgen rays are equally effective.

The conditions under which clouds are formed have been examined very carefully by C. T. R. Wilson, whose most recent form of apparatus is shown in Fig. 343. The expansion is produced in the space S between two glass plates. The upper one

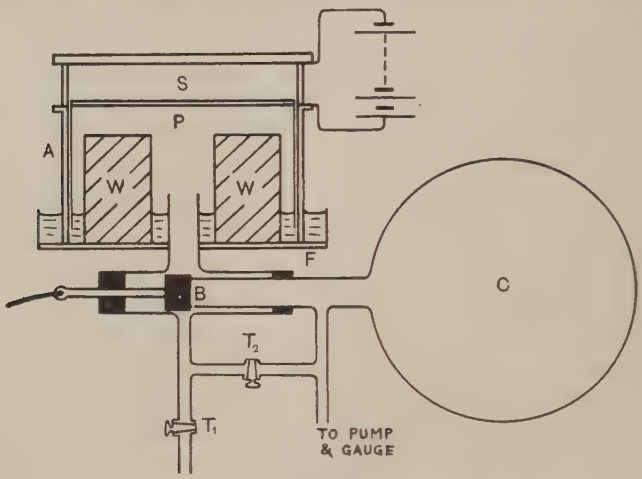


Fig. 343

forms the ceiling of the cylindrical glass vessel A: the lower one P is fixed on the top of a thin-walled brass cylinder so as to form a moveable plunger. It dips into a little water resting on the floor F of the outer vessel, which is covered on the top with a layer of indiarubber. In working the apparatus, the pressure of the air below the plunger is first adjusted by means of the taps T_1, T_2, of which one goes to the air and the other to an air-pump, until the volume of S has an assigned value v_1. T_1 and T_2 are then closed and the large vessel C exhausted. On pulling off the valve B the pressure below P is suddenly reduced: the plunger falls and becomes tightly jammed against the indiarubber floor. The volume v_2 of S in the new position is known, and thus we can produce at will a sudden expansion of the air with a given expansion-ratio v_2/v_1. The hollow wooden cylinder shown in section at WW serves to reduce the volume of the "dead space" below P.

The results of Wilson's experiments are as follows. No cloud appears on expansion in dust-free air as long as v_2/v_1 is less than 1·25. Between 1·25 and 1·38 there is a thin cloud, and when v_2/v_1 exceeds 1·38 dense clouds are always formed. The latter seems a clear case of condensation without extraneous nuclei. Negative ions from Röntgen rays produce dense clouds when v_2/v_1 exceeds 1·28; with positive ions very few drops appear until v_2/v_1 is equal to 1·31, and dense clouds are not obtained until v_2/v_1 has the value 1·35. It is obvious, therefore, that the method allows of very considerable control, which is useful in what follows.

The above apparatus is that used by Wilson in his recent experiments for making visible the paths of ionising particles in gases. Its particular advantage is that the motion of the air in the expansion chamber is in the same direction at all points, and disturbing eddy currents are avoided. Wilson had a falling ball device which first created the expansion, then sent a flash through the Röntgen ray bulb, and finally illuminated the cloud thus formed for the short time necessary to take a photograph. In this case the gas is supersaturated before the rays pass, and the ions have very little time in which to move before they are immobilised by the condensation. A vertical electric field applied to S serves to remove the charged drops after they are done with: a difference of potential of 4 volts has no observable effect on the photographs. It was found that a layer of gelatine on either plate made sufficient electrical contact, while remaining transparent. The expansion ratio was generally about 1·34.

Wilson's experiments give strong support to some of the conclusions already arrived at by other methods, and there is much interesting information to be gained from them. Fig. 344 shows the ionisation due to a narrow beam of Röntgen rays. As far as can be seen, the ions all lie on curled tracks representing the paths of high-speed electrons emitted from the molecules of the gas (Art. 231), and there is no general ionisation other than this. This agrees with Bragg's view that the greater part of the ionisation is due to these high-speed electrons. With ordinary (heterogeneous) Röntgen rays the length of path is of the order of 1 cm., and the total number of ions generated by one electron

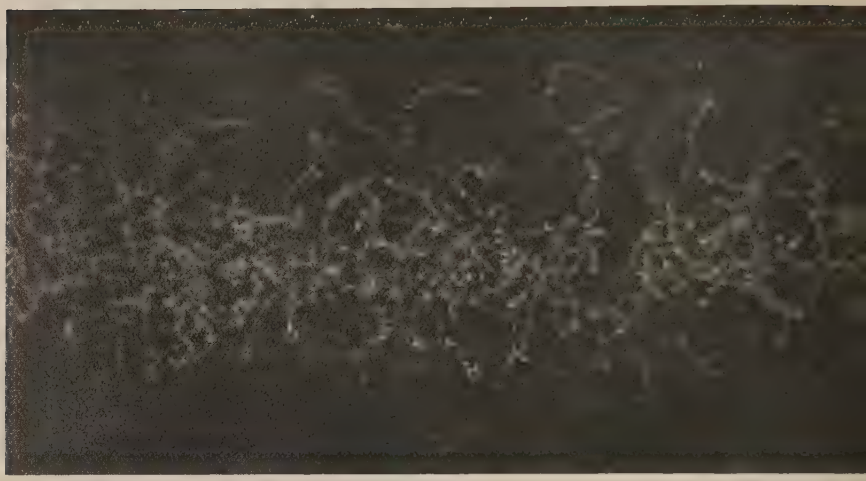


Fig. 344

Fig. 345

is of the order of 1000. The increase of ionisation as the electron slows down, referred to in Art. 220, is shown by the greater distinctness of the cloud near the end of most of the tracks. At the same time the tracks become more curled, showing that the power of the electron to penetrate through the gas without deflexion diminishes as the velocity grows less.

Fig. 345 shows the passage of Röntgen rays through a thin sheet of copper. Sufficient supersaturation to produce a cloud was not obtained in the immediate neighbourhood of the sheet, as its temperature does not fall as rapidly as that of the air. The figure shows clearly the absorption of the rays by the sheet, the emission of large numbers of electrons from it, and the general ionisation produced by the secondary rays. Most of these latter tracks are short, and due probably to the characteristic radiation of copper; one or two, which are longer, must arise from harder radiation scattered by the metal.

CHAPTER XIII

RADIOACTIVITY

233. Discovery of radioactivity. In 1896 H. Becquerel observed that crystals of the double sulphate of uranium and potassium, $K(UO)SO_4 . H_2O$, gave a well-marked impression on a photographic plate after a day's exposure, when the plate was entirely screened from the light by wrapping it in two thicknesses of black paper. This was at first attributed to the fluorescence of the crystals; but it was afterwards found that the action took place equally well when the crystals were kept for a long time in darkness, and moreover that this was a general property of all uranium salts, fluorescent or not. The most strongly fluorescent non-uranium salts give no effect whatever. Becquerel found that the determining factor was the amount of uranium present, and that pure uranium was more active, weight for weight, than any of its compounds. The effect still takes place when a thin sheet of glass is interposed between the metal and the photographic plate to cut off possible vapours rising from the uranium. Thick objects, such as coins, cast shadows on the plate, but the radiations pass easily through thin sheets of light substances, such as aluminium or mica.

Becquerel also found that uranium and its compounds would discharge an electroscope when brought near it. This electrical effect can be examined by any of the methods used in the conduction of electricity through gases. Fig. 346 shows the simplest form of apparatus. The active matter is spread in a thin layer on the lower plate C_2 of a parallel-plate condenser, giving rise to a steady current through the gas, which rises with the applied electromotive force and tends towards a saturation value. This saturation current, as determined by the rate at which the

spot of light moves over the electrometer scale, is taken as a measure of the activity of the substance. It is important to

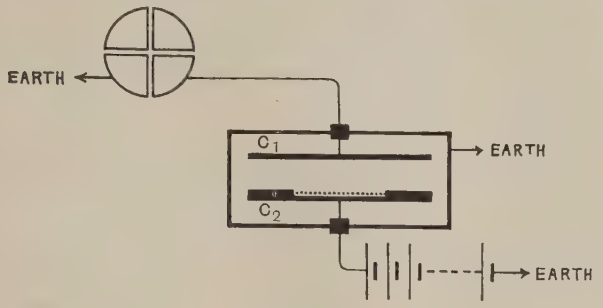


Fig. 346

notice that the current is not undirectional, as in the photo-electric effect, but similar to that produced by weak Röntgen rays.

In accurate work it is desirable to abandon the simple method of observing the rate of deflexion, and to measure the charge acquired by the insulated system in a given time. Since the rays cannot be conveniently put on and off, a special *electrometer key* has to be used, such as that shown in Fig. 347. Here P_1 and

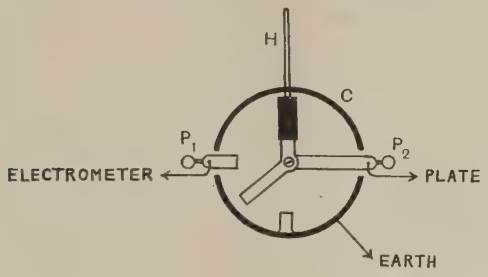


Fig. 347

P_2 are two insulated brass pieces contained in the earthed cylindrical case C, so that by moving the handle P_2 can be connected either to earth or to the electrometer at will, and without disturbing the charges on the insulated system. Sensitive electroscopes have also been used extensively for measuring weak ionisations.

The electric action, like the photographic, can be produced

across thin sheets of matter, but is much reduced by thick ones. The absorption of the radiations by the active substance itself may also become serious. Using very thin layers, however, McCoy found that the activity of uranium salts is accurately proportional to the amount of uranium present, as shown in the following table:

Salt	Activity of unit mass	Percentage of uranium present	Ratio
Uranium oxide U_3O_8	16·69	84·9	·197
Uranium oxalate	11·59	57·6	·201
Ammonium uranate	14·95	74·4	·201

This atomic property is of the utmost importance. The activity of a thin layer depends only on the number of uranium atoms present, and is entirely independent of their state of chemical combination. Moreover, physical influences, such as ordinary changes of temperature, daylight and darkness, are without effect. Mme Curie measured the activity of uranium electrically over an interval of three years, and found it constant within the limits of experimental error, and certainly within 1 per cent.

In 1898 G. C. Schmidt found that thorium and its compounds had properties similar to those of uranium; and at the same time Mme Curie was engaged in examining a large number of substances in order to find whether any of them behaved in the same way. The sensitiveness of her apparatus was such that an activity of one-hundredth of that of uranium could have been detected; but no effect was found with substances not containing uranium or thorium. Certain uranium *minerals*, however, were found to be abnormally active, in apparent disagreement with the atomic law. Thus while the activity of a layer of uranium was represented by 2·3 (unit 10^{-11} ampere), that of a similar layer of chalcolite, a crystallised phosphate of copper and uranium, was 5·2. An artificial chalcolite, prepared from the pure components, had an activity of about 1, corresponding to the amount of uranium present; from which Mme Curie concluded that the most active minerals, such as pitchblende, carnotite and chalcolite,

owe their activity to an unknown but very potent element present in very small quantities, probably so small as to escape detection by ordinary means. By subjecting the mineral to a regular chemical analysis it should be possible to separate out the unknown substance together with some element to which it is chemically allied, and thus obtain preparations of much greater activity than that of uranium. These conclusions were entirely justified, and Pierre and Mme Curie were able to extract from pitchblende a highly active preparation of bismuth, which they regarded as containing a new element called *polonium*. Later, in collaboration with Bémont, they obtained a second element called *radium*, chemically allied to barium. This time it was found possible to separate the radium from the barium, on account of a slight difference in solubility of the chlorides, and obtain the pure salts of the new element. Polonium has not been separated in a weighable quantity.

Radium has an atomic weight of about 226 and a characteristic spectrum of lines not belonging to any other element. Its activity is very great, and its physical properties exceedingly striking (see following article). About 5 kilograms of pitchblende are required to yield 1 milligram of radium, and the price of radium is correspondingly high. The most that has ever been collected together at one time is a little over 1 gram, which was used by Hönigschmid in his determination of the atomic weight. The term "radium" is often used to denote the amount of radium present in a radium salt*; for it is not convenient to use the pure metal which is difficult to prepare and undergoes chemical change when exposed to the atmosphere. There is no doubt that the radioactivity of a radium salt is due entirely to the radium contained in it.

234. The three types of radiation. It was recognised very early, in connexion with uranium, that the radiation could not all be of one quality. Thus while the strength of the photographic image is hardly altered when a sheet of aluminium $\frac{1}{20}$ mm. thick is placed over an uncovered uranium compound, the electrical

* One gram of radium chloride contains 0·761 grams of radium, one gram of radium bromide 0·586 grams.

action is much reduced. We shall now describe experiments which show that in all probability three distinct kinds of rays are emitted by radioactive substances.

The simplest type of activity is that of polonium. It gives rise to a brilliant fluorescence on a screen of zinc sulphide, which is resolved by a magnifying glass into countless scintillations, or bright flashes of short duration: but the luminosity falls off abruptly at a distance of 4 cm. from the active surface, and nothing whatever is visible beyond that distance. This, as Mme Curie first pointed out, suggests a type of radiation consisting of particles projected with a high velocity, and only capable of penetrating a distance of 4 cm. in air. But until more definite evidence of this has been obtained we shall use the non-committal term of α rays. The α rays penetrate only to very short distances in solid substances. Thus when a sheet of aluminium ·02 mm. thick is placed over the active layer the fluorescence ceases, as abruptly as before, 1 cm. away from the layer. A sheet of thickness ·03 mm. suffices to cut off the rays altogether. With a zinc sulphide screen on glass it is very curious to see a brilliant fluorescence when the zinc sulphide side is presented to the polonium plate, and none at all when the glass side is presented.

The behaviour of radium is more complicated*. There is always a residual fluorescence which is visible at 10 cm. or so from the source, and which is produced through considerable thicknesses of aluminium. These more penetrating rays are best observed on a screen of barium platinocyanide mounted on cardboard, which cuts off the α rays. The residual radiation is profoundly affected by a magnetic field of the order of 2000 units. To show this place the radium between the poles of a small electromagnet, and fix the screen horizontally about 1 cm. above the radium. On exciting the electromagnet part of the fluorescence is spread out into a bright band at right angles to the magnetic field, as shown in Fig. 348, and its direction changes with that of the magnetic field. These magnetically deviable rays are called β rays. An examination of the figure shows that they behave like

* Most of the experiments can be carried out with $\frac{1}{4}$ mg. of radium bromide, contained in a capsule with a thin mica cover, to let through the α rays. The greater the quantity of radium, the easier it is to demonstrate its effects.

negatively charged particles emitted by the radium, with velocities extending over a large range. We shall see later that they are electrons.

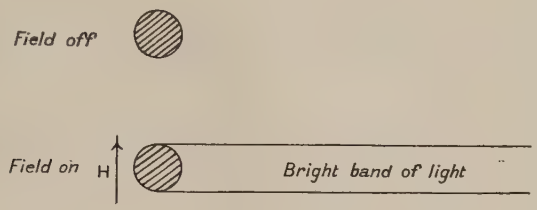


Fig. 348

Most of the β rays will pass through $\frac{1}{2}$ mm. of aluminium. As the thickness of the absorbing sheet increases they are cut off more and more, beginning with the most deviable rays, until finally only a faint undeflected spot remains. The rays coming through 5 mm. of aluminium or $1\frac{1}{2}$ mm. of lead are no longer appreciably affected by a magnetic field. It is probable, therefore, that there is a third type of radiation, very penetrating and not deflected by a magnetic field. These rays are called γ rays. They produce visible fluorescence through 4 mm. of lead; with care their electrical effect can be detected through 8 or 10 centimetres.

The α rays produce intense ionisation close up to the active body. This is illustrated by Rutherford's experiments, showing the effect of placing thin sheets of aluminium over a layer of uranium oxide (Fig. 349). Here the ionisation falls rapidly until a thickness of ·02 mm. is reached, after which further sheets produce a comparatively small effect. The complexity of the radiation was first shown in this way, before the discovery of the magnetic deflexion of the β rays. The difference between the β and γ rays can similarly be shown with sheets of lead, about 2 millimetres being required to cut off the former.

It is difficult to give exact data as to the fraction of the total ionisation to be ascribed to each kind of ray, since that depends very much on the size of the ionisation chamber and the thickness of the layer of active matter. With a thin layer of radium, and plates 5 cm. or so apart, the ionisation due to the α rays may be

100 times that due to the β rays, and that of the γ rays 100 times as small again. A thick layer tends to absorb its own α rays and

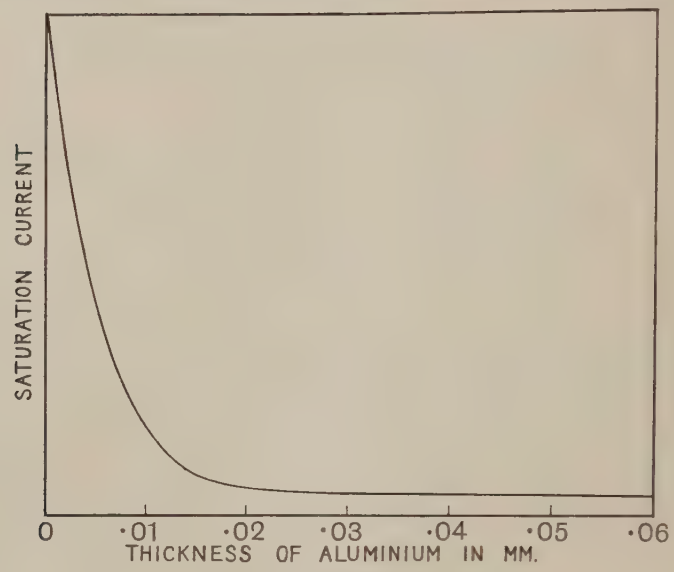


Fig. 349

give greater prominence to the β and γ rays. The γ rays produce comparatively little ionisation per unit volume; but they will travel almost any distance, and to measure them it is advantageous to use a large ionisation chamber. The photographic effect of a uranium compound seems to be largely due to the β rays emitted by it.

It must be admitted that the evidence of complexity furnished by absorption measurements is not as clear as it might be. In particular the γ rays might well be very fast β rays, which would be but little affected by a magnet and would also have high penetrating power. Nevertheless, the correctness of the classification has been proved by other experiments, which will be described later, and which have settled definitely the nature of the three kinds of rays.

In the following pages we shall give a short account of the radioactivity of radium. For the properties of uranium and thorium, and of actinium, a radioactive element separated from pitchblende by Debierne, reference may be made to Mme Curie,

Traité de Radioactivité, or Rutherford, *Radioactive Substances and their Radiations*.

235. Radioactive transformations. The accepted theory of radioactivity is that it consists in a disintegration of the atoms of the radioactive substance itself. These atoms are not absolutely stable, but there are always some of them in the act of breaking up, with the emission of α or β rays. (The γ rays, being as we shall see waves in the ether, have not here the same importance.) The result is an internal redistribution of some kind within the atom, in which it changes all its physical and chemical properties. The new substance thus formed may itself be unstable, and break up with the formation of a third substance, and so on.

The most precise form of this theory is that given by Rutherford and Soddy in 1902. They assume that if there are P atoms of a radioactive substance present at any time, the number breaking up per second is λP, where λ is a constant called the *decay constant* of the substance. Thus for a single substance decaying by itself we have $dP = -\lambda P dt$, which gives on integration

$$P = P_0 e^{-\lambda t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

P_0 being the number of atoms present at time $t = 0$. This law is found to be satisfied accurately by polonium, for which $\lambda = 5{\cdot}9 \times 10^{-8}$. Instead of the decay constant λ we may use the *half-value period*, or time T required for the activity to fall to half its original value. Then from (1) we have

$$T = \frac{\log_e 2}{\lambda} = \frac{0{\cdot}693}{\lambda}.$$

For polonium $T = 136$ days.

On the above theory the amount of a radioactive substance, when left to itself, is constantly diminishing, and if it appears to be constant it is only because the loss is inappreciable in the interval of time over which the observations extend. This is the case with uranium, thorium and radium. The example of polonium, whose activity largely disappears in the course of a year, is to be regarded as typical, and the behaviour of the very long-lived elements as exceptional.

We shall now consider the mathematical theory of more than one transformation. The simplest case is that of a radioactive substance A generated at a constant rate, say $\lambda_0 P_0$, by a substance of very slow decay. Let P be the number of atoms of A present at time t, λ_1 its decay constant. In time dt, $\lambda_0 P_0 dt$ atoms are produced by the primary substance, and $\lambda_1 P dt$ disappear by decay. Hence

$$dP/dt = \lambda_0 P_0 - \lambda_1 P \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Integrating, we have

$$P = Ce^{-\lambda_1 t} + \frac{\lambda_0 P_0}{\lambda_1},$$

where C is a constant. If no atoms of A are present at time $t = 0$, $C = -\lambda_0 P_0/\lambda_1$, and

$$P = \frac{\lambda_0 P_0}{\lambda_1}(1 - e^{-\lambda_1 t}) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

After a sufficient length of time the term $e^{-\lambda_1 t}$ becomes negligible, and we have

$$P = \frac{\lambda_0 P_0}{\lambda_1}.$$

It follows that a steady state is ultimately reached, in which the loss of the substance A by decay is exactly counterbalanced by the gain from the source. This is known as *radioactive equilibrium*. The quicker the decay of the substance A, the less the amount of it present in the equilibrium state, and the less also the time required to attain it, as equation (3) shows. The theory can be extended to cover the case of any number of successive products. If P, Q, R, ... are the numbers of atoms of the various products present at time t, and λ_1, λ_2, λ_3, ... their decay constants, we have

$$\begin{aligned} dP/dt &= \lambda_0 P_0 - \lambda_1 P, \\ dQ/dt &= \lambda_1 P - \lambda_2 Q, \\ dR/dt &= \lambda_2 Q - \lambda_3 R, \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

In the equilibrium state the differential coefficients on the left vanish, and we have

$$\lambda_1 P = \lambda_2 Q = \lambda_3 R = \ldots = \lambda_0 P_0 \ldots\ldots\ldots\ldots (4).$$

The numbers of atoms of the different substances are therefore

proportional to their half-value periods—a result which has an important bearing on the amounts of various radioactive elements to be expected in minerals.

The following theory will be useful to us later in discussing the "active deposit of radium." Let there be three substances A, B, C of decay constants λ_1, λ_2, λ_3, A producing B and B producing C; and initially let all the matter be of one kind A. It is required to find the numbers of atoms of A, B, C present at all subsequent times. The equations are

$$dP/dt = -\lambda_1 P,$$
$$dQ/dt = \lambda_1 P - \lambda_2 Q,$$
$$dR/dt = \lambda_2 Q - \lambda_3 R.$$

These equations can be integrated in succession, the first giving P, the second then giving Q, and the third R. Adjusting the constants of integration so that $P = P_0$, $Q = R = 0$ at time $t = 0$, we find

$$\left.\begin{aligned} P &= P_0 e^{-\lambda_1 t} \\ Q &= -\frac{\lambda_1 P_0}{\lambda_1 - \lambda_2}\,(e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\ R &= \frac{\lambda_1 \lambda_2 P_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \\ &\qquad \times \{(\lambda_2 - \lambda_3) e^{-\lambda_1 t} - (\lambda_1 - \lambda_3) e^{-\lambda_2 t} + (\lambda_1 - \lambda_2) e^{-\lambda_3 t}\} \end{aligned}\right\} \ldots (5).$$

The numbers of atoms of A, B, C breaking up per second are $\lambda_1 P$, $\lambda_2 Q$, $\lambda_3 R$ respectively; hence if each substance in decaying emitted rays of the same quality the activity would be represented by $\lambda_1 P + \lambda_2 Q + \lambda_3 R$. This, however, is not the case, and one of the products may not emit any rays of the type considered. The activity at any time is represented by $k_1 P + k_2 Q + k_3 R$, where k_1, k_2, k_3 are constants depending on the size and shape of the apparatus and the kind of rays that are being observed. For large values of t the activity is proportional to $e^{-\lambda t}$, where λ is the least of the quantities λ_1, λ_2, λ_3; hence the law of decay of the activity is ultimately that of the slowest of the separate constituents.

236. Radium and its emanation. Although radium itself is a substance of very slow decay, it can be seen from its behaviour

that it has short-lived products. For example, pure radium on preparation emits only α rays, together with weak β rays. At the end of a month the α ray activity has quadrupled, the β ray activity increased fifty-fold, and in addition there is a strong γ ray activity. Henceforth the activity remains very nearly constant, though it continues to increase slightly for years. The problem of finding how many products there are, and their relation to one another, will occupy us in this and the next articles.

The following experiment shows that radium is constantly giving off a radioactive gas, called the *radium emanation*. A vessel R (Fig. 350), closed by a tap T_1, contains a solution of a radium salt. Another vessel, communicating with it through a U-tube S, contains a zinc sulphide screen Z, which fluoresces whenever α rays fall on it. The tap T_1 is first closed and the remainder of the apparatus exhausted with a pump, and the radium solution is then boiled. On opening the tap T_1 a bright fluorescence immediately appears on the screen Z. Since this takes place equally well when the vessel R is surrounded by thick lead screens, it cannot be due to any direct action of the rays emitted by the radium. Nor is it due to the passage of ionised gases from the radium chamber to that containing the fluorescent screen: for it is not stopped by filling the space C with cotton-wool, which would remove all the ions from the stream. But the luminosity is stopped by immersing the tube S in liquid air, either before or after opening the tap T_1. These facts point to the emission of a radioactive gas, which is liquefied at the temperature of liquid air. A great increase in the luminosity occurs when the right-hand tube itself is immersed in liquid air, as the liquid emanation then collects at the bottom of the tube.

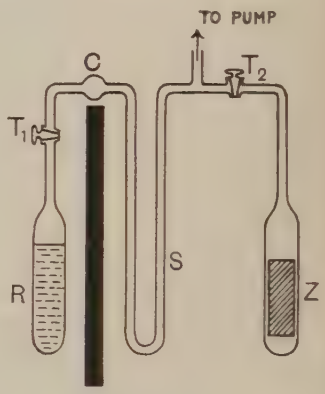


Fig. 350

The gas given off from radium solutions does not consist of the pure emanation, but chiefly of hydrogen and oxygen, arising

from the action of radium on water. Helium and carbon dioxide are also present. After removal of these impurities a small amount of pure emanation is left, which may be condensed by pumping it above mercury into a fine capillary tube. In this way enough has been collected to determine its physical and chemical properties. The volume obtained is never more than a fraction of a cubic millimetre at atmospheric pressure; but most of its properties can be examined without isolating a weighable quantity of it, since its presence is everywhere indicated by its intense radioactive effect. Like radium, the emanation has a characteristic spectrum of lines belonging to no other element. It belongs to the argon group of inert gases, and is the heaviest gas known. Its atomic weight has been found by density measurements to be about 223, on the assumption that the molecule is monatomic.

Pure radium emanation, contained in a sealed tube, emits at first only α rays, the β and γ ray activity appearing in the course of a few hours. There is thus evidence of the existence of further transformation products, which, as we shall see, are solid and deposited on the walls of the tube. In order to find the decay of the emanation itself it is necessary to extract a known fraction of the gas at intervals, and measure its activity either immediately or at a definite time after removal. For convenience of handling, the emanation is kept mixed with air in a small reservoir over mercury. In this way it has been found to diminish in time according to an exponential law, with a half-value period of 3·85 days; a number which is probably correct to about 1 part in 1000. The corresponding decay constant is $2{\cdot}085 \times 10^{-6}$.

The volume of emanation in equilibrium with one gram of radium is about 0·60 cubic millimetres at 0° C. and 760 mm. pressure. From this it is possible to find the half-value period of radium itself. For the above volume contains $1{\cdot}63 \times 10^{16}$ atoms of emanation, and it is in equilibrium with $2{\cdot}71 \times 10^{21}$ atoms of radium. Using the equation $\lambda_0 P_0 = \lambda_1 P_1$, and putting $\lambda_1 = 2{\cdot}085 \times 10^{-6}$, we have $\lambda_0 = 1{\cdot}25 \times 10^{-11}$. The half-value period of radium is therefore about 1750 years.

It is very important to decide whether the emanation is produced directly by radium, or whether there is another transformation product in between. The method of solving such

problems is as follows. It is known that the equilibrium amount of emanation can be collected from radium regularly at intervals of one month, which, as equation (3) shows, is to be expected if radium changes directly into the emanation; the rate of growth would be much slower if a product of long life intervened. The same method of reasoning has been applied in other cases to prove that one substance could not be the direct parent of another.

In 1903 Ramsay and Soddy made the important discovery that helium is continually being produced by radium emanation. When purified emanation is pumped into a small vacuum tube and allowed to decay *in situ*, its spectrum grows gradually fainter, while the helium spectrum appears and becomes gradually more and more distinct. This fact has an important bearing on the nature of the α rays (see Art. 239).

237. Radium A, radium B, radium C. It was discovered by P. and Mme Curie in 1899 that substances left for some time in the presence of uncovered radium salts acquire a temporary activity, which largely disappears in the course of a few hours. The same is true of the walls of a vessel that has contained radium emanation. This was at first called "induced radioactivity": but since the activity is, under similar circumstances, the same for all non-porous substances, it can hardly be due to any secondary action on the substance itself. Further, the activity is confined to a very thin layer on the surface, which can be removed by friction. The most natural explanation is that there is a radioactive substance deposited on the surface of the body. The *active deposit*, as it is called, is only produced in places accessible to the emanation: thus its formation can be prevented by blowing away the emanation with a current of air as it rises, or by enclosing the radium in an air-tight case, even if the walls are thin enough to let through all the three kinds of rays. On the other hand, active deposits can be formed at great distances from the source by blowing a stream of emanation down a long tube. The view adopted by Rutherford, to whom the explanation of these effects is chiefly due, is that radium emanation, in decaying, gives rise to products which are solid at ordinary temperatures, and are therefore deposited on the walls of the tube.

It is found that while the activity of exposed wires is very much reduced in the course of 12 hours, it does not actually vanish, but there is always a residue which persists for many years. It is thus convenient to distinguish between the *active deposit of quick decay* and the *active deposit of slow decay*. The activity of the latter, which never constitutes more than one-thousandth part of the whole, is considered separately in the next article, and what follows refers to the active deposit of quick decay.

If a wire is exposed for a few seconds in the presence of radium emanation and then removed, its activity falls off as shown in Fig. 351, curve I representing the α ray activity and curve II that

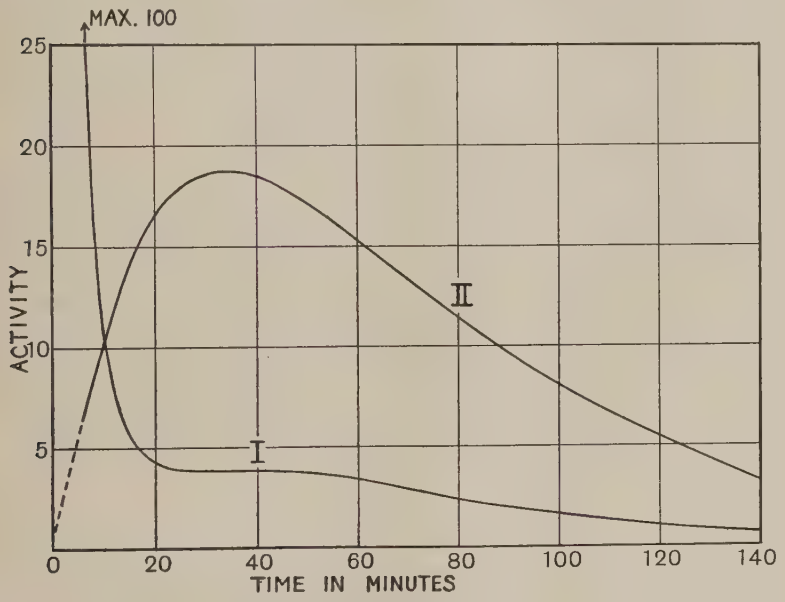


Fig. 351

due to the β and γ rays together. It is clear from this that the active deposit of quick decay is itself complex. A full account of the experiments necessary to disentangle the various products, and to assign to each its proper radiation, would occupy too much space: but we may form some idea of these as follows.

The deposit, just after a short exposure, consists almost entirely of the first transformation product of the emanation, which we

may call radium A. Now initially the α ray activity falls according to an exponential law, with a half-value period of three minutes. It is therefore probable that radium A emits α rays and has a period of about three minutes; and curve II shows that it has at most a very weak β ray activity. For large values of t all the activity curves tend to the exponential form, with a half-value period of 26·7 minutes; which gives the period of the product of slowest decay (not necessarily the last product of the series). It is obviously useless to attempt to separate any of the intermediate products by chemical means, for even if once prepared they will not remain pure, but are constantly changing into other substances. Hence if any pure product is obtained it must be the last product of the series. By dipping a copper or nickel plate into a solution of the active deposit von Lerch obtained a deposit which decayed accurately according to an exponential law, with a half-value period of 19·5 minutes.

These facts are most simply explained by supposing that there are three successive products in the active deposit of quick decay —radium A with half-value period 3 minutes, radium B with period 26·7 minutes, and radium C with period 19·5 minutes. The corresponding decay constants are ·00385, ·000433 and ·000593 respectively. The evidence from the activity curves can be used to test this assumption and to find what rays are given off by the different products in decaying. For example, the curve I of Fig. 351 can be represented fairly accurately by the equation

$$y = \frac{100P + 12{\cdot}3R}{P_0},$$

where P, R are taken from equation (5), Art. 235, and λ_1, λ_2, λ_3 have the values given above. Since good agreement can be obtained by putting $k_2 = 0$ in the formula $k_1P + k_2Q + k_3R$, it is probable that radium B emits no α rays. Indirect evidence of this kind must be accepted with caution; its cogency depends very much on the number of constants available for adjustment, and if the latter are numerous the information gained may be quite valueless. In the present case there are additional checks on the theory, since the values of k_1, k_3, when once chosen, have to serve for all times of exposure to the emanation, from a few

seconds to several hours. Information as to the rays emitted by the different products is afforded by experiments with various thicknesses of lead or aluminium over the active layer. In this way it has been shown, beyond reasonable doubt, that radium A emits α rays only, radium B emits β rays and weak γ rays, and radium C emits α, β and γ rays. The transformations occurring between radium and radium C are shown in the following scheme:

$$\text{Radium} \xrightarrow{\alpha,\ \text{weak}\ \beta} \text{emanation} \xrightarrow{\alpha} \text{radium A} \xrightarrow{\beta,\ \text{weak}\ \gamma} \text{radium B} \xrightarrow{\alpha,\ \beta,\ \gamma} \text{radium C.}$$

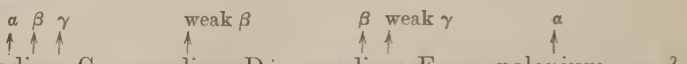
We can now give a more precise account than before of the origin of the different rays from radium, say one month after preparation. By this time radium is in equilibrium with the emanation and with the products A, B, C. The α rays are given off by radium, radium emanation, radium A and radium C. The β rays arise mainly from radium B and radium C, and the γ rays from radium C. Since most products give off either α rays or β rays, but not both, it has been conjectured that radium C consists of two successive products, one of which is of very short life. There is also evidence to show that the atoms of radium C can break up in two distinct ways: but the consideration of this would lead us too far.

Radium C, in decaying, gives rise to the active deposit of slow decay, which we now consider.

238. Radium D, radium E and polonium. The behaviour of the active deposit of slow decay somewhat resembles that of radium immediately after preparation. At first there is only a weak β ray activity, which rises gradually for about two months, and afterwards remains nearly constant. An α ray activity is also developed, but more slowly, requiring two years to reach its maximum value. It is probable, therefore, that the first product of radium C is a substance of slow decay, which we may call radium D, and which itself gives rise to further products.

The separation of these products is comparatively simple. If a polished plate of bismuth or copper is immersed in a solution of the active deposit of slow decay, a deposit is obtained which emits only α rays, and decays exponentially with a period of about 140 days. This exact coincidence with the properties of polonium

can hardly be accidental; hence we conclude that polonium is a transformation product of radium D, and it is the *last* product. If a nickel plate is used instead of bismuth or copper, the deposit at first emits no α rays, but only β rays and weak γ rays, and both activities decay exponentially with a period of 5 days. These facts are exactly accounted for by the following scheme of transformation:



As before, we can test for intermediate products by means of the curves giving the rise of α and β ray activity of radium D with time. No evidence of any such product has been obtained, and there is certainly none of long life.

Equilibrium between radium D, radium E and polonium is practically attained in two years, after which the product shows α, β and γ ray activity, all of which fall off exponentially with the period of radium D. Mme Curie, by direct observations over a period of five years, estimated the period of radium D at 17 years. The chemical properties of radium D closely resemble those of lead, from which it cannot be separated by any known means. It was discovered at an early period that the lead extracted from pitchblende had a very considerable activity, which was ascribed to the presence of a radioactive constituent known as radio-lead. This is now known to be identical with radium D.

The main transformation series of radium, as far as it is necessary to consider it here, is shown in the following table:

	Radiation	Period
Radium	α + weak β	1750 years
Radium emanation ...	α	3·85 days
Radium A	α	3 minutes
Radium B	β + weak γ	26·7 minutes
Radium C	$\alpha + \beta + \gamma$	19·5 minutes
Radium D	weak β	17 years
Radium E	β + weak γ	5 days
Polonium	α	136 days
?	—	—

It can be calculated that over 100 years must elapse before radium is in equilibrium with *all* its products, the delay arising from the long period of radium D. Neither radium D nor its products emit any very penetrating rays, the γ ray activity when measured through 3 mm. of lead being due entirely to radium B and radium C, and most of it to the latter. Hence the γ ray activity is proportional to the amount of radium present, provided that at least one month has elapsed after preparation. This is the basis of an accurate method of comparing quantities of radium, which has the further advantage that, since only very penetrating rays are used, the absorption by the walls of the containing vessel and the radium preparation itself is small and need not generally be allowed for.

In handling radium great care must be taken to avoid what is known as *radioactive contamination*. Any very considerable or prolonged escape of emanation in a laboratory will result in the formation of the active deposit of slow decay on the walls of the building, and give rise to a continual disturbing leak. No chemical operations on radium should be performed in a building in which electrometer work is done, nor should radium emanation be used in other than sealed tubes.

239. The α particles. The α rays were at first thought to be unaffected by magnetic fields; and certainly they are but little deflected in fields strong enough to produce a very great deflexion of the β particles. Rutherford was the first to show that the α rays behave like positively charged particles projected with a great velocity. The exact measurement of e/m and v is made more difficult by the fact that radium sends out several sets of rays with different initial velocity, and if the active layer is at all thick there is an additional loss of velocity by transmission through the layer. In one of Rutherford's experiments the source of the α rays was a deposit of radium C

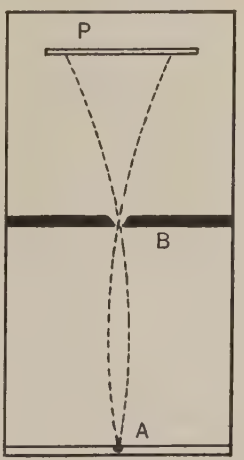


Fig. 352

on a thin wire, laid in a groove at A in the figure. Above it was a metal plate B pierced with a slit parallel to the wire, and higher still a photographic plate P. A strong and uniform magnetic field was applied parallel to the wire, and the whole space inside the case was highly exhausted. If a is the distance from A to B, b the distance from B to P, and ρ the radius of the circle into which the rays are bent, the deflexion δ (supposed small) is given by $2\rho\delta = b\,(a+b)$. But from Art. 206, $\rho = mcv/He$. Hence

$$\frac{e}{mv} = \frac{2c\delta}{Hb\,(a+b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The double deflexion caused by reversing the magnetic field was measured on the plate. It was found that the width of the deflected band was the same as that of the undeflected band: hence *with a homogeneous product like radium C all the rays are sent off with the same velocity*. The decrease in the velocity of the rays after traversing a thin sheet of aluminium was clearly shown by an increase in the magnetic deflexion. The order of magnitude of the latter is seen from the fact that ρ is about 40 cm. in a field of 10,000 gausses.

Strong electric fields are required if the electric deflexion is to be measured accurately. Rutherford used two plates C_1, C_2 (Fig. 353), $\frac{1}{4}$ mm. apart, placed immediately over the active matter. In most of the experiments the ray $ABCD$ which experienced the greatest deflexion left the active matter at A, grazed the left-hand plate at B, and passed over the edge C of the right-hand plate, falling on the photographic plate P at D. On reversing the field a similar extreme ray $A'B'C'D'$ is obtained. The path of either ray is parabolic between the plates, and straight outside. If d is the distance between the plates C_1, C_2, g the distance from C to the photographic plate, and 2ϵ the double deflexion DD', supposed small, it is not difficult to show that

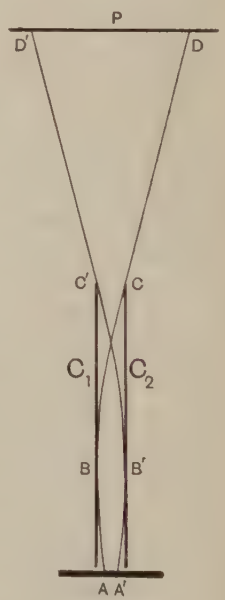


Fig. 353

$$\frac{e}{mv^2} = \frac{(\epsilon - \frac{1}{2}d)^2}{2g^2Ed} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Equations (6) and (7) give e/m and v separately. For radium C, Rutherford found $e/m = +\ 1{\cdot}52 \times 10^{14}$, and for polonium $e/m = +\ 1{\cdot}59 \times 10^{14}$. These values are a little too high. Subsequent experiments have shown that the values of e/m for the α particles from the emanation, radium A and radium C agree to within 1 part in 400, and are equal to $1{\cdot}446 \times 10^{14}$; which shows that the α particle is in all probability the same no matter from what source it is derived. The velocity of projection of the particles from an uncovered preparation of radium C is $1{\cdot}92 \times 10^9$ cm. per second.

The charge on an α particle, if at all comparable with that of the electron, is too small to be measured directly; but it can be found indirectly as follows. It is very probable that the scintillations, of which mention has been made in Art. 234, are produced by the separate impacts of the α particles on the fluorescent screen. If therefore these scintillations are counted on a small screen placed in various positions with respect to the active layer, it is possible to estimate the total number of α particles emitted by the layer per second; after which it is only necessary to measure the total charge carried away by the α particles in a given time. The scintillations were first systematically counted by Regener. Rutherford and Geiger used another method of counting, which consisted in magnifying the ionising effect of the α particles by collisions (cf. Arts. 220–223) until the entry of a single α particle into the ionisation chamber produced a measurable deflexion of an electrometer. They found that the number of particles, as counted electrically, was nearly the same as that found from observations on a carefully-made zinc sulphide screen.

The measurement of the total charge on the α particles from a given thin layer presents some difficulty. Sir J. J. Thomson found that a metallic plate covered with polonium, and placed in a high vacuum, emits a negative and not a positive charge. The negative leak is reversed when the whole apparatus is placed between the poles of a powerful electromagnet, so that it is probably due to the emission of slow-moving electrons as well

as α particles from the polonium plate. It is now generally believed that slow electrons are emitted whenever α rays fall on matter. But even in the highest attainable vacua, and in strong magnetic fields, the current in an apparatus such as that shown in Fig. 346 is slightly different according as the active plate is positive or negative. Rutherford assumed that the difference arises from the ionisation of the residual gas between the plates, and that the charge carried by the α particles per second is equal to the mean of the two currents.

In this way Rutherford and Geiger obtained values of e ranging from $8{\cdot}3 \times 10^{-10}$ to 10×10^{-10}, mean $9{\cdot}3 \times 10^{-10}$ E.S.U. Shortly afterwards Regener completed his counting experiments by measuring the total charge, and found that $e = 9{\cdot}58 \times 10^{-10}$ E.S.U. Comparing these numbers with Millikan's value $4{\cdot}77 \times 10^{-10}$ it becomes extremely probable that the α particle carries a charge numerically double that of the electron. Conversely, if this is assumed, the experiments of Rutherford and Geiger and of Regener give new values $4{\cdot}65 \times 10^{-10}$ and $4{\cdot}79 \times 10^{-10}$ for the charge of the electron. The value $4{\cdot}75 \times 10^{-10}$ has been adopted in this book, giving $9{\cdot}50 \times 10^{-10}$ for the charge on the α particle.

The mass of the α particle can now be found. Putting $e = 9{\cdot}50 \times 10^{-10}$ and $e/m = 1{\cdot}446 \times 10^{14}$, we have $m = 6{\cdot}57 \times 10^{-24}$ grams. Since the mass of a hydrogen atom is $1{\cdot}65 \times 10^{-24}$ grams, it is simplest to assume that *the α particle is an atom of helium*, positively charged and travelling with a high velocity.

The conclusions of the present article receive independent support as follows. One gram of radium in equilibrium with the emanation and with radium A, B, C emits $1{\cdot}43 \times 10^{11}$ α particles per second; and since there are *four* α ray products the number emitted by radium itself is $3{\cdot}57 \times 10^{10}$. If, as is probable, each corresponds to the disintegration of one radium atom, this number is equal to λP, where λ is the decay constant of radium and $P = 2{\cdot}71 \times 10^{21}$ the number of atoms in one gram. This gives $\lambda = 1{\cdot}32 \times 10^{-11}$, which agrees well with the value $1{\cdot}25 \times 10^{-11}$ previously found. The present estimate is more reliable, since the measurement of small volumes of emanation is difficult. If each α particle is a helium atom, this would correspond to an emission of 0·175 c.c. of helium per year by one

gram of radium in equilibrium. The observed amount is about 0·170 c.c.

240. Ionisation produced by α particles. The α particles, as we have seen, produce intense ionisation close up to the active substance. The sudden cessation of the ionising effect at a certain distance from the source, first observed by Mme Curie for polonium, shows that the α particles move straight on through the gas, losing energy as they go, until finally their velocity is insufficient to enable them to ionise the molecules by collision. The deviation from the straight line is in all probability slight, except near the end of the path: hence if a nearly parallel beam of α rays is obtained and the ionisation per unit length measured at various points, this will tell us approximately how the ionising power of a single α particle varies along its path.

The preceding considerations are due to W. H. Bragg, who first measured the ionisation at various points of a beam of α rays.

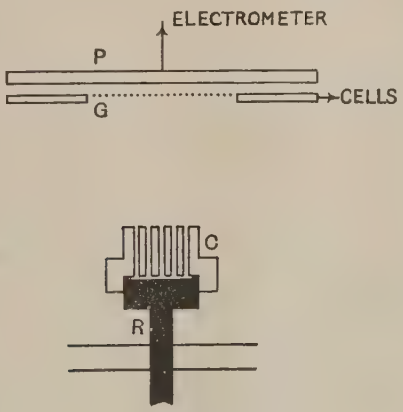


Fig. 354

A thin layer of polonium deposited on a disc of copper (Art. 238) gives a convenient source of α rays, and this is attached to a rod R passing through a hole in a fixed brass plate. In order to render the beam parallel a cap C is placed over the disc, consisting of a thick piece of brass perforated with holes about 1 mm. in diameter, so that practically only rays travelling in a vertical direction can escape from the polonium. The ionisation is measured in the

space between a wire grating G and a brass plate P, and the distance x from the polonium to the centre of the ionisation chamber can be varied by sliding the rod R up and down. At atmospheric pressure the effect of electrons emitted from the plate P is negligible in comparison with the current in the gas itself, as is easily seen by reversing the field. Hence with a narrow gap the saturation current is nearly proportional to the ionisation per millimetre of the path of an α particle at distance x from the source. The results of this experiment are shown in Fig. 355. With an uncovered

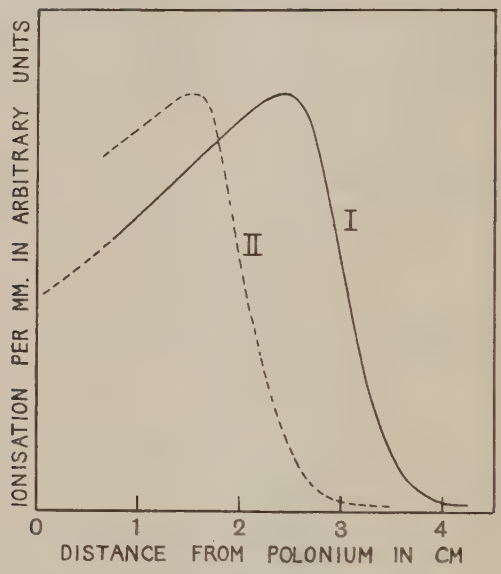


Fig. 355

polonium layer (curve I) the ionisation per unit length first rises, attains a maximum at about 2·5 cm. from the source, and then sinks rapidly to zero. It is probable that if diffusion could be prevented the ionisation would entirely cease at a distance of about 3·8 cm. from the polonium. This distance is called the *range* of the α particles, and marks the stage at which they are no longer capable of ionising the gas. Towards explaining the maximum in the ionisation curve Bragg points out that the faster the particles the less the time occupied in moving over the diameter of a molecule of the gas, and therefore also the time available

for the detachment of electrons. On the other hand, if the velocity is too small the α particle does not get near enough to the centre of the atom or molecule to produce an appreciable effect.

Curve II is obtained with a sheet of aluminium ·006 mm. thick placed over the polonium. The effect is merely to move curve I back as a whole 1 cm. nearer to the active layer; which confirms the view that the α particles, though retarded, are very little deflected by passing through light substances. It will be noticed that the ionisation at certain distances is *increased* by the interposition of the aluminium sheet.

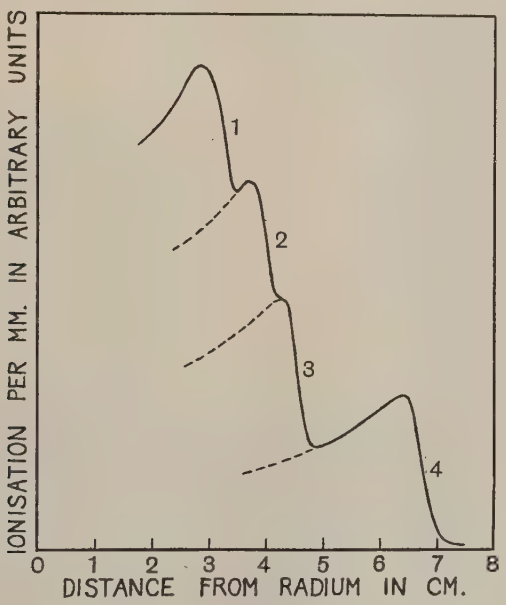


Fig. 356

Experiments with radium give a more complicated curve (Fig. 356) which strongly suggests an analysis into four simple curves, as shown. It is known that radium in equilibrium with its products of quick decay contains four substances which emit α rays—radium, radium emanation, radium A and radium C. The numbers of the different atoms breaking up per second are equal (see Art. 235), and therefore presumably also the numbers

of α particles emitted by the four products. The curve in Fig. 356 can be exactly explained on this hypothesis, as Bragg showed. Towards settling the particular ray to be assigned to each of the separate products we proceed as follows. A freshly prepared layer gives a single curve corresponding to portion 1 of the figure: a deposit of radium C, prepared by von Lerch's method, gives curve 4 alone. As regards the remaining two, it is known that the α rays from radium emanation are more easily absorbed than

Ranges in cm. in air at 15° C. and 760 mm. pressure.

Radium ...	3·30
Emanation ...	4·16
Radium A ...	4·75
Radium C ...	6·94
Polonium ...	3·77

those from radium A: hence curve 1 is to be ascribed to radium, curve 2 to the emanation, curve 3 to radium A and curve 4 to radium C. An old preparation of radium would show a disturbance between 1 and 2 due to the α rays from polonium; but about 100 years would be required for its full development.

The ranges of the various α particles, as measured by Geiger, are shown above.

Fig. 357

The actual paths of the α particles from radioactive substances have been made visible by C. T. R. Wilson, using the cloud method described in Art. 232. Fig. 357 shows a photograph obtained with a small quantity of radium on the end of a wire. For the most part the tracks are straight and quite continuous, no resolution into separate drops being possible on account of the large number of ions produced per centimetre of the path. Near the end there is nearly always some curvature, and the rays are often deflected through finite angles, as in Fig. 358, which gives an

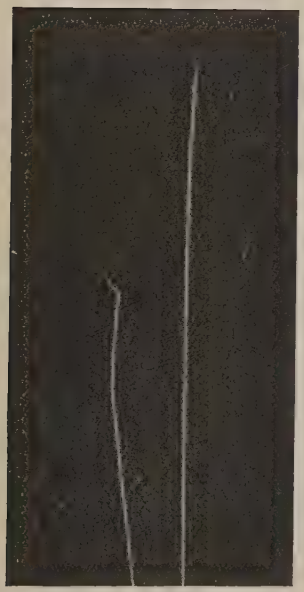


Fig. 358

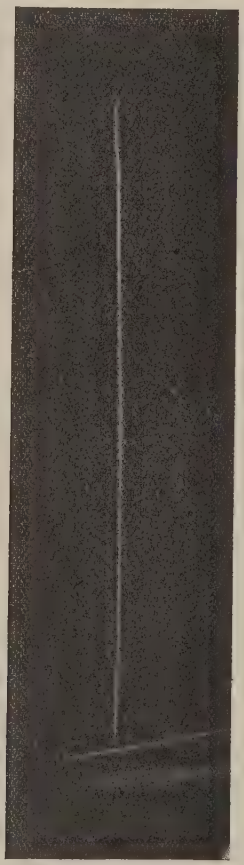


Fig. 359

enlarged view of the ends of two tracks. The "spur" on the left-hand track should be noticed. It is probably due to an atom recoiling with such a velocity as to ionise for a short distance by collision. The ionising power of the α particle falls off very abruptly near the end of its range, and the comparatively gradual drop in Fig. 355 is to be ascribed to scattering and to slight inequalities in the ranges of individual particles.

Wilson's experiments, like Bragg's, confirm in a striking way the theory of successive transformations. At first all the α rays radiate from the radium at the end of the wire; but as the emanation diffuses into the expansion chamber more and more tracks originate *in the gas itself*, and others, due to the active deposit, start from various points on the walls of the chamber. The track of a complete α particle from radium emanation is shown in Fig. 359. Here we notice the characteristic hook at the end, the spot at the beginning due to the recoil of the atom of the emanation, and the gradual increase of the density of ionisation as the ray slows down. One of these tracks measured by Wilson had a length of 4·3 cm., reduced to 15° C. and 760 mm. pressure: which agrees well with the accepted value 4·16 cm. of the range of the α particle.

The average number of ions produced by an α particle from radium C is about 3000 per millimetre of its path, or about 200,000 in all. It is obvious from the initial distribution of these ions that there must be a great deal of recombination, and most of all when the electric force is parallel to the direction of the α particles, since then an ion is constantly meeting other ions produced by the same particle. Experiments by Moulin show that the electric force required for saturation does, in fact, vary with the direction of the rays, as much as 1000 volts per cm. being necessary when the rays are parallel to the electric force.

241. The β particles. The magnetic deflexion of the β rays was discovered by Giesel in 1899. Early experiments showed that the rays were complex, and that the slowest rays were most easily absorbed by matter. It was also found possible to detect the negative charge carried by the β particles, provided that the receiving plate was completely surrounded by insulators

in order to prevent dissipation of the charge by the neighbouring gas.

The charge on the β particles has never been directly measured, but many experiments have been made to determine the velocity and the ratio of the charge to the mass. Perhaps the most accurate are those of Bucherer, which have since been repeated by G. Neumann. The principle of Bucherer's experiment is shown

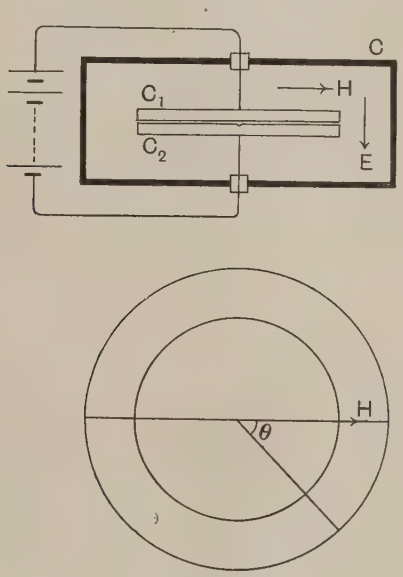


Fig. 360

in Fig. 360. A cylindrical box C contains two parallel plates C_1, C_2, about 20 cm. in diameter and $\frac{1}{4}$ mm. apart. A strong and uniform electric field is applied by connecting the plates to the terminals of a battery of small accumulators. The whole box is enclosed in a solenoid long enough to give a uniform magnetic field parallel to the plates, as shown in the figure, and the pressure inside the box is kept as low as possible. A very small cavity in the lower plate contains radium fluoride, and the inner curved wall of the box is covered with a photographic film pressed against it by springs. In the absence of electric and magnetic forces

β rays spread straight out from the radium, and those which are projected horizontally escape, giving rise to a straight band on the photographic film. To find what rays escape when the electric and magnetic fields are both applied, consider those projected horizontally in a direction making an angle θ with the magnetic field (Fig. 360). The electric field exerts a force eE downwards, and the magnetic field a force $eHv \sin \theta/c$ upwards, where v is the velocity. If these forces are not equal the particle is drawn to one or other of the plates; hence only those particles escape for which $E = Hv \sin \theta/c$. The effect of the combined fields is therefore to sort out the particles of different velocity, so that each corresponds to a particular value of θ. After leaving the condenser the rays are acted on by the magnetic field only, and are bent into a circle of radius ρ, where $\rho = mcv/He$. Bucherer actually only considered the rays experiencing the maximum deflexion, for which $\theta = \frac{1}{2}\pi$. Hence we have

$$\frac{v}{c} = \frac{E}{H} \text{ and } \frac{e}{m} = \frac{cv}{\rho H} = \frac{c^2 E}{\rho H^2}.$$

Since ρ is known from observations of the deflexion, both e/m and v are known.

In these experiments no trace has ever been obtained with fields such that $E > H$. Hence *the velocity of the β particles is always less than that of light.* Another remarkable fact, first observed by Kaufmann, is that *the value of e/m is not constant, but decreases with increase of v.* The smaller the velocity the more e/m approximates to the value $5{\cdot}30 \times 10^{17}$ which it has for cathode rays and other comparatively slow electrons. Thus, though there is no reason for believing that the β particles are not electrons, yet it must be recognised that a new difficulty arises in that e/m appears to be a function of the velocity. The experimental facts might be explained in several ways; for example, that the mass of an electron depends on its velocity, that its charge does, or that the law of action of a magnetic field on a moving charge (Art. 95) has to be modified for very high velocities of the particle. It has been found most convenient to assume that the *mass* alone varies: the theoretical aspects of this are discussed fully in the next chapter.

The experiments of Bucherer and Neumann agree very accurately with the formula

$$\frac{e}{m} = \frac{e}{m_0}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

over a range $v/c = 0{\cdot}3$ to $0{\cdot}8$, e/m_0 having the normal value $5{\cdot}30 \times 10^{17}$. This formula has a theoretical foundation (Art. 257), but may at present be regarded as empirical. It follows that the mass m of an electron at velocity v is given by

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where $m_0 = 8{\cdot}96 \times 10^{-28}$ is the mass of an electron with a velocity small compared with that of light. The values of m/m_0 and of $\rho H = mcv/e$ for various values of v are shown in the following table. Hence by interpolation it is possible to find the velocity of a β particle by magnetic deflexions alone.

v	m/m_0	ρH	v	m/m_0	ρH
10^9	1·000	56·8	$2{\cdot}15 \times 10^{10}$	1·434	1745
2×10^9	1·002	113	2·20 ,,	1·471	1832
3 ,,	1·005	171	2·25 ,,	1·512	1925
4 ,,	1·009	229	2·30 ,,	1·557	2027
5 ,,	1·014	287	2·35 ,,	1·609	2140
6 ,,	1·021	346	2·40 ,,	1·667	2264
7 ,,	1·028	408	2·45 ,,	1·733	2403
8 ,,	1·038	470	2·50 ,,	1·809	2561
9 ,,	1·048	534	2·55 ,,	1·899	2740
10^{10}	1·061	600	2·60 ,,	2·00	2949
$1{\cdot}1 \times 10^{10}$	1·075	669	2·65 ,,	2·13	3200
1·2 ,,	1·091	741	2·70 ,,	2·29	3507
1·3 ,,	1·110	817	2·75 ,,	2·50	3891
1·4 ,,	1·131	896	2·80 ,,	2·79	4416
1·5 ,,	1·155	980	2·85 ,,	3·20	5165
1·6 ,,	1·182	1071	2·90 ,,	3·91	6412
1·7 ,,	1·214	1168	2·95 ,,	5·50	9177
1·8 ,,	1·250	1273	2·96 ,,	6·14	10290
1·9 ,,	1·292	1390	2·97 ,,	7·08	11910
2·0 ,,	1·342	1519	2·98 ,,	8·69	14640
2·05 ,,	1·370	1585	2·99 ,,	12·2	20730
2·10 ,,	1·400	1665	3 ,,	∞	∞

Fig. 361 shows one of Wilson's photographs of the tracks of β particles in gases. The complete trails are too long to be seen

on a single photograph. Sudden deflexions through finite angles are very rare, most tracks showing a continuous curvature due

Fig. 361

to a succession of small deflexions, each in itself inappreciable. At the very end the tracks closely resemble those of electrons emitted under the action of Röntgen rays (Art. 232).

242. Homogeneous groups of β particles. The β particles from a thick layer of a radioactive substance have all velocities over a considerable range; but with a thin layer a number of the particles, at least, form homogeneous groups of definite velocity. This was first shown by von Baeyer, Hahn and Frl. Meitner, by means of the magnetic deflexion in a vacuum. Their apparatus closely resembled that used for the magnetic deflexion of the α particles (Fig. 352), except that much smaller magnetic fields were used. A thin film of radium deprived of its transformation products gave two weak bands on the photographic plate; a deposit of radium C gave four strong bands, and so on. The velocity of the rays can be found by interpolation from the table in the last article, or by direct calculation from the equations

$$\rho H = \frac{mcv}{e}, \quad \frac{e}{m} = 5{\cdot}30 \times 10^{17}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}.$$

The results of these experiments are shown in Fig. 362, which

gives the velocities of the most prominent groups of rays from radium and its products. Some of the rays from radium C have velocities closely approaching that of light. The rays from radium

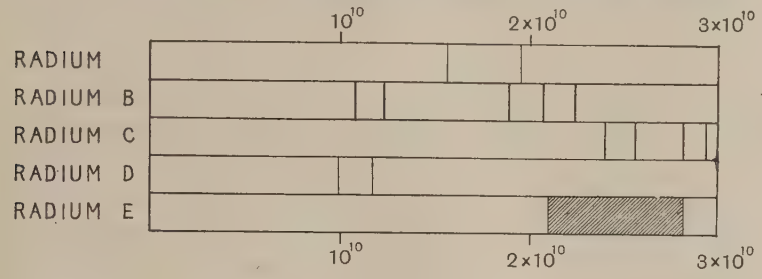


Fig. 362

D are both weak and feebly penetrating, and were, in fact, actually discovered by the photographic method. So far radium E, which gives a continuous spectrum of β rays with velocities between 2·1 and 2·8 $\times$ 10^{10} cm. per second, has not been found to emit any homogeneous groups.

More detailed observations by Danysz and others have shown that the β ray spectra are often very complicated. Rutherford and Robinson were able to measure the velocity of 64 sets of rays in the radiation from radium B + C, and other faint bands were also visible on the photographic plate.

243. The γ rays. The γ rays were discovered by Villard in 1900. They are chiefly remarkable for their great penetrating power. After the γ rays from radium have passed through 3 mm. of lead, every additional thickness of 1·5 cm. of lead, or 6 cm. of aluminium, cuts them down to about one-half of their original intensity. The electrical effect of a strong radium preparation can be measured through 20 cm. of lead, and might possibly be detected through 1 metre of aluminium. The absorption of the air is such that a thickness of about 100 metres at atmospheric pressure is required to cut down the rays by one-half.

From the fact that γ rays are not deflected in magnetic fields, and that they are always found together with fairly swift β rays, it has been concluded that they are similar to Röntgen rays, only harder. Since the discovery of the diffraction of Röntgen

rays by crystals it has become possible to test this assumption and find out definitely whether the γ rays are waves in the ether or not. Diffraction effects with γ rays were first observed by Shaw in 1913, and subsequently Rutherford and Andrade measured the wave-lengths of the γ rays from radium B and C together, using as a source some radium emanation in a small glass tube. It was known that both radium B and radium C emit hard γ rays, and the former also emits some rays of about the same penetrating power as ordinary Röntgen rays. This suggested a search for wave-lengths of the order of 10^{-8} cm., and also for shorter wave-lengths, probably of the order 10^{-9} cm. Those actually found

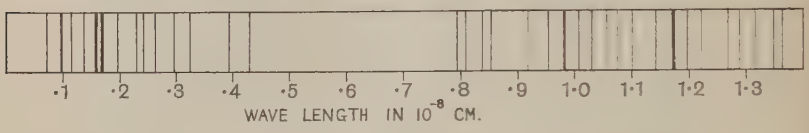


Fig. 363

are shown in Fig. 363. It is clear that the γ ray spectra, like the β ray spectra, are extremely complicated. For this reason experiments on the absorption of the γ rays from various substances by matter give only the roughest indication of the type of rays present. It seems, however, certain that the longer wave-lengths (exceeding 7×10^{-9} cm.) belong to radium B, as radium C is not known to emit any rays of this quality. They may very well constitute the L radiation of radium B (cf. Art. 229). Similarly it has been conjectured that some of the hard γ rays belong to the K radiation of radium B or radium C: but the evidence is very fragmentary.

The process of ionisation by γ rays is similar to that by Röntgen rays (Arts. 231, 232); that is, electrons are ejected from the atoms of the gas under the action of the rays, and these subsequently move through the gas and produce many more ions by collision. What is so remarkable is the very high velocity with which the electrons are ejected. They are capable of penetrating through many centimetres of air at atmospheric pressure; and C. T. R. Wilson's photographs show that their tracks are indistinguishable from those of β particles*. Similarly high-speed

* The track reproduced in Fig. 361 is actually that of an electron liberated by γ rays.

electrons are emitted when γ rays fall on metals, and the velocity of emission can be measured by magnetic deflexion in a vacuum. In this way Rutherford, Robinson and Rawlinson found that the γ rays from radium B + C in passing through lead give rise to a number of different sets of electrons, with velocities of the order of 2×10^{10} cm. per second; i.e. comparable with that of the primary β rays from radium B and C.

The fact that both β and γ rays occur in homogeneous groups suggests problems connected with the structure of the atoms of radioactive substances, just as the Röntgen ray and ordinary spectra do with respect to the structure of ordinary atoms. Little progress has hitherto been made with these. The question of the origin of the β and γ rays is further complicated by the fact that the two are to a certain extent interchangeable: a β particle can give rise to γ rays when it experiences a sudden stoppage or rapid acceleration, and as we have seen γ rays falling on matter liberate electrons of much the same velocity.

244. Radioactive elements in general. A knowledge of the exact nature of the α, β and γ rays makes it possible to specify more precisely what is going on when one radioactive substance changes into another. The atomic weight of the ejected α particle or helium atom being 4, every α ray transformation results in the reduction of the atomic weight of the radioactive element by 4 units. The loss of a β particle, on the other hand, can only affect the atomic weight by very little. On this view the atomic weights of the elements of the radium family are as follows:

	Radiation	Atomic weight
Radium ...	α + slow β	226
Emanation ...	α	222
Radium A ...	α	218
Radium B ...	β + weak γ	214
Radium C ...	$\alpha + \beta + \gamma$	214
Radium D ...	slow β	210
Radium E ...	β + weak γ	210
Polonium ...	α	210
?	—	206

The only direct evidence bearing on this is Gray and Ramsay's determination of the atomic weight of the emanation, which gave the value 223. Each of these substances has a very definite chemical behaviour: thus both radium B and radium D are chemically allied to lead, radium C and radium E to bismuth, and the emanation to the group of inert gases.

The reader cannot have failed to notice the remarkable fact that radium is only found in minerals which contain considerable quantities of uranium. This receives a natural explanation if we suppose that radium is a transformation product of uranium. On this view, sufficiently old uranium minerals should contain a definite proportion of radium corresponding to the equilibrium amounts of the two substances. Boltwood showed that the masses of uranium and radium in many minerals are in a constant ratio, which is about $3{\cdot}3 \times 10^{-7}$. Since the atomic weights of uranium and radium are so nearly equal, this is approximately the ratio of the half-value period of radium to that of uranium. Hence the half-value period of uranium is about 5×10^9 years. Such a long period is what would be expected from the weak activity of uranium. There is reason to believe that all the stages of the transformation from uranium to radium are now known, and that there are *three* products emitting α particles. Since the atomic weight of uranium, according to Hönigschmid, is 238·2, and that of radium 226, the agreement is fairly satisfactory.

The question of the end product of the radium series, after the decay of polonium, is settled somewhat similarly. It is known that lead occurs in all uranium minerals, and the Pb–Ur ratio is often fairly constant. Since the atomic weight of the end product is 206, and that of lead 207, it has been conjectured that the two are identical. The force of this conclusion has, however, been weakened by other considerations.

Considerable light has been thrown on the chemistry of the radioactive elements by Fajans' discovery that all the elements can be fitted into the periodic table by means of certain hypotheses. These are:

(1) A simple radioactive element emits either α particles, or β particles, but not both;

(2) The emission of an α particle involves a shift of two places in the periodic table;

(3) The emission of a β particle involves a shift of one place in the opposite direction to that produced by the α particle.

If we except the weak β radiation of radium, all the transformations can be accounted for by the above rules, provided that in certain cases (radium C) two consecutive products are assumed, one of which has such a short life that its separate existence cannot be proved experimentally.

CHAPTER XIV

THE THEORY OF ELECTRONS

245. Introduction. Electron theory of metallic conduction. The last two chapters should have made it sufficiently clear what is meant by an *electron*, or fundamental unit of negative electricity. It has been shown that while negative electricity may be associated with a mass much smaller than that of a hydrogen atom, positive electricity is never found on anything of less than atomic mass. The present chapter is devoted chiefly to matters of theoretical interest. In the first place it is desirable to inquire more closely into the process of conduction of electricity in metals, which, as it is not accompanied by any chemical change or transfer of matter, is probably to be ascribed to the motion of electrons only. On the other hand, phenomena such as the emission of Röntgen and γ rays, and the remarkable change of the mass of the β particle with speed, show that the relation of the electron to the surrounding ether requires to be more carefully examined, in order that some insight may be gained into the underlying principles. It does not seem to be an exaggeration to say that at one time the idea of an electric medium like the ether appeared to be antagonistic to the other idea of the atomic constitution of electricity. Discrete electric particles, by association with the Newtonian ideas, suggested action at a distance; and it was only in 1895, when the work of H. A. Lorentz became known, that the two ideas were clearly seen to be reconcilable. These theories will occupy most of the present chapter, from Art. 247 onwards.

The theory of metallic conduction is mainly due to Drude. He supposed that the free electrons in a metal are constantly in a state of thermal agitation, like the molecules in the kinetic theory of gases. There is no reason for supposing that the electrons

are few in number, but as their diameter is probably very small collisions among themselves are rare, and we need only consider collisions between the electrons and the atoms of the metal. The latter are supposed to be capable of some slight vibratory motion about fixed positions, but not of migration as a whole. For the sake of simplicity the electrons may all be supposed to have the same free path l between collisions with atoms of the metal, and the same velocity of agitation V. Under these conditions the theory of Art. 174 and of Art. 210 may be applied unaltered, and the mean velocity of drift of the electrons in an electric field E is given by $v = eEl/2mV$.

Consider the case of a straight wire carrying a current. If n is the number of electrons per c.c., the volume-density ρ of the moving charges is ne. Hence the current-density in the wire in electromagnetic units is $j = \rho v/c = nev/c = ne^2El/2mcV$. Here E is measured in electrostatic units, so that if σ is the conductivity of the metal in electromagnetic units, $j = \sigma cE$. Comparing the two last formulae, we have

$$\sigma = \frac{ne^2l}{2mc^2V} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The kinetic energy of a gaseous molecule at ordinary temperatures is $5{\cdot}9 \times 10^{-14}$ (Art. 213). Assuming that the kinetic energy of agitation of an electron in a metal has the same value, we have $mV^2 = 1{\cdot}18 \times 10^{-13}$. Taking $m = 8{\cdot}96 \times 10^{-28}$, we find $V = 1{\cdot}15 \times 10^7$. The mean velocity of agitation is therefore of the order of one-thousandth of that of light. The number of electrons per c.c. for a given metal, say copper, may be estimated as follows. The energy required to raise the temperature of 1 c.c. of copper through 1° is 0·8 calories, or $3{\cdot}4 \times 10^7$ ergs. Part of this goes to increase the energy of vibration of the atoms of the metal, part to increase the energy of agitation of the electrons. Assuming that the mean energy is the same for each of the N atoms in one cubic centimetre as it is for each of the n electrons, the increase of kinetic energy of a single particle of either kind in one degree rise of temperature is $\frac{3{\cdot}4 \times 10^7}{n + N}$ ergs. But this is equal to Boltzmann's constant $\alpha = 2 \times 10^{-16}$. Hence

$$n + N = 1{\cdot}7 \times 10^{23};$$

and since $N = 8{\cdot}6 \times 10^{22}$, we have $n = 8{\cdot}4 \times 10^{22}$. The number of free electrons in a metal is therefore comparable with the number of atoms.

Equation (1) now gives us an estimate of l, the mean free path of the electrons in the metal. Putting $\sigma = 6 \times 10^{-4}$ for copper, and making the other substitutions, we find $l = 5{\cdot}9 \times 10^{-7}$ cm. This rather large result would seem to show that only a few of what would ordinarily be called collisions with atoms result in appreciable deflexion of the electron. But it is doubtful whether the path of an electron is ever really "free."

The mean velocity of drift is connected with the current-density by the equation $j = nev/c$, so that for a current of 100 amperes per square centimetre v is only $7{\cdot}5 \times 10^{-3}$ cm. per second. The considerable transfer of charge obtained in ordinary experiments is therefore due to the very large number of electrons concerned rather than to any great speed of each.

246. The Hall effect. Contact and thermal potentials. It was discovered by Hall in 1877 that the lines of current-flow in a metal are altered in a magnetic field. Suppose that a current enters a thin metal plate at A and leaves it at B, and that C, D

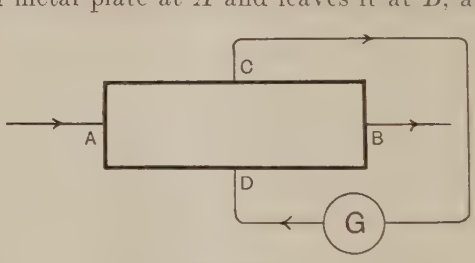


Fig. 364

are points at equal potentials on the sides, so that no deflexion occurs when they are joined to a sensitive galvanometer. If a magnetic field is applied pointing vertically upwards from the paper, a current will usually flow through the galvanometer in the direction shown in the figure. This is the case with bismuth, copper, nickel, platinum and silver; with antimony, cobalt, iron and zinc the current is in the opposite direction. The effect in bismuth is extraordinarily great, antimony and iron coming next in order of magnitude. With other metals it is necessary to use

very thin sheets and a sensitive galvanometer, otherwise the effect will escape observation altogether.

It is easy to show that the electron theory, in its ordinary form, gives the same direction for the Hall effect in all metals, namely that shown in the figure. In the magnetic field the carriers of the current, whatever their sign, are deflected laterally from C towards D; and if they are electrons the current within the sheet is from D to C, as shown. The theory does not therefore account for the observed variability in the sign of the Hall effect, although most non-magnetic metals seem to give the sign required by it. It is clear from this that the electron theory of metallic conduction, in the above form, is not free from serious defects. Considerable modifications would be required to treat the case of a magnetic metal, but it is not certain what advantages would be gained thereby.

It is interesting to notice the explanation given by electron theory of the mechanical action of magnets on currents. A magnetic field is regarded as acting primarily on the free electrons in the metal according to the rule of Art. 95, and these in turn react on the conductor by colliding with the atoms of the metal. In this sense the actions are secondary.

The electron theory also gives a qualitative explanation of certain other phenomena, although it is clear from what has been said that exact numerical agreement is not to be looked for. Consider first of all what happens when two different metals are in contact with one another. Let n_1 be the number of electrons per c.c. in the first metal, n_2 in the second, and suppose that $n_1 > n_2$. At first electrons will diffuse from metal 1 to metal 2; but their accumulation in the second metal gives rise to an electric force from 1 to 2 which tends to prevent further electrons from crossing the boundary. A steady state will ultimately occur in which the boundary region is a seat of electric force, which however does not penetrate far into either metal. There will therefore be a difference of electrostatic potential between the two metals, and if $n_1 > n_2$ the first metal is at a higher potential than the second.

Take axes such that the metal 1 is to the left of the plane $x = 0$ and the metal 2 to the right, and let n be the number of

electrons per c.c. at abscissa x. In the steady state the total flux of electrons vanishes, so that $Kdn/dx = nuE$, u being the mobility of the electrons in the metal considered, E the electric force and K the coefficient of diffusion. Since $u/K = Ne/\Pi$, we have in either metal

$$\frac{dn}{n} = \frac{Ne}{\Pi} E\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When $x = -\infty$, $n = n_1$, and when $x = \infty$, $n = n_2$. Hence if V is the difference of potential between the main masses of the two metals,

$$V = \int_{-\infty}^{\infty} E\,dx = \frac{\Pi}{Ne} \log_e \frac{n_1}{n_2}.$$

For copper and zinc at ordinary temperatures we have $n_1 = 8{\cdot}4 \times 10^{22}$, $n_2 = 5{\cdot}4 \times 10^{22}$, and $Ne\,\Pi = 1{\cdot}2 \times 10^4$. Hence $V = 3{\cdot}6 \times 10^{-5}$ electrostatic units $= 1/100$ volt nearly. This may be called the true contact potential. It is not the same as the ordinary contact potential between plates close to one another in air, which is of the order of one volt. The latter is not accounted for by the simple theory. The evidence drawn from the emission of electrons by hot bodies points to a surface-layer at the boundary of air and metal, hindering the emission of electrons considerably at ordinary temperatures. It is probable that the difference of potential between the metal and a point in the gas just outside varies a good deal with the state of the surface, as is known to happen with the contact potential in air.

Since the conductivity of a metal varies with the temperature, the product nl does also, and therefore probably both n and l. Such a variation is also required by the Kelvin effect, which shows that differences of potential exist in a single unequally heated conductor. Equation (2) may be written

$$E = \frac{\Pi}{nNe} \frac{dn}{d\theta} \frac{d\theta}{dx},$$

so that no electric forces can arise in an ideal conductor unless n varies with θ. Similarly the electron theory regards the electromotive forces of the thermopile (Seebeck effect) as due to the difference of true contact potential at the temperatures of the hot and cold junctions, and the potential differences arising are actually of the right order of magnitude.

247. Lorentz's equations. The foundations of electrical theory have been very lucidly treated by Lorentz, who was the first to consider in detail the relations between electricity and ether. He finds it convenient to regard the conception introduced in Art. 190 in order to obtain an expression for the ether-current as representing the actual state of affairs in the universe when analysis is pushed to extremes. If we imagine a being of indefinitely keen powers of observation to look into the interior of molecules, atoms and electrons, he will find everywhere ether and moving electricity. Thus the universe is regarded as completely specified by the values of ρ, v_x, v_y, v_z at all points, the difficulty as to the presence of both electricity and ether in the same volume-element being ignored. The ordinary observer will merely reckon with the mean values of the electrical quantities over regions of space and time which are small but not infinitely small. Some common electrical phenomena have no place in the fundamental Lorentz scheme: for example, a polarised dielectric is resolved into its constituent charges; and further there are no magnetic poles, but only magnetic force caused by the motion of electric charges. It may be mentioned that the general tendency of recent speculation is to give up the analogy that appears to exist between electric and magnetic quantities.

With these assumptions the total current-density is now given by three equations of the type

$$j_x = \frac{1}{c}\left(\frac{1}{4\pi}\frac{\partial E_x}{\partial t} + \rho v_x\right),$$

and the first three of Lorentz's equations become

$$\left.\begin{aligned} \frac{1}{c}\left(\frac{\partial E_x}{\partial t} + 4\pi\rho v_x\right) &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{1}{c}\left(\frac{\partial E_y}{\partial t} + 4\pi\rho v_y\right) &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{1}{c}\left(\frac{\partial E_z}{\partial t} + 4\pi\rho v_z\right) &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots (3).$$

The second set of equations of Art. 190 are left unaltered, namely

$$\left.\begin{aligned} -\frac{1}{c}\frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ -\frac{1}{c}\frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ -\frac{1}{c}\frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4).$$

Further, assuming that Gauss' theorem is generally true, we have

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho \quad \ldots\ldots\ldots\ldots (5),$$

and similarly

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots (6).$$

We must add to these equations some expression for the mechanical force experienced by electricity per unit charge. In fundamental matters of this kind what is important is not so much to analyse the well-nigh inextricable concepts of force and mass as to formulate some workable scheme which shall not obviously conflict with the facts of observation. We shall denote by F the mechanical force per unit charge on the electricity near the point (x, y, z), arising from all causes. In accordance with Art. 95 we may say that the magnetic field H at the point (x, y, z) exerts on the electricity in the vicinity a force

$$\frac{1}{c}(v_y H_z - v_z H_y, \quad v_z H_x - v_x H_z, \quad v_x H_y - v_y H_x)$$

per unit charge. Applying this to the ultimate moving charges considered by Lorentz, and adding in the effect of the electric force, we have

$$\left.\begin{aligned} F_x &= E_x + \frac{1}{c}(v_y H_z - v_z H_y) \\ F_y &= E_y + \frac{1}{c}(v_z H_x - v_x H_z) \\ F_z &= E_z + \frac{1}{c}(v_x H_y - v_y H_x) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7).$$

This is Lorentz's complete set of eleven equations. The question of what mass, if any, is to be ascribed to the moving electricity is discussed in Art. 250.

248. The electromagnetic potentials. The fundamental problem of Lorentz's theory is to determine the electric and magnetic field when the charges and their motion are specified beforehand; that is, when ρ, v_x, v_y, v_z are given functions of x, y, z, t. The work is simplified by the introduction of two functions which resemble the electrostatic potential and the vector potential of magnetic force respectively. Equation (6) is satisfied identically if we put $H = \text{curl } A$, or

$$H_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad H_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad H_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad \ldots(8).$$

Substituting in equations (4) we have

$$\frac{\partial \Phi_z}{\partial y} = \frac{\partial \Phi_y}{\partial z}, \quad \frac{\partial \Phi_x}{\partial z} = \frac{\partial \Phi_z}{\partial x}, \quad \frac{\partial \Phi_y}{\partial x} = \frac{\partial \Phi_x}{\partial y},$$

where

$$\Phi_x \equiv E_x + \frac{1}{c}\frac{\partial A_x}{\partial t},$$

$$\Phi_y \equiv E_y + \frac{1}{c}\frac{\partial A_y}{\partial t},$$

$$\Phi_z \equiv E_z + \frac{1}{c}\frac{\partial A_z}{\partial t}.$$

It follows that a function V of x, y, z, t can be found such that

$$\Phi_x = -\frac{\partial V}{\partial x}, \quad \Phi_y = -\frac{\partial V}{\partial y}, \quad \Phi_z = -\frac{\partial V}{\partial z}.$$

Substituting, we have

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \\ E_y &= -\frac{\partial V}{\partial y} - \frac{1}{c}\frac{\partial A_y}{\partial t} \\ E_z &= -\frac{\partial V}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9).$$

Here V is called the *scalar electromagnetic potential*, and A_x, A_y, A_z are regarded as the components of an *electromagnetic vector potential*. Substituting in (3), we find three equations of the type

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_x$$

$$- \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c}\frac{\partial V}{\partial t}\right) = -\frac{4\pi\rho v_x}{c},$$

and equation (5) may be written in the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{1}{c}\frac{\partial V}{\partial t}\right)=-4\pi\rho.$$

We shall assume, subject to subsequent verification, that

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}+\frac{1}{c}\frac{\partial V}{\partial t}=0 \quad \ldots\ldots\ldots (10).$$

Then writing Δ for Laplace's operator, the preceding equations become

$$\left.\begin{aligned}\Delta A_x-\frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2}&=-\frac{4\pi\rho v_x}{c}\\ \Delta A_y-\frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2}&=-\frac{4\pi\rho v_y}{c}\\ \Delta A_z-\frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2}&=-\frac{4\pi\rho v_z}{c}\end{aligned}\right\} \quad \ldots\ldots\ldots (11)$$

and

$$\Delta V-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=-4\pi\rho \quad \ldots\ldots\ldots (12).$$

These equations are all of the same form, which is a generalisation of Poisson's equation. If V and ρ are independent of t the solution would be given by the ordinary formula $V=\int\rho\, d\tau/r$. In the general case in which $\rho=f(x, y, z, t)$ we shall show that the solution of (12) is

$$V=\int\frac{f\left(\xi, \eta, \zeta, t-\frac{r}{c}\right)d\tau}{r} \quad \ldots\ldots\ldots (13),$$

where $d\tau=d\xi\, d\eta\, d\zeta$, $r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2$,

and the integration is extended through the whole of space. That is, V can be calculated by the ordinary rules of electrostatics, provided that the volume-density at any point Q is taken to be, not that actually existing at time t, but that existing at time $t-r/c$, where r is the distance from Q to (x, y, z). Such potentials are called *retarded potentials*. Roughly speaking, the potentials may themselves be said to be propagated with the velocity of light.

To verify (13), we notice that the function

$$\theta = \frac{f\left(t - \frac{r}{c}\right)}{r},$$

where r is the distance of (x, y, z) from any fixed point, satisfies the equation

$$\Delta\theta = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} \quad \text{(Art. 6).}$$

It follows that $\dfrac{f\left(\xi, \eta, \zeta, t - \frac{r}{c}\right)}{r}$, considered as a function of (x, y, z, t), does so also. It cannot, however, be concluded that $\Delta V = \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2}$, on account of the behaviour of the integral infinitely near the point (x, y, z). Let V_0 be the value of the integral (13) extended over a small region S_0 enclosing the point (x, y, z). In this region r/c is small and the "retarded" volume-density may be taken as that actually existing at the points considered. The behaviour of V_0 is now known from the discussion in Art. 32. V_0 is itself small, and so also is $\partial^2 V_0/\partial t^2$; but the second derivates $\partial^2 V_0/\partial x^2$, $\partial^2 V_0/\partial y^2$, $\partial^2 V_0/\partial z^2$ are finite and have the sum $-4\pi\rho$. Hence

$$\Delta V_0 - \frac{1}{c^2}\frac{\partial^2 V_0}{\partial t^2} = -4\pi\rho.$$

If V_1 is the value of the integral (13) extended over the rest of space, $\Delta V_1 = \dfrac{1}{c^2}\dfrac{\partial^2 V_1}{\partial t^2}$, and the complete integral is $V = V_0 + V_1$. Hence equation (12) is satisfied.

A "retarded" function of the form $f\left(\xi, \eta, \zeta, t - \frac{r}{c}\right)$ may be conveniently abbreviated into $[f]$. Hence, since the equations (11) are of the same form as (12), we have the solutions

$$\left.\begin{aligned} A_x &= \frac{1}{c}\int\frac{[\rho v_x]}{r}\,d\tau \\ A_y &= \frac{1}{c}\int\frac{[\rho v_y]}{r}\,d\tau \\ A_z &= \frac{1}{c}\int\frac{[\rho v_z]}{r}\,d\tau \end{aligned}\right\} \qquad \text{.....(14)}$$

and $$V = \int \frac{[\rho]}{r} d\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

To verify that the solution given by (14) and (15) actually satisfies the condition (10) we proceed as follows. Instead of calculating A_x at the point (x, y, z) calculate it at the point $(x + h, y, z)$, where h is a small quantity, and consider the contribution of the electricity in the neighbourhood of $(\xi + h, \eta, \zeta)$. Here ρv_x is replaced by $\rho v_x + h \dfrac{\partial}{\partial \xi}(\rho v_x)$, and since r is unaltered $[\rho v_x]$ is replaced by $[\rho v_x] + h \left[\dfrac{\partial}{\partial \xi}(\rho v_x)\right]$. Hence

$$A_x + h \frac{\partial A_x}{\partial x} = \int \left\{ [\rho v_x] + h \left[\frac{\partial}{\partial \xi} (\rho v_x) \right] \right\} d\tau,$$

or $$\frac{\partial A_x}{\partial x} = \int \left[\frac{\partial}{\partial \xi} (\rho v_x) \right] d\tau.$$

Similarly $$\frac{\partial A_y}{\partial y} = \int \left[\frac{\partial}{\partial \eta} (\rho v_y) \right] d\tau,$$

$$\frac{\partial A_z}{\partial z} = \int \left[\frac{\partial}{\partial \zeta} (\rho v_z) \right] d\tau,$$

and $$\frac{\partial V}{\partial t} = \int \left[\frac{\partial \rho}{\partial t} \right] d\tau.$$

But since we have at all points

$$\frac{\partial}{\partial \xi}(\rho v_x) + \frac{\partial}{\partial \eta}(\rho v_y) + \frac{\partial}{\partial \zeta}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0$$

it is clear that (10) is satisfied. A_x, A_y, A_z and V being known, E and H are found from equations (8) and (9), so that the field is completely determined.

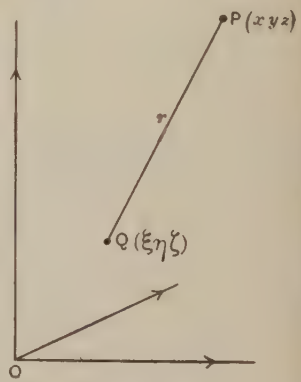


Fig. 365

249. Electromagnetic field of an electron moving with a small uniform velocity. As an example of the preceding formulae we shall find the electromagnetic field round an electron moving with a uniform

velocity v small compared with that of light. Consider in the first place an arbitrary distribution of electricity moving parallel to itself along the z-axis with the velocity v. Let Q (Fig. 365) be a point of the moving system referred to fixed axes, P the point at which the potentials are to be calculated; then ρ is given by an equation of the form

$$\rho = f(\xi, \eta, \zeta - vt) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Hence

$$[\rho] = f\left(\xi, \eta, \zeta - vt + \frac{rv}{c}\right)$$

$$= \rho + \frac{rv}{c}\frac{\partial \rho}{\partial \zeta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

approximately; and similarly we have, correct to the first order of small quantities,

$$\frac{[\rho v_x]}{c} = \frac{[\rho v_y]}{c} = 0, \quad \frac{[\rho v_z]}{c} = \frac{\rho v}{c} \quad \ldots\ldots\ldots\ldots (18).$$

From (17) we have

$$V = V_0 + \frac{v}{c}\int \frac{\partial \rho}{\partial \zeta}\, d\tau,$$

where $V_0 = \int \rho\, d\tau / r$ is the electrostatic potential at P, calculated by the ordinary rules. But if ρ vanishes at infinity the second term is zero, and we have

$$\left.\begin{aligned} V &= V_0 \\ A_x = A_y &= 0, \quad A_z = \frac{vV_0}{c} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (19).$$

In calculating the electric and magnetic forces from equations (8) and (9) it must be remembered that $\partial V_0/\partial t$ is itself of the first order in v, and therefore $\partial A_z/\partial t$ is of the second order. Hence the electric force E at any point can be calculated by the ordinary rules of electrostatics, and the magnetic field is given by

$$H_x = -\frac{vE_y}{c}, \quad H_y = \frac{vE_x}{c}, \quad H_z = 0 \quad \ldots\ldots\ldots (20).$$

It follows that if O is a small electron moving with a small velocity v along the z-axis, and P a point whose co-ordinates

with respect to O are x, y, z, the components of magnetic force at P are $(-evy/cr^3, evx/cr^3, 0)$. Hence its direction is perpendicular to the plane containing OP and the direction of motion, and the magnitude of the resultant is

$$\frac{ev \sin \theta}{cr^2},$$

where θ is the angle between OP and the direction of motion.

To find the force at points inside the electron it is necessary to make some assumption as to the distribution of charge. We shall assume that the charge is spread through the volume with uniform volume-density ρ, so that if a is the radius of the electron, $e = \frac{4}{3}\pi\rho a^3$. The electrostatic field has already been found (Art. 31), and thus we have at an internal point

$$\left.\begin{aligned} E_x = \frac{ex}{a^3}, \quad E_y = \frac{ey}{a^3}, \quad E_z = \frac{ez}{a^3} \\ H_x = -\frac{evy}{ca^3}, \quad H_y = \frac{evx}{ca^3}, \quad H_z = 0 \end{aligned}\right\} \ldots\ldots\ldots (21),$$

the origin being taken at the centre of the electron.

In the case of slow motion the mechanical force F per unit charge is, to the first order, the same as the electric force E. It follows that the resultant force exerted by the electromagnetic field on the electron is zero. More generally, any electrified system can move through the ether with slow uniform velocity in a straight line without requiring any external force to maintain the motion, though internal forces are necessary to keep the different parts in the same relative position. This corresponds to Newton's first law of motion.

250. Accelerated motion of a slow electron. Electromagnetic mass. The preceding theory can be extended to the case in which the velocity is not uniform, provided that it, together with the acceleration and other derivates, is always small. If the system moves parallel to itself so that the distance traversed in time t is $\phi(t)$, equation (16) is replaced by

$$\rho = f\{\xi, \eta, \zeta - \phi(t)\}.$$

Hence

$$[\rho] = f\left\{\xi, \eta, \zeta - \phi\left(t - \frac{r}{c}\right)\right\}$$

$$= f\left\{\xi, \eta, \zeta - \phi(t) + \frac{r}{c}\phi'(t) - \frac{r^2}{2c^2}\phi''(t) + \frac{r^3}{6c^3}\phi'''(t) - \ldots\right\}$$

$$= \rho + \frac{r\phi'(t)}{c}\frac{\partial\rho}{\partial\zeta} - \frac{r^2\phi''(t)}{2c^2}\frac{\partial\rho}{\partial\zeta} + \frac{r^3\phi'''(t)}{6c^3}\frac{\partial\rho}{\partial\zeta} - \ldots,$$

provided that $\phi'(t)$, $\phi''(t)$, ... are so small that squares and products can be neglected. If we stop at the term involving $\phi''(t)$, we find

$$V = V_0 + \frac{\phi'(t)}{c}\int\frac{\partial\rho}{\partial\zeta}\,d\tau - \frac{\phi''(t)}{2c^2}\int r\,\frac{\partial\rho}{\partial\zeta}\,d\tau$$

$$= V_0 + \frac{\phi''(t)}{2c^2}\int\rho\,\frac{\partial r}{\partial\zeta}\,d\tau$$

or

$$V = V_0 + \frac{\phi''(t)}{2c^2}\int\rho\,\frac{\zeta - z}{r}\,d\tau.$$

Similarly

$$A_x = A_y = 0, \quad A_z = \frac{\phi'(t)}{c}\int\frac{\rho\,d\tau}{r} + \frac{\phi''(t)}{c^2}\int\rho\,d\tau.$$

What we are chiefly interested in is the total force exerted by the electromagnetic field on the electron in the direction of the axis of z. To the present degree of approximation we have

$$F_z = E_z = -\frac{\partial V}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t},$$

or

$$F_z = -\frac{\partial V_0}{\partial z} - \frac{\phi''(t)}{2c^2}\int\left\{\frac{1}{r} + \frac{(\zeta - z)^2}{r^3}\right\}\rho\,d\tau \quad \ldots\ldots(22).$$

The integral $\int\frac{(\zeta - z)^2}{r^3}\rho\,d\tau$ is troublesome, but its evaluation may be avoided by an artifice. Since F_z is to be multiplied by the element of charge near (x, y, z) and integrated once more over the system, there will be no error in the final result for a spherical electron if $(\zeta - z)^2/r^3$ is replaced by $1/3r$. Further, the term $-\partial V_0/\partial z$ may be omitted since it contributes nothing to the total force. Hence instead of (22) we may take

$$F_z = -\frac{2\phi''(t)}{3c^2}V_0.$$

Taking axes such that the centre of the electron at time t is at the origin, we may write for an internal point

$$F_z = -\frac{2\phi''(t)}{3c^2}\frac{e}{2a}\left(3 - \frac{x^2+y^2+z^2}{a^2}\right)$$

(cf. Art. 31). Integrating, we have for the total force exerted by the field on the electron $-\frac{4e^2}{5ac^2}\phi''(t)$, or $-\frac{4e^2}{5ac^2}f$, where f is the acceleration. The accelerated motion of an electron therefore calls forth a force tending to stop the acceleration. This is known as the *reaction of the field.*

Now suppose that the electron has a mass M in the ordinary sense. If an external force P is applied, the equation of motion, to the present degree of approximation, is

$$Mf = P - \frac{4e^2}{5ac^2}f.$$

Writing this in the form

$$\left(M + \frac{4e^2}{5ac^2}\right)f = P,$$

we see that the motion is the same as if the ether was absent and the mass of the electron increased by $4e^2/5ac^2$. This added mass is known as the *electromagnetic mass.*

The idea that electric charges might possess inertia in virtue of their motion was first put forward by Sir J. J. Thomson in 1881. The simplest hypothesis that we can make is that $M = 0$, which is expressed by the statement that all the mass of the electron is electromagnetic. Adopting this, the mass of the electron at slow speeds becomes

$$m_0 = \frac{4e^2}{5ac^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

From this it is possible to make an estimate of the radius of the electron, considered as a sphere filled with electricity at uniform volume-density. Putting $m_0 = 8{\cdot}96 \times 10^{-28}$, $e = 4{\cdot}75 \times 10^{-10}$ and $c = 3 \times 10^{10}$ in equation (23), we find $a = 2{\cdot}2 \times 10^{-13}$ cm. Other distributions of charge would give different results, but always of the same order. We conclude that in any case the linear dimensions of the electron are extremely small (of the order of

one hundred-thousandth of the diameter of an atom); which fits in well with other facts, for example their power of moving bodily through solid conductors, where an atom or molecule would be unable to get past.

251. A model source of light and its radiation. The simplest model of a source of light in the electromagnetic theory is a vibrating point-charge or electron. In order to make it electrically neutral we may, if we wish, add an equal positive charge at a fixed point. Now as such an electron is constantly sending out waves in the ether its vibrations must be damped out to some extent. This is not accounted for by the theory of the last article, and we must go further by including in the equations terms involving $\phi'''(t)$. We thus find, instead of (22),

$$F_z = -\frac{\partial V_0}{\partial z} - \frac{\phi''(t)}{2c^2}\int\left\{\frac{1}{r} + \frac{(\zeta - z)^2}{r^3}\right\}\rho\, d\tau + \frac{2\phi'''(t)}{3c^3}\int \rho\, d\tau.$$

The first two terms are allowed for by the theory of "electromagnetic mass" just sketched; the last term gives the additional force $\frac{2e^2}{3c^3}\phi'''(t)$. If the electron is vibrating along the z axis under a restraining force of magnitude $a\zeta$ at distance ζ, its equation of motion, to this degree of approximation, is

$$m\frac{d^2\zeta}{dt^2} = -a\zeta + \frac{2e^2}{3c^3}\frac{d^3\zeta}{dt^2},$$

where m is the mass of the electron. The motion is therefore given by a differential equation of the third order, which can be shown to represent a damped simple-harmonic vibration. If the last term is neglected the motion is simple-harmonic with period $2\pi/n$, where $n^2 = a/m$. In general write $\zeta = Ae^{ipt}$; then we have

$$-mp^2 = -mn^2 - ip^3\frac{2e^2}{3c^3},$$

or

$$p^2 = n^2 + ip^3\frac{2e^2}{3mc^3}.$$

If the damping term is small we may put $p = n$ in it, and obtain

$$p^2 = n^2 + in^3\frac{2e^2}{3mc^3},$$

or

$$p = n + \frac{in^2e^2}{3mc^3}.$$

The motion is therefore given by an equation of the form

$$\zeta = Ae^{-\frac{n^2e^2t}{3mc^3}} \cos(nt + \epsilon),$$

representing a damped vibration with decrement $\delta = 2\pi ne^2/3mc^3$. For the frequency of sodium light ($n = 3 \times 10^{15}$) we have $\delta = 6 \times 10^{-8}$, and δ is still very small for the frequency of the hardest known γ rays. It is obvious, therefore, that the effect of the additional term is negligible in almost all practical cases of motion of an electron, nor is it probable that terms involving $\phi^{(4)}\, t$ or subsequent derivates have any appreciable effect.

The electric field of a vibrating electron at distant points is most conveniently calculated as follows. Let $\zeta = \phi(t)$ be the distance of the electron from the origin at time t, e its charge. If a compensating charge $-e$ is fixed at the origin, the two form a Hertzian oscillator whose moment at time t is $f(t) = e\zeta$. Then from Art. 192 the field at a distant point (x, y, z) is given by

$$\left.\begin{aligned} E_x = \frac{exz}{c^2r^3}\phi''\left(t - \frac{r}{c}\right), \quad E_y = \frac{eyz}{c^2r^3}\phi''\left(t - \frac{r}{c}\right), \\ E_z = -\frac{e(x^2 + y^2)}{c^2r^3}\phi''\left(t - \frac{r}{c}\right) \\ H_x = -\frac{ey}{c^2r^2}\phi''\left(t - \frac{r}{c}\right), \quad H_y = \frac{ex}{c^2r^2}\phi''\left(t - \frac{r}{c}\right), \quad H_z = 0 \end{aligned}\right\}.$$

As in the case of the Hertzian oscillator, the other terms occurring are of a lower order of magnitude, and the field due to the fixed charge $-e$ is also negligible in comparison with that due to the vibrating charge. It follows that the determining factor in the force at a distance at time t is the *acceleration* of the electron at time $t - r/c$.

The preceding theory can be extended to the case of an electron moving so as to remain permanently in the vicinity of the origin, its coordinates at time t being ξ, η, ζ. We thus find for the field at large distances

$$\left.\begin{aligned} E_x &= -(y^2 + z^2)\lambda + xy\mu + xz\nu \\ E_y &= xy\lambda - (z^2 + x^2)\mu + yz\nu \\ E_z &= zx\lambda + yz\mu - (x^2 + y^2)\nu \\ H_x = r(z\mu - y\nu), \quad H_y &= r(x\nu - z\lambda), \quad H_z = r(y\lambda - x\mu) \end{aligned}\right\} \ldots (24),$$

where
$$\lambda, \mu, \nu = \frac{e}{c^2 r^2}\left(\frac{d^2\xi}{dt^2}, \frac{d^2\eta}{dt^2}, \frac{d^2\zeta}{dt^2}\right)_{t-\frac{r}{c}} \quad \ldots\ldots\ldots(25).$$

252. The Zeeman effect. In 1896 Zeeman observed that the sodium lines were broadened slightly when the flame was placed between the poles of a powerful electromagnet. He suspected that it was due to the action of the field on moving charges within the atom, and at Lorentz's suggestion examined the edges of the broadened line for traces of circular or plane polarisation, according to whether the observation was made along or at right angles to the lines of magnetic force. These were actually observed. Later, in 1897, Zeeman found that *three* spectral lines could be produced from a single line, having certain remarkable and definite polarisations. The normal Zeeman effect, as it is called, is as follows:

Let A be the original spectral line, as shown diagrammatically in the figure. The light emitted along the lines of magnetic force can be resolved into two spectral lines B, C as shown. Of these B is circularly polarised in a right-handed direction with respect to the magnetic field, and C shows left-handed circular polarisation. When viewed at right angles to the magnetic field three equidistant spectral lines D, E, F occur, of which D and F are in the same position in the spectrum as B and C, while E is in the position of the original spectral line A. All three lines are plane-polarised, E along the magnetic field and D, F at right angles to it. These effects are called the longitudinal and transverse effects respectively. If the separation is insufficient we get a broadened line with peculiar polarisations at the edges, as in Zeeman's original observations.

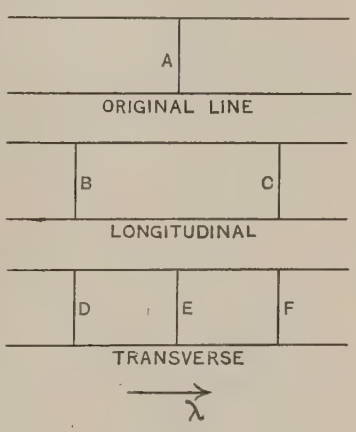


Fig. 366

For accurate measurement of the Zeeman effect we require strong magnetic fields and great resolving power, since the

separation in a field of 30,000 units is only of the order of one-tenth of the distance between the D lines of sodium. A suitable electromagnet is that shown in Fig. 151; the pole-pieces must be perforated for the observation of the longitudinal effect. A Rowland concave grating is generally used to resolve the lines.

If it is desired merely to demonstrate the effect without measurement, a small Lummer-Gehrcke interference plate* may

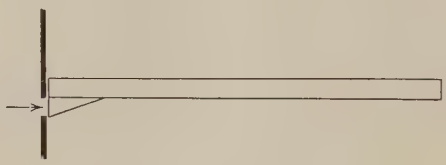


Fig. 367

be conveniently used in conjunction with an ordinary spectroscope. Fig. 367 shows a side view of the Lummer-Gehrcke plate, Fig. 368 a vertical view of the whole apparatus. Light from a spectrum tube, preferably filled with helium and lying

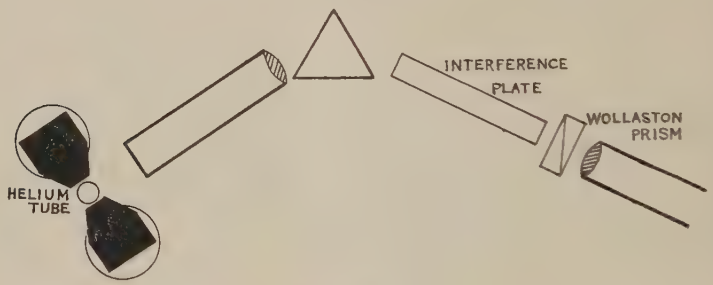


Fig. 368

between the poles of an ordinary electromagnet, passes through a collimator and prism and falls on the interference plate. The light emerging at nearly grazing angle passes into the telescope, and on looking into the eyepiece the usual spectrum is seen, but each line is crossed by a series of horizontal fringes produced by the Lummer-Gehrcke plate. Let us consider one spectral

* See Wood's *Physical Optics*, 2nd ed., p. 282.

line only, for example the yellow line of helium. If a Wollaston double-image prism is inserted as shown in the figure, the sets of fringes are doubled horizontally, since the prism gives a relative displacement to the spectra for rays polarised in planes perpendicular and parallel to the plane of the paper. On applying the magnetic field one set of fringes remains unaltered, while each fringe of the other set is doubled. A separate experiment with the double-image prism is required to determine the directions of polarisation, and thus it can be verified that the unaltered rays are polarised along the lines of magnetic force.

With helium all the lines are split up in the way here described; but the simple triplet is not the only type of resolution. In many cases more than three lines occur: thus the D_2 line of sodium is split into six by the magnetic field. For an account of these abnormal Zeeman effects reference may be made to Zeeman, *Researches in Magneto-optics*. In spite of their irregularity, however, the vast majority of separations come under the following rules:

(1) All lines belonging to the same spectral series show the same type of resolution, and moreover if n is the frequency of the original spectral line and $n + \delta n$ that of one of the displaced lines, δn is the same for all lines of the same series (Preston's law).

(2) The value of δn in a given field is either the same as the change of frequency δn_0 in the helium triplets or p/q times as large, where p and q are *small* integers (Runge's law).

253. Theory of the normal Zeeman effect. We shall first of all consider the motion of the vibrating electron described in Art. 251 in a uniform magnetic field of strength H parallel to the axis of z. We have already seen that its motion is not affected in any practical case by the presence of the ether, provided that the electromagnetic mass of the electron is included in its total mass m. Let the electron be at the point (ξ, η, ζ) at time t, and let the restorative force necessary to make it vibrate be $(-a\xi, -a\eta, -a\zeta)$. Then in the absence of the magnetic field the motion in all coordinates is simple-harmonic with period $2\pi/n$, where $n^2 = a/m$.

The equations of motion in the magnetic field are

$$m\frac{d^2\xi}{dt^2} = -a\xi + \frac{He}{c}\frac{d\eta}{dt},$$

$$m\frac{d^2\eta}{dt^2} = -a\eta - \frac{He}{c}\frac{d\xi}{dt},$$

$$m\frac{d^2\zeta}{dt^2} = -a\zeta.$$

Hence the motion in the direction of the z axis is unaffected by the field, so that one possible mode of vibration is along the lines of magnetic force with the normal period $2\pi/n$. For the x and y components try the solution

$$\xi = A\cos pt, \quad \eta = A\sin pt,$$

and write also for shortness $h = He/mc$. Then putting these values in the equations we find

$$(p^2 - n^2)\,A = -hpB,$$
$$(p^2 - n^2)\,B = -hpA.$$

Hence either $A = B$ and $p^2 - n^2 = -hp$ .....(26)

or $A = -B$ and $p^2 - n^2 = hp$ .....(27).

The first case gives as a possible mode of vibration a circular motion in the plane $z = 0$, described in a right-handed screw direction with respect to the axis of z. The second case gives a left-handed circular vibration.

We have thus found *three* types of simple-harmonic vibration in a magnetic field, one period being that of the original spectral line and the other two respectively greater and less. This is precisely what is found in the normal Zeeman effect, and we shall see presently that the polarisations are correctly accounted for.

Writing $p = n + \delta n$ in equation (26) and assuming that δn is small, we have

$$\delta n = -\tfrac{1}{2}h = -\frac{He}{2mc} \quad \text{.....(28).}$$

If λ is the wave-length corresponding to the original vibration, $\lambda + \delta\lambda$ that corresponding to the right-handed circular polarisation, $n = 2\pi c/\lambda$ and $\delta n = -2\pi c\,\delta\lambda/\lambda^2$. Hence from (28)

$$\frac{\delta\lambda}{\lambda^2} = \frac{H}{4\pi c^2}\frac{e}{m} \quad \text{.....(29).}$$

Hence by measurement of the Zeeman effect it is possible to find the value of e/m of the vibrating particles giving rise to a particular spectral line. Zeeman himself in 1896 showed that the particle must have a negative charge, and that e/m was of the order of -10^{17}. Lohmann, in a series of concordant experiments with helium, obtained the value $\delta\lambda/\lambda^2 = -4{\cdot}70 \times 10^{-4}$, giving $e/m = -5{\cdot}32 \times 10^{17}$. The remarkable agreement between this value and the value $-5{\cdot}30 \times 10^{17}$ adopted for the slow-moving electron leaves no doubt that the vibrating particles are electrons. Other accurate experiments (on zinc and cadmium lines) have given values almost exactly double the above. It has been conjectured that the vibrating particles are still electrons, but that the separation is double the normal, in accordance with Runge's rule. But though Runge's rule suggests a theoretical problem no answer to it has yet been given; and generally little progress has been made towards explaining the abnormal Zeeman effects.

When the motion of the electron is known, the field at a distant point is given by equations (24) and (25). With the present axes the longitudinal Zeeman effect corresponds to an observer at the point $(0, 0, z)$, the transverse effect to the point $(x, 0, 0)$.

(1) *Effect of the vibration of the electron along the axis of z with period* $2\pi/n$.

Here $$\lambda = \mu = 0, \quad \nu = \frac{e}{c^2 r^3}\left(\frac{d^2\zeta}{dt^2}\right)_{t-\frac{r}{c}}.$$

At the point $(0, 0, z)$ $E_x = E_y = E_z = 0$. Hence there is no central line in the longitudinal position. At the point $(x, 0, 0)$ $E_x = E_y = 0$, $E_z = -x^2\nu$, or the central line in the transverse position is polarised in the direction of the impressed field.

(2) *Effect of the right-handed circular vibration in the* x, y *plane.*

Taking $\xi = A\cos pt$, $\eta = A\sin pt$, we have

$$\lambda : \mu : \nu = \cos p\left(t - \frac{r}{c}\right) : \sin p\left(t - \frac{r}{c}\right) : 0.$$

At the point $(0, 0, z)$ equations (24) give

$$E_x : E_y : E_z = \lambda : \mu : 0 = \cos p\left(t - \frac{z}{c}\right) : \sin p\left(t - \frac{z}{c}\right) : 0.$$

This represents a right-handed circular polarisation in a plane parallel to $z = 0$. Similarly at $(x, 0, 0)$ we find $E_x = E_z = 0$, so that the direction of polarisation is the axis of y.

The left-handed circular vibration gives similar results, and thus the polarisations in the normal Zeeman effect are completely explained.

254. Theory of electrical and optical phenomena in moving media. In all our previous work, such as the formulation and discussion of the equations of the free ether or of Lorentz's equations, we have referred everything to a system of coordinate axes which for the sake of brevity may be described as fixed. Now just as in mechanics there is no reason to suppose that any particular set of axes is better or more fundamental than another, so here it is idle to discuss whether the set of axes chosen is "fixed in space" or not. But a peculiar difficulty arises in connexion with the theory of electricity, as hitherto developed. In mechanics, if one set of axes A is selected for referring everything to in the first place, the fundamental laws are not altered by taking a set of axes B instead, provided that the relative motion of B and A is a uniform motion in a straight line. In other words, the equations of mechanics are not changed by the transformation

$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t \ldots\ldots\ldots (30),$$

where v is some constant. In considering the effect of this transformation on Lorentz's equations it must be borne in mind that the electric and magnetic forces are not the same in the fixed and moving systems. The charges which the moving observer would consider at rest have a velocity v with respect to the "fixed" observer, and the new electric force E' is the mechanical force per unit charge; that is, $E + \frac{1}{c}$ times the vector product of v and H. Hence we should write

$$E_x' = E_x - \frac{vH_y}{c}, \quad E_y' = E_y + \frac{vH_x}{c}, \quad E_z' = E_z \ldots (31).$$

Further, in estimating the magnetic field (as revealed, for example, by the mutual action of electric currents) the moving observer would not allow for the fact that the charges have an additional

motion of translation with velocity v. This motion has however a magnetic effect, and the components of the "true" magnetic force H differ from those of H' registered by the moving observer by the quantities in equation (20). Hence

$$H_x = H_x' - \frac{vE_y}{c}, \quad H_y = H_y' + \frac{vE_x}{c}, \quad H_z = H_z',$$

or
$$H_x' = H_x + \frac{vE_y}{c}, \quad H_y' = H_y - \frac{vE_x}{c}, \quad H_z' = H_z \quad ..(32).$$

The only remaining change is to write v_z' for $v_z - v$. Making all the transformations, it is easy to see that the form of Lorentz's equations is altered.

As far as this goes, therefore, there would seem to be one, and only one*, set of axes to which electrical phenomena can be referred without upsetting the ordinary laws of electricity. The difficulty thus arising is very grave. It would appear that electrical laws must be different, for example, in the morning and in the evening, in spring and in autumn, since the motion of the frame of reference changes with all these conditions. As no such difference has been observed it is imperative that the matter should be discussed further and that the rights of all axes to be considered equally fundamental should be as far as possible vindicated. The theory centering round this is known as the theory of relativity.

Let us first of all view everything from the standpoint of a "fixed" observer. Suppose that we have a medium moving with velocity v along the axis of z, and that there are observers in it engaged in measuring the velocity of light. Two points A, B (Fig. 369) are chosen, and signals are sent from A to B and back again. The signal which arrives at B at time t came from a position A' occupied by A at some previous time $t - \tau$, where $A'A = v\tau$ and $A'B = c\tau$. If $AB = r$ and $\angle A'AB = \alpha$, the trangle $AA'B$ gives

$$c^2\tau^2 = v^2\tau^2 + r^2 - 2vr\tau \cos\alpha \quad \ldots\ldots\ldots(33).$$

* Here we are only considering axes as distinct when they are in relative motion. The distinction between the possible sets of axes at rest with respect to each other is trivial.

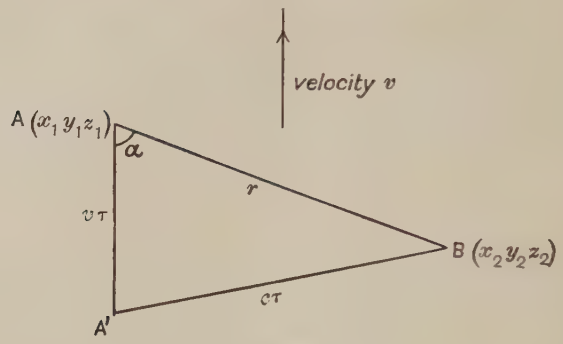


Fig. 369

If v is so small that v^2/c^2 is negligible, we have approximately

$$\tau^2 = \frac{r^2}{c^2} - \frac{2vr\tau\cos\alpha}{c^2} = \frac{r^2}{c^2} - \frac{2vr^2\cos\alpha}{c^3},$$

or

$$\tau = \frac{r}{c} - \frac{vr\cos\alpha}{c^2}$$

$$= \frac{r}{c} - \frac{v}{c^2}(z_1 - z_2).$$

The return signal from B similarly requires a time

$$\tau' = \frac{r}{c} + \frac{v}{c^2}(z_1 - z_2).$$

The total time $\tau + \tau' = 2r/c$, which is the same as if both A and B were at rest. Hence the moving observers would conclude that the velocity of light in all directions is c, and they would have no reason to suspect that they are in motion at all. We shall suppose that the clocks in the moving system are set by optical or electrical signals (e.g. wireless telegraphy) from a central point A. If the clock at A is started off and a signal sent at the same time, it is received at B at a time

$$\tau - \frac{r}{c} - \frac{v}{c^2}(z_1 - z_2).$$

The observer at B, allowing for the velocity of light, apparently determined as c in all directions, will set his clock so that it marks a time r/c on receipt of the signal. Hence when A's clock marks a time τ, B's clock marks a time r/c. Adding $t - \frac{r}{c} - \frac{vz_2}{c^2}$ to each,

we see that when A's clock marks $t - \frac{vz_1}{c^2}$, B's clock marks $t - \frac{vz_2}{c^2}$.

The time $t' = t - \frac{vz}{c^2}$ is called the *local time* at the point (x, y, z). Since the only way of correlating times at different places is by signalling, the local time has a perfect right to be considered as the true time by a moving observer.

Now consider what Lorentz's equations become when referred to moving axes and local time. Writing

$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t - \frac{vz}{c^2} \ldots\ldots (34),$$

we have to find what the moving observer will regard as the velocity and volume-density of electricity. We have

$$v_x' = \frac{dx'}{dt'} = \frac{dx}{dt - \frac{v\,dz}{c^2}} = \frac{v_x}{1 - \frac{vv_z}{c^2}}.$$

Similarly
$$v_y' = \frac{v_y}{1 - \frac{vv_z}{c^2}}, \quad v_z' = \frac{v_z - v}{1 - \frac{vv_z}{c^2}}.$$

The electrical density ρ' for the moving observer will be defined so that $\rho' v_x'$ is still the x-component of current-density with reference to his moving axes. This gives

$$\left.\begin{aligned} \rho' v_x' = \rho v_x, \quad \rho' v_y' = \rho v_y, \quad \rho' v_z' = \rho (v_z - v) \\ \rho' = \rho \left(1 - \frac{vv_z}{c^2}\right) \end{aligned}\right\} \ldots (35).$$

Equations (31), (32), (34), (35) give all the data for transforming Lorentz's equations. If v^2/c^2 is neglected throughout, we may write

$$\left.\begin{aligned} \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - \frac{v}{c^2}\frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial z'} \\ E_x = E_x' + \frac{vH_y'}{c}, \quad E_y = E_y' - \frac{vH_x'}{c}, \quad E_z = E_z' \\ H_x = H_x' - \frac{vE_y'}{c}, \quad H_y = H_y' + \frac{vE_x'}{c}, \quad H_z = H_z' \\ \rho v_x = \rho' v_x', \quad \rho v_y = \rho' v_y', \quad \rho v_z = \rho' v_z' + \rho' v \\ \rho = \rho' \left(1 + \frac{vv_z'}{c^2}\right) \end{aligned}\right\}.$$

Carrying out the substitutions and suppressing the accents, we find that Lorentz's equations (3), (4), (5), (6) become

$$\left.\begin{aligned}
\frac{1}{c}\left(\frac{\partial E_x}{\partial t}+4\pi\rho v_x\right) &= \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z} \\
\frac{1}{c}\left(\frac{\partial E_y}{\partial t}+4\pi\rho v_y\right) &= \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x} \\
\frac{1}{c}\left(\frac{\partial E_z}{\partial t}+4\pi\rho v_z\right) &= \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}+\frac{v}{c}\left(\frac{\partial E_x}{\partial x}-\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}-4\pi\rho\right)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
-\frac{1}{c}\frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} \\
-\frac{1}{c}\frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x} \\
-\frac{1}{c}\frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}-\frac{v}{c}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)
\end{aligned}\right\},$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}-4\pi\rho=\frac{v}{c}\left(\frac{1}{c}\frac{\partial E_z}{\partial t}+\frac{4\pi\rho v_z}{c}-\frac{\partial H_y}{\partial x}+\frac{\partial H_x}{\partial y}\right),$$

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=\frac{v}{c}\left(\frac{1}{c}\frac{\partial H_z}{\partial t}+\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right).$$

To the first order, however, these equations are the same as before. Hence Lorentz's equations preserve their form unaltered in the moving system, provided that the time is taken to be the *local* time at every point. Since Lorentz's equations can be made the basis of all electrical theory the argument is general, and thus we see that there is no insuperable difficulty about reconciling electric phenomena in a moving system with the axiom of relativity, as long as we only consider velocities which are small compared with the velocity of light.

The above theory is due essentially to Lorentz and Einstein. Lorentz was the first to introduce the idea of *local time*: Einstein is responsible for the view that Lorentz's local time is, to all intents and purposes, the *true* time at any point, since from what has been said above the observer cannot detect his own motion by measurements of the velocity of light.

255. Theory of Fizeau's experiment. The merit of the above theory, however, does not lie merely in the fact that it

explains the negative results of certain experiments (which might conceivably be done in other ways): it explains at the same time a remarkable experiment of Fizeau's. By observing a shift in a set of interference fringes caused by the motion of a stream of water in the path of a ray of light, Fizeau was able to estimate the speed at which light travelled within the moving water itself. If c is the velocity of light *in vacuo*, μ the refractive index and v the velocity of the water, then the observed velocity of the waves was not $\frac{c}{\mu} + v$, as we might expect, but $\frac{c}{\mu} + kv$, where k was certainly less than unity.

The electrons of a dielectric are attached to their atoms by forces whose origin is uncertain; but we shall assume that they are not, to the first order, affected by a uniform motion of translation of the whole system. Under these circumstances the velocity of propagation of the waves, as registered by the moving observer, will be $c/\sqrt{K}$, where K is the dielectric constant when at rest. That is, the points at which any particular phase of a wave travelling along the z axis occurs are connected by the equation $z' - ct'/\sqrt{K} = \text{constant}$. In terms of z and t, this becomes

$$z - vt - \frac{c}{\sqrt{K}}\left(t - \frac{vz}{c^2}\right) = \text{const.},$$

or

$$z\left(1 + \frac{v}{c\sqrt{K}}\right) - t\left(v + \frac{c}{\sqrt{K}}\right) = \text{const.}$$

It follows that the velocity *as perceived by an observer at rest* is

$$\frac{v + \frac{c}{\sqrt{K}}}{1 + \frac{v}{c\sqrt{K}}},$$

or, as only first order quantities are being considered,

$$\frac{c}{\sqrt{K}} + \left(1 - \frac{1}{K}\right)v.$$

Putting $K = \mu^2$, the velocity becomes $\frac{c}{\mu} + \left(1 - \frac{1}{\mu^2}\right)v$. In this it has been assumed that K and μ are constants; but the theory

will hold approximately for water, for which μ is nearly constant over the visible spectrum. Putting $\mu = \frac{4}{3}$, the coefficient $1 - \frac{1}{\mu^2}$ becomes 0·438. Michelson and Morley, who repeated Fizeau's experiment, found the value $0{\cdot}434 \pm 0{\cdot}02$, which agrees well with the above.

Experiments like Fizeau's, in which the velocity of light in a moving medium is measured by a fixed observer, should, of course, not be confused with those in which the observer and all his apparatus move with the same velocity. It is to these latter that the null experiments apply.

256. The Lorentz-Einstein transformation. Difficulties arise in the theory of moving media when the velocity is comparable with that of light. If AB (Fig. 369) is parallel to the direction of motion, and if no approximations are made, the time $\tau + \tau'$ for a direct and return signal is found from equation (33) to be

$$\frac{2rc}{c^2 - v^2} = \frac{2r}{c} + \frac{2rv^2}{c^3} + \ldots,$$

while if AB is at right angles to the direction of motion it is

$$\frac{2r}{(c^2 - v^2)^{\frac{1}{2}}} = \frac{2r}{c} + \frac{rv^2}{c^3} + \ldots.$$

The difference is rv^2/c^3, correct to terms of the second order in v/c. Hence it would seem that an observer at A must obtain different values for the velocity of light in different directions, which would contradict the axiom of relativity. Michelson and Morley's celebrated experiment* is really one to detect a difference in the velocities of propagation in different directions with respect to the direction of the earth's motion, if it exists. No such difference was found, although the method was sensitive enough to reveal effects of the order v^2/c^2, where v is the velocity of the earth in its orbit and c the velocity of light.

It follows that as long as the equations

$$x' = x, \quad y' = y, \quad z' = z - vt$$

are retained no transformation can be found which will leave the

* See Wood's *Physical Optics*, 2nd ed., p. 672; or Michelson, *Light Waves and their Uses*, p. 158.

fundamental equations unaltered. Einstein, following Lorentz, proposed the transformation

$$\left.\begin{aligned} x' = x, \quad y' = y, \quad z' = \beta\,(z - vt) \\ t' = \beta\left(t - \frac{vz}{c^2}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots (36),$$

where

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (37).$$

If we take in addition

$$\left.\begin{aligned} E_x' = \beta\left(E_x - \frac{vH_y}{c}\right), \quad E_y' = \beta\left(E_y + \frac{vH_x}{c}\right), \quad E_z' = E_z \\ H_x' = \beta\left(H_x + \frac{vE_y}{c}\right), \quad H_y' = \beta\left(H_y - \frac{vE_x}{c}\right), \quad H_z' = H_z \\ \rho' v_x' = \rho v_x, \quad \rho' v_y' = \rho v_y, \quad \rho' v_z' = \beta\rho\,(v_z - v) \\ \rho' = \beta\rho\left(1 - \frac{vv_z}{c^2}\right) \end{aligned}\right\} \ldots (38),$$

Lorentz's equations are left unaltered *to all orders of approximation*. For velocities so small that v^2/c^2 is negligible, $\beta = 1$ and the transformation reduces to the one in Art. 254. The more general transformation, with certain implications, includes the negative result of all experiments made to detect the optical effect of the earth's motion by experiments carried out entirely on its surface, without restriction as to the magnitude of its velocity. Since the velocity of the whole solar system "in space" is probably at least as great as that of the earth in its orbit, the cogency of the null experiments is greater than might appear.

If equations (36) and (38) are solved for $x, y, z, \ldots$ in terms of $x', y', z', \ldots$, we find

$$\left.\begin{aligned} x = x', \quad y = y', \quad z = \beta\,(z' + vt') \\ t = \beta\left(t' + \frac{vz'}{c^2}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots (39),$$

$$\left.\begin{aligned} E_x = \beta\left(E_x' + \frac{vH_y'}{c}\right), \quad E_y = \beta\left(E_y' - \frac{vH_x'}{c}\right), \quad E_z = E_z' \\ H_x = \beta\left(H_x' - \frac{vE_y'}{c}\right), \quad H_y = \beta\left(H_y' + \frac{vE_x'}{c}\right), \quad H_z = H_z' \\ \rho v_x = \rho' v_x', \quad \rho v_y = \rho' v_y', \quad \rho v_z = \beta\rho'\,(v_z' + v) \\ \rho = \beta\rho'\left(1 + \frac{vv_z'}{c^2}\right) \end{aligned}\right\} \ldots (40),$$

which are of precisely the same form as the above, with $-v$ written for v. The relation of the two sets of axes to each other is therefore perfectly reciprocal, so that either set can be regarded as fixed and the other moveable, or neither fixed, at will.

To illustrate the formulae of transformation consider an electrical system moving with velocity v in the direction of the axis of z. This body appears at rest to the moving observer. If P is a point whose coordinates in the two systems at a given time t are (x_1, y_1, z_1) and (x_1', y_1', z_1') respectively, and Q a point whose coordinates are (x_2, y_2, z_2), (x_2', y_2', z_2'), we have from (36)

$$x_2' - x_1' = x_2 - x_1, \quad y_2' - y_1' = y_2 - y_1, \quad z_2' - z_1' = \beta (z_2 - z_1).$$

It follows that lengths measured in the direction of motion are differently judged by the two observers, the estimate of the moving observer being β times that of the fixed observer. Corresponding elements of volume are related by the equation $d\tau' = \beta d\tau$. Moreover, (38) gives in this case $\rho' = \beta\rho \left(1 - \frac{v^2}{c^2}\right) = \frac{\rho}{\beta}$, so that $\rho' d\tau' = \rho d\tau$. This shows that it is the same charge which is being considered in both cases.

The interpretation of the above theory has given rise to much discussion. Lorentz considered it best to take one particular set of axes as the fundamental one, and to assume that any material system moving with velocity v contracts in the direction of motion in the ratio β^{-1} or $(1 - v^2/c^2)^{\frac{1}{2}}$ to unity. Thus an electron which is spherical when at rest becomes an ellipsoid when in motion, and for the swiftest β particles approximates to a flat plate. The flattening of ordinary bodies could not be detected by an observer carried along with them, since his rulers and callipers would be distorted according to the same law, and would continue to fit at all speeds. The difference in the estimates of length, already referred to, is a consequence of the contraction of the measuring instruments.

Lorentz's representation has the merit of clearly fixing one's ideas, which a general theory of relativity does not; but for the reasons given above it cannot be said that the contraction actually takes place. Recognising this, Einstein preferred to begin with the dictum that all motion is relative, and merely to seek a

transformation which leaves Lorentz's equations unaltered. Any such transformation is then an expression of the principle of relativity, provided that the transformed space and time are considered to have objective reality.

It is sometimes said that the theory of relativity does away with the ether. But since it does not do away with electromagnetic disturbances propagated with a velocity $c = 3 \times 10^{10}$ cm. per second, it is difficult to see what is meant by the statement.

257. Electromagnetic mass of an electron at any speed. Suppose that an electron is moving with any velocity less than that of light, and that its acceleration and other derivates of the velocity are small. Take axes so that at time t it has a velocity v along the axis of z. To an observer moving with velocity v it will appear like an accelerated electron instantaneously at rest, and therefore will behave like a particle of constant mass m_0. Hence the equations of motion are

$$\left.\begin{aligned} m_0 \frac{d^2x'}{dt'^2} &= eE_x' = e\beta\left(E_x - \frac{vH_y}{c}\right) \\ m_0 \frac{d^2y'}{dt'^2} &= eE_y' = e\beta\left(E_y + \frac{vH_x}{c}\right) \\ m_0 \frac{d^2z'}{dt'^2} &= eE_z' = eE_z \end{aligned}\right\},$$

where E, H are the fields other than those produced by the motion of the electron itself. Now

$$\frac{dx'}{dt'} = \frac{dx}{\beta\left(dt - \frac{v\,dz}{c^2}\right)} = \frac{\dot{x}}{\beta\left(1 - \frac{v\dot{z}}{c^2}\right)},$$

and

$$\frac{dt'}{dt} = \beta\left(1 - \frac{v\dot{z}}{c^2}\right).$$

Hence

$$\frac{d^2x'}{dt'^2} = \frac{d}{dt}\left(\frac{dx'}{dt'}\right)\frac{dt}{dt'} = \frac{\left(1 - \frac{v\dot{z}}{c^2}\right)\ddot{x} + \dot{x}\,\frac{v\ddot{z}}{c^2}}{\beta^2\left(1 - \frac{v\dot{z}}{c^2}\right)^3}.$$

Since $\dot{x} = \dot{y} = 0$, $\dot{z} = v$ at the time considered, we have

$$\frac{d^2x'}{dt'^2} = \beta^2\ddot{x}.$$

Similarly $$\frac{d^2y'}{dt'^2} = \beta^2 \ddot{y}, \quad \frac{d^2z'}{dt'^2} = \beta^3 \ddot{z}.$$

The equations of motion, referred to the "fixed" axes, are therefore

$$\left.\begin{aligned} m_0 \beta\, \ddot{x} &= e\left(E_x - \frac{vH_y}{c}\right) \\ m_0 \beta\, \ddot{y} &= e\left(E_y + \frac{vH_x}{c}\right) \\ m_0 \beta^3 \ddot{z} &= eE_z \end{aligned}\right\}.$$

The terms on the right are the external forces on the moving electron, and the coefficients on the left are different according as the acceleration is along or at right angles to its instantaneous direction of motion. For transverse accelerations the electron behaves as if it had a *transverse electromagnetic mass*

$$m_t = m_0\beta = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots (41).$$

For accelerations in the direction of motion it behaves as if it had a *longitudinal electromagnetic mass*

$$m_l = m_0\beta^3 = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots (42).$$

If the acceleration makes an angle θ with the direction of motion, the mass is $m_l \cos\theta + m_t \sin\theta$.

Abraham was the first to point out that a swiftly moving electron, such as a β particle, has two distinct electromagnetic masses. In most cases the accelerations produced experimentally have been at right angles to the direction of motion, so that only the transverse mass comes into play. This is the case with Bucherer's experiments (Art. 241). How a particle with other than electromagnetic mass, subjected to non-electromagnetic forces, would behave when moving with a velocity comparable with that of light, is of course uncertain. But since Lorentz's formula (41) agrees accurately with Bucherer's experiments, we conclude that the particular form of the principle of relativity here used is a correct one, and that *the mass of the electron is entirely electromagnetic.*

258. The theory of quanta. There are a number of phenomena connected with short electric waves in the ether (ultra-violet light, Röntgen rays, γ rays) which seem sufficiently persistent to justify their being brought provisionally under a general rule.

The maximum velocity of emission of electrons under the action of ultra-violet light is determined not by the intensity but by the frequency of the light. Thus when a source of ultra-violet light is moved away from a metal plate fewer electrons are emitted, but what are have the same velocity as before. This peculiarity, as Bragg pointed out, suggests rather a flight of particles than a wave which becomes weaker as it spreads out from the source. The greater the frequency of the light, the greater the velocity of the fastest electrons. Hughes' experiments (Art. 218) might be explained by supposing that when light of frequency n falls on matter energy is transferred to an electron, if at all, in finite portions of hn ergs at a time, of which at least eV_0 is lost by the electron in escaping from the metal. The constant h is nearly the same for all metals, and approximately $5{\cdot}6 \times 10^{-27}$.

The above would seem a very slender basis for speculation, were it not that the same considerations apply to the emission of electrons by Röntgen rays and γ rays. Applying them to the case of the K radiation of nickel falling on iron (Art. 231) we find that the velocity of emission of electrons should be of the order of 5×10^9 cm. per second, which is actually what is observed. The relation of wave and electron is also to a certain extent reciprocal. An electron must have a certain minimum velocity if it is to excite the characteristic Röntgen rays of a given element, and the velocity does not differ much from that of the electrons set free when the rays fall on a metal plate. Similarly (Art. 243) γ rays are always found together with fairly swift β rays, and themselves produce β rays of similar velocity when they fall on matter.

These facts have led to what is known as the *quantum hypothesis*, which for our present purposes may be enunciated as follows: A simple-harmonic ether wave of frequency n communicates energy to matter in finite portions, or *quanta*, of hn ergs at a time,

where h is a universal constant which is about 6×10^{-27}. Such a wave cannot be excited by the impact of a body of kinetic energy less than hn.

There is no doubt that this hypothesis represents the experimental facts over a considerable range. Its physical interpretation is more obscure; and in particular nothing is said about waves of other than simple-harmonic form. The diffraction phenomena lend no support to any corpuscular theory of light or Röntgen rays. It would seem most probable, therefore, that the cause of the finite transfer of energy is to be found in the relation of ether to matter rather than in any discontinuous structure of the ether itself.

The quantum hypothesis, first suggested by Planck, can also be approached from other directions, for example from the law of variation of the specific heats of bodies at low temperatures. The consideration of these cases would lead us too far. From what has been said it is clear that no appreciable disturbance would be caused by introducing the hypothesis into mechanics, since the quantity hn is exceedingly small for all ordinary values of n. But the exact bearing of the theory on the fundamental laws of mechanics and electricity is at present uncertain, and further developments on this point must be awaited.

APPENDIX

UNITS AND NOTATION

The *electrostatic system* is based on the unit charge of electricity, which repels a similar charge placed 1 cm. away with a force of 1 dyne.

The *electromagnetic system* is based on the unit magnetic pole, which repels a similar pole placed 1 cm. away with a force of 1 dyne.

The ratio of the electromagnetic to the electrostatic unit of charge is denoted by c. Its value is very nearly 3×10^{10} C.G.S. units.

The *derived electromagnetic system* is obtained by taking multiples or submultiples of the true electromagnetic units as new units, as shown in the following scheme:

1 ampere	$= \frac{1}{10}$	E.M.U.	of current
1 coulomb	$= \frac{1}{10}$	,,	charge
1 volt	$= 10^8$	,,	potential
1 ohm	$= 10^9$	,,	resistance
1 farad	$= 10^{-9}$	,,	capacity
1 henry	$= 10^9$	,,	inductance.

The next table shows the notation used in this book for the commoner electrical quantities. The use of a single letter indicates the system of units in which any given quantity is usually expressed. Thus unless otherwise stated, *charge* is expressed in electrostatic units, *current* in electromagnetic units, and so on. The usage for potential and capacity varies. The table shows how the measure of a given quantity changes when the system of units is changed; thus 1 E.S.U. of capacity $= 1/9 \times 10^{11}$ farads, 1 E.S.U. of potential = 300 volts, and so on.

Quantity	Measure in E.S.U.	Measure in E.M.U.	Measure in derived E.M.U.	Name of derived unit
Charge	e	e/c	$10\,e/c = e/3\times10^9$	Coulomb
Volume- and surface-density	ρ, σ	—	—	—
Electric force ...	$E\,(E_x, E_y, E_z)$	cE	—	—
Potential (E.S.U.) ...	V	cV	$cV/10^8 = 300\ V$	Volt
Capacity (E.S.U.) ...	C	C/c^2	$10^9 C/c^2 = C/9\times10^{11}$	Farad
Electric polarisation	$P\,(P_x, P_y, P_z)$	—	—	—
Electric induction	$D(D_x, D_y, D_z)$	—	—	—
Dielectric constant	K	—	—	—
Magnetic pole ...	—	m	—	—
Magnetic force ...	—	$H(H_x, H_y, H_z)$	—	—
Magnetic potential	—	Ω	—	—
Intensity of magnetisation	—	$I\,(I_x, I_y, I_z)$	—	—
Magnetic induction	—	$B\,(B_x, B_y, B_z)$	—	—
Current	ci	i	$10i$	Ampere
Current-density ...	—	$j\,(j_x, j_y, j_z)$	—	—
Electromotive force (potential)	V/c	V	$V/10^8$	Volt
Capacity (E.M.U.) ...	c^2C	C	$10^9\,C$	Farad
Resistance	—	R	$R/10^9$	Ohm
Specific resistance	—	k	$k/10^9$	—
Conductivity ...	—	σ	$10^9\,\sigma$	—
Flux of magnetic induction	—	N	—	—
Self-inductance ...	—	L	$L/10^9$	Henry
Mutual inductance	—	M	$M/10^9$	Henry

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